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Univalence Criteria for some General Integral Operators

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Abstract

The main object of this paper is to extend the univalent condition for two general integral operators. A number of known univalent condition would follow upon specializing the parameters involved in our main results.

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.

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We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

In this paper, we obtain new univalence conditions for the general integral operators $\mathcal{M}_n$ and $\mathcal{C}_n$.

We consider the integral operator:

$$\mathcal{M}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left({g_i}'(t) \right)^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}}, \quad (1)$$

where f_i, g_i are analytic in $\mathbb{U}$, and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

Remark 1.1. *The integral operator $\mathcal{M}_n$ defined by (1), introduced by Bărbatu and Breaz in the paper [1], is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:*

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [6]

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18]

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t) \right)^\alpha dt.$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [4]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, S. Owa and N. Breaz [5]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[f_i'(t) \right]^{\alpha_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [16] .

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was defined and studied by Frasin [7]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [11].

vi) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = 0$ we obtain the integral operator which was studied by Ularu in [21]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(g_i{}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $\mathcal{M}_n$, introduced here by the formula (1), can be considered as an extension and a generalization of these operators above mentioned.

Now we consider the integral operator:

$$\mathcal{C}_n(z) = \left\{\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)}\right)^{\alpha_i - 1} \left(h_i{}'(t)\right)^{\beta_i} \left(\frac{h_i(t))}{t}\right)^{\gamma_i}\right] \mathrm{dt}\right\}^{\frac{1}{\delta}}, \tag{2}$$

where f_i, g_i, h_i are analytic in $\mathbb{U}$ and $\delta, \alpha_i, \beta_i, \gamma_i$ are complex numbers for all $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$..

This general integral operator was introduced by Bărbatu and Breaz in the paper [2], is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

Remark 1.2. *i) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \beta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [6]*

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18]

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t)\right)^{\alpha} dt.$$

iii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [4]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [5]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[f_i'(t)\right]^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [16] .

v) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Frasin [7]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left({f_i}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [11].

vi) For $n = 1$, $\delta = \beta$ and $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by Stanciu in [20]

$$\mathcal{H}_1(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha} dt\right]^{\frac{1}{\beta}}.$$

Thus, the integral operator $\mathcal{C}_n$, introduced here by the formula (2), can be considered as an extension and a generalization of these operators above mentioned.

The following univalent conditions was derived by Pescar.

Theorem 1.3. *(Pescar [14]) Let γ be complex number, $Re\gamma > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. If*

$$|c| \, |z|^{2Re\gamma} + \frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)\,dt\right)^{\frac{1}{\gamma}},$$

is regular and univalent in $\mathbb{S}$.

Theorem 1.4. *(Pescar [14]) Let γ be complex number, $Re\gamma > 0$ and c a complex number, $|c| \le 1$, $c \ne -1$, and $f \in \mathcal{A}$. If*

$$|c|\,|z|^{2Re\gamma} + \frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zf''(z))}{f'(z)}\right| \le 1,$$

for all $z \in \mathbb{U}$, then for any complex γ with $Re\gamma \ge Re\delta$, the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)\,dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

In [22], it is defined the class $\mathcal{S}(p)$, which for $0 < p \le 2$, includes the functions $f \in \mathcal{A}$ which satisfy the conditions:

$$f(z) \ne 0 \quad for \quad 0 < |z| < 1 \tag{3}$$

and

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le p \tag{4}$$

for all $z \in \mathcal{U}$.

Theorem 1.5. *(Singh [19]) If $f \in \mathcal{S}(p)$, then the following inequality is true*

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| \le p\,|z|^2, z \in \mathbb{U} \tag{5}$$

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma.

Lemma 1.6. *(General Schwarz Lemma [8]) Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ with $|f(z)| < M$ for a fixed number* M > 0 *fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m} z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2 Main results

Our main results give sufficient conditions for the general integral operators $\mathcal{M}_n$ and $\mathcal{C}_n$ to be univalent in the open disk $\mathbb{U}$.

Theorem 2.1. *Let $f_i, g_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i are real positive numbers and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1, n}$, with*

$$Re\delta > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[(1 + p_i) M_i + 1 \right] + |\beta_i| + |\gamma_i| \left[(1 + p_i) N_i + 1 \right] \right\}, \tag{6}$$

where $|c| \leq 1$, $c \neq -1$. If

$$|f_i(z)| < M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq 1,$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[(1 + p_i) M_i + 1 \right] + |\beta_i| + |\gamma_i| \left[(1 + p_i) N_i + 1 \right] \right\} \tag{7}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ then, the integral operator $\mathcal{M}_n$, defined by (1) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$M_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(g_i{}'(t) \right)^{\beta_i} \left(\frac{g_i(t)}{t} \right)^{\gamma_i} \mathrm{dt},$$

for all $f_i, g_i \in \mathcal{A}$, $i = \overline{1, n}$.

The function M_n is regular in $\mathbb{U}$ and satisfies the following normalization condition $M_n(0) = M_n'(0) - 1 = 0$.

We easily find that

$$\frac{zM_n''(z)}{M_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} + \gamma_i \left(\frac{zg_i'(z)}{g_i(z)} - 1 \right) \right],$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, which readily shows that

$$|c|\,|z|^{2\mathrm{Re}\delta} + \frac{1-|z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \le |c|\,|z|^{\mathrm{Re}\delta} +$$

$$+\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right| + |\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right| + |\gamma_i|\left|\frac{zg_i'(z)}{g_i(z)}-1\right|\right] \le$$

$$\le |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right) + |\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right| + |\gamma_i|\left(\left|\frac{zg_i'(z)}{g_i(z)}\right|+1\right)\right]. \tag{8}$$

By applying the General Schwarz Lemma to the functions f_i, g_i, $i = \overline{1,n}$ we obtain:

$$|f_i(z)| \le M_i\,|z|, \quad |g_i(z)| \le N_i\,|z|.$$

Since g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \le 2$ for all $i = \overline{1,n}$ from (8) and hypothesis we obtain:

$$|c|\,|z|^{2\mathrm{Re}\delta} + \frac{1-|z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \le$$

$$\le |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)\right] +$$

$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right| + |\gamma_i|\left(\left|\frac{z^2g_i'(z)}{[g_i(z)]^2}\right|\left|\frac{g_i(z)}{z}\right|+1\right)\right] \le$$

$$\le |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right|+1\right)M_i+1\right]\right\} +$$

$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\beta_i|\cdot 1 + |\gamma_i|\left[\left(\left|\frac{z^2g_i'(z)}{[g_i(z)]^2}-1\right|+1\right)N_i+1\right]\right\} \le |c| +$$

$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(p_iM_i\,|z|^2+M_i+1\right) + |\beta_i| + |\gamma_i|\left(p_iN_i\,|z|^2+N_i+1\right)\right] +$$

$$\le |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[(1+p_i)\,M_i+1\right] + |\beta_i| + |\gamma_i|\left[(1+p_i)\,N_i+1\right]\right\}. \tag{9}$$

So, using (7) and (9), we have

$$|c|\,|z|^{2\mathrm{Re}\delta} + \frac{1-|z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta}\left|\frac{zM_n''(z)}{M_n'(z)}\right| \le 1$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Finally, by applying Theorem 1.4, we conclude that, the general integral operator $\mathcal{M}_n$ given by (1) is in the class $\mathcal{S}$. □

Theorem 2.2. *Let $f_i, g_i, h_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i, P_i are real positive numbers and $\delta, \alpha_i, \beta_i, \gamma_i, c$ be complex numbers for all $i = \overline{1,n}$, with*

$$Re\delta > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1 \right] + |\beta_i| + |\gamma_i| (P_i + 1) \right\}, \tag{10}$$

where $|c| \leq 1$, $c \neq -1$. If

$$\left| \frac{z f_i'(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} \right| \leq P_i$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1 \right] + |\beta_i| + |\gamma_i| (P_i + 1) \right\} \tag{11}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then the integral operator $\mathcal{C}_n$, defined by (2) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$C_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(h_i{}'(t) \right)^{\beta_i} \left(\frac{h_i(t)}{t} \right)^{\gamma_i} \right] \mathrm{dt},$$

for all $f_i, g_i, h_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function C_n is regular in $\mathbb{U}$ and satisfies the following normalization condition $C_n(0) = C_n'(0) - 1 = 0$.

We easily find that

$$|c| |z|^{2\mathrm{Re}\delta} + \frac{1 - |z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta} \left| \frac{z C_n''(z)}{C_n'(z)} \right| \leq$$

$$\leq |c| |z|^{\mathrm{Re}\delta} + \frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \sum_{i=1}^{n} |\alpha_i - 1| \left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| +$$

$$+ \frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \sum_{i=1}^{n} \left[\left| z g_i'(z) \right| + |\beta_i| \left| \frac{z h_i''(z)}{h_i'(z)} \right| + |\gamma_i| \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \right] \leq$$

$$\leq |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + \left|\frac{z^2 g_i^{'}(z)}{[g_i(z)]^2}\right|\left|\frac{[g_i(z)]^2}{z}\right|\right] +$$
$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}\right| + 1\right)\right]. \quad (12)$$

By applying the General Schwarz Lemma to the functions g_i, $i = \overline{1,n}$ we obtain

$$|g_i(z)| \leq N_i |z|,$$

Since g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$ for all $i = \overline{1,n}$ from (12) and hypothesis we obtain:

$$|c| |z|^{2\mathrm{Re}\delta} + \frac{1 - |z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta}\left|\frac{zC_n''(z)}{C_n'(z)}\right| \leq$$
$$\leq |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[M_i + 1 + \left(\left|\frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1\right| + 1\right)N_i^2\right]\right\} +$$
$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left(\left|\frac{z^2 h_i'(z)}{[h_i(z)]^2}\right|\left|\frac{zh_i(z)}{z}\right| + 1\right)\right] \leq$$
$$\leq |c| + \frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[M_i + N_i^2(p_i + 1) + 1\right] + |\beta_i| + |\gamma_i|(P_i + 1)\right\}. \quad (13)$$

So, using (11) and (13), we have

$$|c| |z|^{2\mathrm{Re}\delta} + \frac{1 - |z|^{2\mathrm{Re}\delta}}{2\mathrm{Re}\delta}\left|\frac{zC_n''(z)}{C_n'(z)}\right| \leq 1$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Finally, by applying Theorem 1.4, we conclude that, the general integral operator $\mathcal{C}_n$ given by (2) is in the class $\mathcal{S}$. $\square$

3 Corollaries and consequences

First of all, upon setting $M_i = N_i = 1$ in Theorem 2.1, we have the following corollary:

Corollary 3.1. *Let $f_i, g_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ and $\delta, \alpha_i, \beta_i, \gamma_i, c$ are complex numbers, for all $i = \overline{1,n}$, with*

$$Re\delta > \sum_{i=1}^{n}\left[(p_i + 2)(|\alpha_i - 1| + |\gamma_i|) + |\beta_i|\right], \quad |c| \leq 1. \quad (14)$$

If

$$|f_i(z)| < 1, \quad |g_i(z)| < 1, \quad \left|\frac{g_i^{''}(z)}{g_i^{'}(z)}\right| \leq 1,$$

and

$$|c| \leq 1 - \frac{1}{Re\delta}\sum_{i=1}^{n}\left[(p_i+2)\left(|\alpha_i - 1| + |\gamma_i|\right) + |\beta_i|\right] \tag{15}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *then, the integral operator* $\mathcal{M}_n$, *defined by (1) is in the class* $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Teorema 2.1, we obtain the next corollary:

Corollary 3.2. *Let* $f, g \in \mathcal{S}(p)$, $0 < p \leq 2$, M, N *are real positive numbers and* α, c *complex numbers, with*

$$Re\alpha > |\alpha - 1|\left[(1+p)M + (1+p)N + 3\right], \quad |c| \leq 1. \tag{16}$$

If

$$|f(z)| < M, \quad |g(z)| < N, \quad \left|\frac{g^{''}(z)}{g^{'}(z)}\right| \leq 1,$$

and

$$|c| \leq 1 - \frac{1}{Re\alpha} > |\alpha - 1|\left[(1+p)M + (1+p)N + 3\right]. \tag{17}$$

for all $z \in \mathbb{U}$ *then, the integral operator* $\mathcal{M}$, *defined by*

$$\mathcal{M}(z) = \left\{\alpha \int_0^z \left[f(t)g'(t)\frac{g(t)}{t}\right]^{\alpha-1} dt\right\}^{\frac{1}{\alpha}}, \tag{18}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 3.3. *Let* $f_i, g_i \in \mathcal{A}$, *where* g_i *be in the class* $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i *are real positive numbers and* α_i, β_i, c *be complex numbers for all* $i = \overline{1,n}$, *with*

$$1 > \sum_{i=1}^{n}\left\{|\alpha_i - 1|\left[(1+p_i)M_i + 1\right] + |\beta_i|\right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{19}$$

If

$$|f_i(z)| < M_i, \quad \left|\frac{g_i^{''}(z)}{g_i^{'}(z)}\right| \leq 1,$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \{|\alpha_i - 1| [(1 + p_i) M_i + 1] + |\beta_i|\} \tag{20}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *then, the integral operator* $\mathcal{F}_n$, *defined by*

$$\mathcal{F}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} (g_i{}'(t))^{\beta_i} \right] dt \tag{21}$$

is in the class $\mathcal{S}$.

Remark 3.4. *The integral operator from Corollary 3.3, given by (21) is a known result proven in [21].*

Letting $\delta = 1$ and $\beta_i = 0$ in Theorem 2.1, we have the following corollary:

Corollary 3.5. *Let* $f_i, g_i \in \mathcal{A}$, M_i, N_i *are real positive numbers and* α_i, γ_i, c *be complex numbers for all* $i = \overline{1, n}$, *with*

$$1 > \sum_{i=1}^{n} \{|\alpha_i - 1| [(1 + p_i) M_i + 1] + |\gamma_i| [(1 + p_i) N_i + 1]\}, \quad |c| \leq 1, \quad c \neq -1. \tag{22}$$

If

$$|f_i(z)| < M_i, \quad |g_i(z)| < N_i,$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \{|\alpha_i - 1| [(1 + p_i) M_i + 1] + |\gamma_i| [(1 + p_i) N_i + 1]\} \tag{23}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *then, the integral operator* $\mathcal{G}_n$, *defined by*

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{24}$$

is in the class $\mathcal{S}$.

Remark 3.6. *On the integral operator from Corollary 3.5, given by (24) if we take* $\alpha_i - 1 = \alpha_i$, *we obtain another known result proven in [10].*

Putting $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 3.7. *Let $g_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, N_i are real positive numbers and β_i, γ_i, c be complex numbers for all $i = \overline{1,n}$, with*

$$1 > \sum_{i=1}^{n} \{|\beta_i| + |\gamma_i| [(1+p_i) N_i + 1]\}, \quad |c| \leq 1, \quad c \neq -1. \tag{25}$$

If

$$|g_i(z)| < N_i, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq 1,$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \{|\beta_i| + |\gamma_i| [(1+p_i) N_i + 1]\} \tag{26}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ then, the integral operator $\mathcal{I}_n$, defined by

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[(g_i'(t))^{\beta_i} \left(\frac{g_i(t))}{t} \right)^{\gamma_i} \right] dt \tag{27}$$

is in the class $\mathcal{S}$.

Remark 3.8. *The integral operator from Corollary 3.7, given by (27) was proven in [7].*

Letting $M_i = N_i = P_i = 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 3.9. *Let $f_i, g_i, h_i \in \mathcal{S}(p_i)$, $0 < p_i \leq 2$ and $\delta, \alpha_i, \beta_i, \gamma_i, c$ are complex numbers, for all $i = \overline{1,n}$, with*

$$Re\delta > \sum_{i=1}^{n} [(p_i+3)|\alpha_i - 1| + |\beta_i| + 2|\gamma_i|], \quad |c| \leq 1. \tag{28}$$

If

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq 1, \quad |g_i(z)| \leq 1, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq 1$$

and

$$|c| \leq 1 - \frac{1}{Re\delta} \sum_{i=1}^{n} [(p_i+3)|\alpha_i - 1| + |\beta_i| + 2|\gamma_i|], \tag{29}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ then, the integral operator $\mathcal{C}_n$, defined by (2) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_1 - 1 = \beta_1 = \gamma_1$ in Theorem 2.2, we have:

Corollary 3.10. *Let $f, g, h \in \mathcal{S}(p)$, $0 < p \leq 2$, M, N, P are real positive numbers and α, c complex numbers, with*

$$Re\alpha > |\alpha - 1| \left[M + N^2 (1 + p) + P + 3\right], \quad |c| \leq 1. \tag{30}$$

If

$$\left|\frac{zf'(z)}{f(z)}\right| \leq M, \quad |g(z)| < N, \quad \left|\frac{h''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zh'(z)}{h(z)}\right| \leq P$$

and

$$|c| \leq 1 - \frac{1}{Re\alpha} |\alpha - 1| \left[M + N^2 (1 + p) + P + 3\right] \tag{31}$$

for all $z \in \mathbb{U}$ then, the integral operators $\mathcal{C}$, defined by

$$\mathcal{C}(z) = \left\{\alpha \int_0^z \left[f(t) e^{g(t)} h'(t) \frac{h(t))}{t}\right]^{\alpha - 1} dt\right\}^{\frac{1}{\alpha}}, \tag{32}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.2, we obtain the next corollary:

Corollary 3.11. *Let $f_i, g_i, h_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i are real positive numbers and α_i, β_i, c complex numbers, for all $i = \overline{1, n}$, with*

$$1 > \sum_{i=1}^{n} \left\{|\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1\right] + |\beta_i|\right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{33}$$

If

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \left\{|\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1\right] + |\beta_i|\right\} \tag{34}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ then, the integral operator $\mathcal{T}_n$ defined by

$$\mathcal{T}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)}\right)^{\alpha_i - 1} \left(h_i'(t)\right)^{\beta_i}\right] dt, \tag{35}$$

is in the class $\mathcal{S}$.

Remark 3.12. *The integral operator from Corollary 3.11, given by (35), if we take $\beta_i = 0$, we obtain a known result proven in [20].*

Putting $\delta = 1$ and $\beta_i = 0$ in Theorem 2.2, we obtain the following corollary:

Corollary 3.13. *Let $f_i, g_i, h_i \in \mathcal{A}$, where g_i be in the class $\mathcal{S}(p_i)$, $0 < p_i \leq 2$, M_i, N_i, P_i are real positive numbers and α_i, γ_i, c complex numbers, for all $i = \overline{1,n}$, with*

$$1 > \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1 \right] + |\gamma_i| (P_i + 1) \right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{36}$$

If

$$\left| \frac{z f_i'(z)}{f_i(z)} \right| \leq M_i, \quad |g_i(z)| < N_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} \right| \leq P_i$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[M_i + N_i^2 (1 + p_i) + 1 \right] + |\gamma_i| (P_i + 1) \right\} \tag{37}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ then, the integral operator $\mathcal{R}_n$, defined by

$$\mathcal{R}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{38}$$

is in the class $\mathcal{S}$.

Remark 3.14. *Putting $\gamma_i = 0$ in (38) we obtain another known result proven in [20].*

Letting $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.2, we obtain:

Corollary 3.15. *Let $h_i \in \mathcal{A}$, P_i are real positive numbers and β_i, γ_i, c complex numbers, for all $i = \overline{1,n}$, with*

$$1 > \sum_{i=1}^{n} \left\{ |\beta_i| + |\gamma_i| (P_i + 1) \right\}, \quad |c| \leq 1, \quad c \neq -1. \tag{39}$$

If

$$\left| \frac{h_i''(z)}{h_i'(z)} \right| \leq 1, \quad \left| \frac{z h_i'(z)}{h_i(z)} \right| \leq P_i$$

and

$$|c| \leq 1 - \sum_{i=1}^{n} \left\{ |\beta_i| + |\gamma_i| (P_i + 1) \right\} \tag{40}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *then, the integral operator* $\mathcal{I}_n$, *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left({h_i}'(t) \right)^{\beta_i} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{41}$$

is in the class $\mathcal{S}$.

Remark 3.16. *The integral operator from Corollary 3.15, given by (41) was proven in [7].*

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