Univalence Criteria for some General Integral Operators

Camelia Bărbatu and Daniel Breaz

Abstract

The main object of this paper is to extend the univalent condition for two general integral operators. A number of known univalent condition would follow upon specializing the parameters involved in our main results.

1 Introduction and preliminaries

Let $A$ denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$$

and satisfy the following usual normalization conditions:

$$f(0) = f'(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.
We denote by $S$ the subclass of $A$ consisting of functions $f \in A$, which are univalent in $U$.

In this paper, we obtain new univalence conditions for the general integral operators $M_n$ and $C_n$.

We consider the integral operator:

$$M_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i} - 1 \right] \left( g_i'(t) \right)^{\beta_i} \left( \frac{g_i(t)}{t} \right)^{\gamma_i} \right\}^{\frac{1}{\delta}}, \quad (1)$$

where $f_i, g_i$ are analytic in $U$, and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i = 1, n$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\text{Re}\delta > 0$.

**Remark 1.1.** The integral operator $M_n$ defined by (1), introduced by Bărbatu and Breaz in the paper [1], is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [6]

$$T_\alpha(z) = \int_0^z \left( \frac{f(t)}{t} \right)^{\alpha} dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18]

$$G_\alpha(z) = \int_0^z (f'(t))^{\alpha} dt.$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [4]

$$D_n(z) = \left[ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, S. Owa and N. Breaz [5]

$$I_n(z) = \left[ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left( f_i'(t) \right)^{\alpha_i} \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [16].
v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was defined and studied by Frasin [7]

$$\mathcal{T}_n(z) = \left[ \delta \int_0^z t^{\delta - 1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \left( f_i'(t) \right)^{\beta_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [11].

vi) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was studied by Ularu in [21]

$$I_n(z) = \left[ \delta \int_0^z t^{\delta - 1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \left( g_i'(t) \right)^{\beta_i} dt \right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $M_n$, introduced here by the formula (1), can be considered as an extension and a generalization of these operators above mentioned.

Now we consider the integral operator:

$$C_n(z) = \left\{ \delta \int_0^z t^{\delta - 1} \prod_{i=1}^n \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \left( g_i'(t) \right)^{\beta_i} \left( h_i(t) \right)^{\gamma_i} \right] dt \right\}^{\frac{1}{\delta}},$$

(2)

where $f_i, g_i, h_i$ are analytic in $U$ and $\delta, \alpha_i, \beta_i, \gamma_i$ are complex numbers for all $i = 1, n, n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\text{Re} \delta > 0$.

This general integral operator was introduced by Bărbatu and Breaz in the paper [2], is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

**Remark 1.2.** i) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \beta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [6]

$$\mathcal{F}_\alpha(z) = \int_0^z \left( \frac{f(t)}{t} \right)^{\alpha} dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18]

$$\mathcal{G}_\alpha(z) = \int_0^z \left( f'(t) \right)^{\alpha} dt.$$
iii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [4]

\[ \mathcal{D}_n(z) = \left[ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \, dt \right]^{\frac{1}{\delta}}, \]

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [5]

\[ \mathcal{I}_n(z) = \left[ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[ f_i'(t) \right]^{\alpha_i} \, dt \right]^{\frac{1}{\delta}}, \]

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [16].

v) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Frasin [7]

\[ \mathcal{F}_n(z) = \left[ \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\alpha_i} \left( f_i'(t) \right)^{\beta_i} \, dt \right]^{\frac{1}{\delta}}, \]

this integral operator is a generalization of the integral operator introduced by Ovesea in [11].

vi) For $n = 1$, $\delta = \beta$ and $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by Stanciu in [20]

\[ \mathcal{H}_1(z) = \left[ \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} e^{\gamma(t)} \right)^{\alpha} \, dt \right]^{\frac{1}{\beta}}. \]

Thus, the integral operator $\mathcal{E}_n$, introduced here by the formula (2), can be considered as an extension and a generalization of these operators above mentioned.

The following univalent conditions was derived by Pescar.

**Theorem 1.3.** (Pescar [14]) Let $\gamma$ be complex number, $\Re \gamma > 0$ and $c$ a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \ldots$. If

\[ |c| |z|^{2\Re \gamma} + \frac{1 - |z|^{2\Re \gamma}}{\Re \gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \]
for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t) \, dt\right)^{\frac{1}{\gamma}},$$

is regular and univalent in $S$.

**Theorem 1.4.** (Pescar [14]) Let $\gamma$ be a complex number, $\Re \gamma > 0$ and $c$ a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in A$. If

$$|c||z|^{2\Re \gamma} + \frac{1}{\Re \gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then for any complex $\gamma$ with $\Re \gamma \geq \Re \delta$, the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t) \, dt\right)^{\frac{1}{\gamma}},$$

is in the class $S$.

In [22], it is defined the class $S(p)$, which for $0 < p \leq 2$, includes the functions $f \in A$ which satisfy the conditions:

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z| < 1 \quad (3)$$

and

$$\left| \frac{z^{m} f'(z)}{f(z)^2} \right| \leq p \quad (4)$$

for all $z \in \mathbb{U}$.

**Theorem 1.5.** (Singh [19]) If $f \in S(p)$, then the following inequality is true

$$\left| \frac{z^2 f'(z)}{|f(z)|^2} - 1 \right| \leq p |z|^2 , z \in \mathbb{U} \quad (5)$$

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma.

**Lemma 1.6.** (General Schwarz Lemma [8]) Let $f$ be the function regular in the disk $\mathbb{U}_R = \{ z \in \mathbb{C} : |z| < R, R > 0 \}$ with $|f(z)| < M$ for a fixed number $M > 0$ fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} z^m , \quad z \in \mathbb{U}_R.$$
The equality for \( z \neq 0 \) can hold only if
\[
f(z) = e^{i\theta} \frac{M}{R^m} z^m,
\]
where \( \theta \) is constant.

2 Main results

Our main results give sufficient conditions for the general integral operators \( M_n \) and \( C_n \) to be univalent in the open disk \( U \).

**Theorem 2.1.** Let \( f_i, g_i \in A \), where \( g_i \) be in the class \( S(p_i) \), \( 0 < p_i \leq 2 \), \( M_i, N_i \) are real positive numbers and \( \delta, \alpha_i, \beta_i, \gamma_i, c \) be complex numbers for all \( i = \overline{1, n} \), with
\[
\text{Re} \delta > \sum_{i=1}^{n} \{|\alpha_i - 1|[(1 + p_i) M_i + 1] + |\beta_i| + |\gamma_i| [(1 + p_i) N_i + 1] \}, \tag{6}
\]
where \( |c| \leq 1 \), \( c \neq -1 \). If
\[
|f_i(z)| < M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{g''_i(z)}{g'_i(z)} \right| \leq 1,
\]
and
\[
|c| \leq 1 - \frac{1}{\text{Re} \delta} \sum_{i=1}^{n} \{|\alpha_i - 1|[(1 + p_i) M_i + 1] + |\beta_i| + |\gamma_i| [(1 + p_i) N_i + 1] \} \tag{7}
\]
for all \( z \in U \), \( i = \overline{1, n} \) then, the integral operator \( M_n \), defined by (1) is in the class \( S \).

**Proof.** Let us define the function
\[
M_n(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \left( g_i' (t) \right)^{\beta_i} \left( \frac{g_i(t)}{t} \right)^{\gamma_i} \, dt,
\]
for all \( f_i, g_i \in A \), \( i = \overline{1, n} \).

The function \( M_n \) is regular in \( U \) and satisfies the following normalization condition \( M_n(\theta) = M_n'(\theta) - 1 = 0 \).

We easily find that
\[
\frac{z M''_n(z)}{M'_n(z)} = \sum_{i=1}^{n} \left[ (\alpha_i - 1) \left( \frac{zf_i(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg''_i(z)}{g'_i(z)} + \gamma_i \left( \frac{zg'_i(z)}{g_i(z)} - 1 \right) \right],
\]
for all \( z \in U, i = 1, \ldots, n \), which readily shows that

\[
|c| |z|^{2Re\delta} + \frac{1 - |z|^{2Re\delta}}{2Re\delta} \left| z M''_n(z) \right| \leq |c| |z|^{Re\delta} + \\
+ \frac{1 - |z|^{2Re\delta}}{Re\delta} \sum_{i=1}^{n} \left[ |\alpha_i - 1| \left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{z g_i''(z)}{g_i'(z)} \right| + |\gamma_i| \left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| \right] \leq \\
\leq |c| + \frac{1}{Re\delta} \sum_{i=1}^{n} \left[ |\alpha_i - 1| \left( \left| \frac{z^2 f_i'(z)}{f_i(z)^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right) \right] + \\
+ \frac{1}{Re\delta} \sum_{i=1}^{n} \left[ |\beta_i| \left| \frac{z g_i''(z)}{g_i'(z)} \right| + |\gamma_i| \left( \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) \right] \leq \\
\leq |c| + \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[ \left( \left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} \right| - 1 \right) M_i + 1 \right] \right\} + \\
+ \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\beta_i| \cdot 1 + |\gamma_i| \left( \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} \right| - 1 \right) N_i + 1 \right\} \leq |c| + \\
+ \frac{1}{Re\delta} \sum_{i=1}^{n} \left[ |\alpha_i - 1| \left( p_i M_i |z|^2 + M_i + 1 \right) + |\beta_i| + |\gamma_i| \left( p_i N_i |z|^2 + N_i + 1 \right) \right] + \\
\leq |c| + \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left( (1 + p_i) M_i + 1 \right) + |\beta_i| + |\gamma_i| \left( (1 + p_i) N_i + 1 \right) \right\}. \quad (9)
\]

So, using (7) and (9), we have

\[
|c| |z|^{2Re\delta} + \frac{1 - |z|^{2Re\delta}}{2Re\delta} \left| z M''_n(z) \right| \leq 1
\]
for all \( z \in U, \ i = 1, n. \)

Finally, by applying Theorem 1.4, we conclude that, the general integral operator \( M_n \) given by (1) is in the class \( S. \)

**Theorem 2.2.** Let \( f_i, g_i, h_i \in \mathcal{A}, \) where \( g_i \) be in the class \( S(p_i), \) \( 0 < p_i \leq 2, \) \( M_i, N_i, P_i \) are real positive numbers and \( \delta, \alpha_i, \beta_i, \gamma_i, c \) be complex numbers for all \( i = 1, n, \) with

\[
Re\delta > \sum_{i=1}^{n} \{ |\alpha_i - 1| [M_i + N_i^2 (1 + p_i) + 1] + |\beta_i| + |\gamma_i| (P_i + 1) \}, \quad (10)
\]

where \( |c| \leq 1, \ c \neq -1. \) If

\[
\left| \frac{zf'_i(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{h''_i(z)}{h'_i(z)} \right| \leq P_i
\]

and

\[
|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \{ |\alpha_i - 1| [M_i + N_i^2 (1 + p_i) + 1] + |\beta_i| + |\gamma_i| (P_i + 1) \} \quad (11)
\]

for all \( z \in U, \ i = 1, n, \) then the integral operator \( C_n, \) defined by (2) is in the class \( S. \)

**Proof.** Let us define the function

\[
C_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i-1} \left( \frac{h'_i(t)}{h_i(t)} \right)^{\beta_i} \left( \frac{h_i(t)}{t} \right)^{\gamma_i} \right] dt,
\]

for all \( f_i, g_i, h_i \in \mathcal{A}, \ i = 1, n. \)

The function \( C_n \) is regular in \( U \) and satisfies the following normalization condition \( C_n(0) = C'_n(0) - 1 = 0. \)

We easily find that

\[
|c||z|^{2 Re\delta} + \frac{1 - |z|^{2 Re\delta}}{Re\delta} \left| \frac{zC''_n(z)}{C'_n(z)} \right| \leq 1
\]

\[
\leq |c||z|^{2 Re\delta} + \frac{1 - |z|^{2 Re\delta}}{Re\delta} \sum_{i=1}^{n} |\alpha_i - 1| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| +
\]

\[
+ \frac{1 - |z|^{2 Re\delta}}{Re\delta} \sum_{i=1}^{n} \left[ |zg'_i(z)| + |\beta_i| \left| \frac{zh''_i(z)}{h'_i(z)} \right| + |\gamma_i| \left| \frac{zh'_i(z)}{h_i(z)} - 1 \right| \right] \leq \]
\[
\begin{aligned}
\leq |c| + \frac{1}{\text{Re} \delta} \sum_{i=1}^{n} & \left[ |\alpha_i - 1| \left( \left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + \left| \frac{z^2 g_i'(z)}{g_i(z)^2} \right| \left| \frac{g_i(z)}{z} \right| \right] + \\
+ \frac{1}{\text{Re} \delta} \sum_{i=1}^{n} \left[ |\beta_i| \left| \frac{zh_i''(z)}{h_i'(z)} \right| + |\gamma_i| \left( \left| \frac{zh_i'(z)}{h_i(z)} \right| + 1 \right) \right].
\end{aligned}
\]

By applying the General Schwarz Lemma to the functions \( g_i, i = 1, n \) we obtain
\[
|g_i(z)| \leq N_i |z|,
\]

Since \( g_i \) be in the class \( S(p_i), 0 < p_i \leq 2 \) for all \( i = 1, n \) from (12) and hypothesis we obtain:
\[
|c| \leq \left| \frac{1}{2 \text{Re} \delta} \left[ \frac{zC_n''(z)}{C_n'(z)} \right] \right| \leq 1
\]

for all \( z \in U, i = 1, n \).

Finally, by applying Theorem 1.4, we conclude that, the general integral operator \( \mathfrak{C}_n \) given by (2) is in the class \( S \).

3 Corollaries and consequences

First of all, upon setting \( M_i = N_i = 1 \) in Theorem 2.1, we have the following corollary:

**Corollary 3.1.** Let \( f_i, g_i \in S(p_i), 0 < p_i \leq 2 \) and \( \delta, \alpha_i, \beta_i, \gamma_i, c \) are complex numbers, for all \( i = 1, n \), with
\[
\text{Re} \delta > \sum_{i=1}^{n} \left( (p_i + 2) \left( |\alpha_i - 1| + |\gamma_i| \right) + |\beta_i| \right), \quad |c| \leq 1.
\]
If
\[ |f_i(z)| < 1, \quad |g_i(z)| < 1, \quad \left| \frac{g''_i(z)}{g'_i(z)} \right| \leq 1, \]
and
\[ |c| \leq 1 - \frac{1}{\text{Res}} \sum_{i=1}^{n} \left[ (p_i + 2) (|\alpha_i - 1| + |\gamma_i|) + |\beta_i| \right] \]
(15)
for all \( z \in U, \ i = \overline{1, n} \) then, the integral operator \( M_n \), defined by (1) is in the class \( S \).

Letting \( n = 1, \delta = \gamma = \alpha \) and \( \alpha_1 - 1 = \beta_1 = \gamma_1 \) in Theorem 2.1, we obtain the next corollary:

**Corollary 3.2.** Let \( f, g \in S(p), \ 0 < p \leq 2, \ M, N \) are real positive numbers and \( \alpha, c \) complex numbers, with
\[ \text{Re} \alpha > |\alpha - 1| [(1 + p) M + (1 + p) N + 3], \quad |c| \leq 1. \] (16)
If
\[ |f(z)| < M, \quad |g(z)| < N, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq 1, \]
and
\[ |c| \leq 1 - \frac{1}{\text{Re} \alpha} > |\alpha - 1| [(1 + p) M + (1 + p) N + 3]. \] (17)
for all \( z \in U \) then, the integral operator \( M \), defined by
\[ M(z) = \left\{ \alpha \int_{0}^{z} \left[ \frac{f(t)g'(t)g(t)}{t} \right]^{\alpha - 1} \frac{dt}{t} \right\}^{\frac{1}{\alpha}}, \] (18)
is in the class \( S \).

Letting \( \delta = 1 \) and \( \gamma_i = 0 \) in Theorem 2.1, we obtain the following corollary:

**Corollary 3.3.** Let \( f_i, g_i \in A \), where \( g_i \) be in the class \( S(p_i) \), \( 0 < p_i \leq 2, \ M_i \) are real positive numbers and \( \alpha_i, \beta_i, c \) be complex numbers for all \( i = \overline{1, n} \), with
\[ 1 > \sum_{i=1}^{n} \{ |\alpha_i - 1| [(1 + p_i) M_i + 1] + |\beta_i| \}, \quad |c| \leq 1, \quad c \neq -1. \] (19)
If
\[ |f_i(z)| < M_i, \quad \left| \frac{g''_i(z)}{g'_i(z)} \right| \leq 1, \]
and

\[ |c| \leq 1 - \sum_{i=1}^{n} (|\alpha_i - 1|[1 + p_i] M_i + 1) + |\beta_i| \]  

(20)

for all \( z \in \mathbb{U} \), \( i = 1, n \) then, the integral operator \( F_n \), defined by

\[ F_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \left( g_i'(t) \right)^{\beta_i} dt \]  

(21)

is in the class \( S \).

**Remark 3.4.** The integral operator from Corollary 3.3, given by (21) is a known result proven in [21].

Letting \( \delta = 1 \) and \( \beta_i = 0 \) in Theorem 2.1, we have the following corollary:

**Corollary 3.5.** Let \( f_i, g_i \in A, M_i, N_i \) are real positive numbers and \( \alpha_i, \gamma_i, c \) be complex numbers for all \( i = 1, n \), with

\[ 1 > \sum_{i=1}^{n} \{|\alpha_i - 1|[1 + p_i] M_i + 1 + |\gamma_i|[1 + p_i] N_i + 1\}, \quad |c| \leq 1, \quad c \neq -1. \]  

(22)

If

\[ |f_i(z)| < M_i, \quad |g_i(z)| < N_i, \]

and

\[ |c| \leq 1 - \sum_{i=1}^{n} (|\alpha_i - 1|[1 + p_i] M_i + 1 + |\gamma_i|[1 + p_i] N_i + 1) \]  

(23)

for all \( z \in \mathbb{U} \), \( i = 1, n \) then, the integral operator \( G_n \), defined by

\[ G_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \left( g_i'(t) \right)^{\gamma_i} dt \]  

(24)

is in the class \( S \).

**Remark 3.6.** On the integral operator from Corollary 3.5, given by (24) if we take \( \alpha_i - 1 = \alpha_i \), we obtain another known result proven in [10].

Putting \( \delta = 1 \) and \( \alpha_i - 1 = 0 \) in Theorem 2.1, we obtain the following corollary:
Corollary 3.7. Let \(g_i \in A\), where \(g_i\) be in the class \(S(p_i)\), \(0 < p_i \leq 2\), \(N_i\) are real positive numbers and \(\beta_i, \gamma_i, c\) be complex numbers for all \(i = 1, n\), with

\[
1 > \sum_{i=1}^{n} \{|\beta_i| + |\gamma_i| \left[(1 + p_i) N_i + 1\right]\}, \quad |c| \leq 1, \quad c \neq -1.
\] (25)

If

\[
|g_i(z)| < N_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1,
\]

and

\[
|c| \leq 1 - \sum_{i=1}^{n} \{|\beta_i| + |\gamma_i| \left[(1 + p_i) N_i + 1\right]\} \quad \text{for all } z \in U, i = 1, n
\] (26)

then, the integral operator \(I_n\), defined by

\[
I_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[(g_i'(t))^\beta_i \left(\frac{g_i(t)}{t}\right)^\gamma_i\right] dt
\] (27)

is in the class \(S\).

Remark 3.8. The integral operator from Corollary 3.7, given by (27) was proven in [7].

Letting \(M_i = N_i = P_i = 1\) in Theorem 2.2, we obtain the following corollary:

Corollary 3.9. Let \(f_i, g_i, h_i \in S(p_i)\), \(0 < p_i \leq 2\) and \(\delta, \alpha_i, \beta_i, \gamma_i, c\) are complex numbers, for all \(i = 1, n\), with

\[
\text{Re} \delta > \sum_{i=1}^{n} \{(p_i + 3) |\alpha_i - 1| + |\beta_i| + 2 |\gamma_i|\}, \quad |c| \leq 1.
\] (28)

If

\[
\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq 1, \quad |g_i(z)| \leq 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq 1
\]

and

\[
|c| \leq 1 - \frac{1}{\text{Re} \delta} \sum_{i=1}^{n} \{(p_i + 3) |\alpha_i - 1| + |\beta_i| + 2 |\gamma_i|\},
\] (29)

for all \(z \in U, i = 1, n\) then, the integral operator \(C_n\), defined by (2) is in the class \(S\).
Letting \( n = 1, \delta = \gamma = \alpha \) and \( \alpha_1 - 1 = \beta_1 = \gamma_1 \) in Theorem 2.2, we have:

**Corollary 3.10.** Let \( f, g, h \in S(p), \) \( 0 < p \leq 2, \) \( M, N, P \) are real positive numbers and \( \alpha, c \) complex numbers, with

\[
\text{Re} \alpha > |\alpha - 1| \left[ M + N^2 (1 + p) + P + 3 \right], \quad |c| \leq 1. \tag{30}
\]

If

\[
\left| \frac{zf'(z)}{f(z)} \right| \leq M, \quad |g(z)| < N, \quad \left| \frac{h''(z)}{h'(z)} \right| \leq 1, \quad \left| \frac{zh'(z)}{h(z)} \right| \leq P
\]

and

\[
|c| \leq 1 - \frac{1}{\text{Re} \alpha} \left| \frac{\alpha - 1}{|\alpha - 1| \left[ M + N^2 (1 + p) + P + 3 \right]} \right| \tag{31}
\]

for all \( z \in \mathbb{U} \) then, the integral operators \( C, \) defined by

\[
C(z) = \left\{ \alpha \int_0^z \left[ f(t)e^{g(t)}h'(t)h(t) \right]^{\alpha-1} dt \right\}^{\frac{1}{\alpha}}, \tag{32}
\]

is in the class \( S. \)

Letting \( \delta = 1 \) and \( \gamma_i = 0 \) in Theorem 2.2, we obtain the next corollary:

**Corollary 3.11.** Let \( f_i, g_i, h_i \in A, \) where \( g_i \) be in the class \( S(p_i), \) \( 0 < p_i \leq 2, \) \( M_i, N_i \) are real positive numbers and \( \alpha_i, \beta_i, c \) complex numbers, for all \( i = 1, n, \)

\[
1 > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[ M_i + N_i^2 (1 + p_i) + 1 \right] + |\beta_i| \right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{33}
\]

If

\[
\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1
\]

and

\[
|c| \leq 1 - \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[ M_i + N_i^2 (1 + p_i) + 1 \right] + |\beta_i| \right\} \tag{34}
\]

for all \( z \in \mathbb{U}, i = 1, n \) then, the integral operator \( T_n \) defined by

\[
T_n(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i-1} (h_i'(t))^{\beta_i} dt, \tag{35}
\]

is in the class \( S. \)
Remark 3.12. The integral operator from Corollary 3.11, given by (35), if we take \( \beta_i = 0 \), we obtain a known result proven in [20].

Putting \( \delta = 1 \) and \( \beta_i = 0 \) in Theorem 2.2, we obtain the following corollary:

**Corollary 3.13.** Let \( f_i, g_i, h_i \in A \), where \( g_i \) be in the class \( S(p_i) \), \( 0 < p_i \leq 2 \), \( M_i, N_i, P_i \) are real positive numbers and \( \alpha_i, \gamma_i, c \) complex numbers, for all \( i = 1, n \), with

\[
1 > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[ M_i + N_i^2 (1 + p_i) + 1 \right] + |\gamma_i| (P_i + 1) \right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{36}
\]

If

\[
\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq P_i
\]

and

\[
|c| \leq 1 - \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[ M_i + N_i^2 (1 + p_i) + 1 \right] + |\gamma_i| (P_i + 1) \right\} \tag{37}
\]

for all \( z \in U, i = 1, n \) then, the integral operator \( R_n \), defined by

\[
R_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left( \frac{h_i(t)}{t} \right)^{\gamma_i} \right] dt, \tag{38}
\]

is in the class \( S \).

**Remark 3.14.** Putting \( \gamma_i = 0 \) in (38) we obtain another known result proven in [20].

Letting \( \delta = 1 \) and \( \alpha_i - 1 = 0 \) in Theorem 2.2, we obtain:

**Corollary 3.15.** Let \( h_i \in A \), \( P_i \) are real positive numbers and \( \beta_i, \gamma_i, c \) complex numbers, for all \( i = 1, n \), with

\[
1 > \sum_{i=1}^{n} \left\{ |\beta_i| + |\gamma_i| (P_i + 1) \right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{39}
\]

If

\[
\left| \frac{h_i''(z)}{h_i(z)} \right| \leq 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq P_i
\]

and

\[
|c| \leq 1 - \sum_{i=1}^{n} \left\{ |\beta_i| + |\gamma_i| (P_i + 1) \right\} \tag{40}
\]
for all $z \in \mathbb{U}$, $i = 1, n$ then, the integral operator $I_n$, defined by

$$I_n(z) = \int_0^z \prod_{i=1}^n \left[ \left( h_i'(t) \right)^\lambda_i \left( \frac{h_i(t)}{t} \right)^\gamma_i \right] dt,$$  

(41)

is in the class $S$.

**Remark 3.16.** The integral operator from Corollary 3.15, given by (41) was proven in [7].

**References**


Camelia Bărbatu,
Faculty of Mathematics and Computer Sciences,
“Babeș-Bolyai” University,
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania.
Email: camipode@yahoo.com

Daniel Breaz,
Department of Mathematics,
“1 Decembrie” University,
5, Gabriel Bethlen Street, 510009 Alba-Iulia, Romania.
Email: dbreaz@uab.ro