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Extension of the Sophomore's Dream

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Abstract

In this article, we are going to look at the convergence properties of the integral $\int_0^1 (ax+b)^{cx+d}\,dx$, and express it in series form, where a, b, c and d are real parameters.

1 Introduction

In this article, we are going to investigate the integral

$$\int_0^1 (ax+b)^{cx+d}\,dx \tag{1}$$

where a, b, c and d are real parameters. We are going to pose limitations on these parameters and express the value of the integral in series form, see section 1.2. This integral can be regarded as the generalization of the integrals

$$\int_0^1 x^x\,dx = 0.783430510\ldots \tag{2}$$

and

$$\int_0^1 x^{-x}\,dx = 1.291285997\ldots \tag{3}$$

which form the base of the identities called the sophomore's dream. In section 1.1, we are going to recall these identities and supply rigorous proofs for their validity.

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The limits of integration could be also generalized in expression (1), but note that if $0 \leq \lambda_1 < \lambda_2 < \infty$ would be the lower and upper limits of the integral, we could still reduce it back to the base case as

$$\int_{\lambda_1}^{\lambda_2} (ax+b)^{cx+d}\,dx = (\lambda_2 - \lambda_1) \int_0^1 (At+B)^{Ct+D}\,dt$$

where we have used the linear substitution $t = (x-\lambda_1)/(\lambda_2-\lambda_1)$, so it is sufficient to examine the integral where we integrate from zero to one. (Although we will start the evaluation of expression (1) by removing the terms a and b from the parentheses using a linear substitution, which will change the limits of integration.)

1.1 Known results

The pair of identities in connection with integrals (2) and (3), which are referred to as sophomore's dream are discovered by Johann Bernoulli, see [4].

Proposition 1.

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n} \tag{4}$$

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} \tag{5}$$

For a recollection about these results in English, see chapter 3 of [5]. We are going to supply two rigorous proofs for these identities.

Proof of proposition 1, first variant. We want to transform the integrand on the left hand side of equation (4) by using the identity $x^x = \exp(x \ln x)$. This is the joint application of the identities $x = \exp(\ln x)$ and $\ln x^y = y \ln x$. The former holds when x is a positive real number; the later holds when x is a positive real number and y is a real number. Because the lower limit of integration is zero, we have to handle the integrand carefully.

The function x^x is continuous on the interval $[\omega, 1] \subset \mathbb{R}$ for every $\omega > 0$, so we can write the integral on the left hand side of equation (4) as

$$\lim_{\omega \to 0^+} \int_\omega^1 x^x\,dx.$$

Because the function $\exp(\ln x)$ is also continuous on the interval $[\omega, 1] \subset \mathbb{R}$ for every $\omega > 0$, and the lower limit of integration is positive, we can write this integral as

$$\lim_{\omega \to 0^+} \int_\omega^1 e^{\ln x^x}\, dx = \lim_{\omega \to 0^+} \int_\omega^1 e^{x \ln x}\, dx.$$

Using the power series form of the exponential function, we can write this last integral as

$$\lim_{\omega \to 0^+} \int_\omega^1 \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \lim_{\omega \to 0^+} \int_\omega^1 x^n \ln^n x\, dx \tag{6}$$

where the order of the integration and the summation can be exchanged due to the uniform convergence of the power series. Here, using integration by parts, the integral can be written as

$$\lim_{\omega \to 0^+} \left[\frac{x^{n+1}}{n+1} \ln^n x \right]_\omega^1 - \frac{n}{n+1} \lim_{\omega \to 0^+} \int_\omega^1 x^n \ln^{n-1} x\, dx \tag{7}$$

when n is a positive integer. The first term disappears at one and also at the limit because

$$\lim_{x \to 0^+} x^m \ln^n x = 0 \tag{8}$$

holds by l'Hospital rule when m is a positive real number and n is a positive integer. From this, by using recursion we get that

$$\lim_{\omega \to 0^+} \int_\omega^1 x^n \ln^n x\, dx = (-1)^n \frac{n!}{(n+1)^{n+1}}$$

holds when n is a positive integer, and by using this in equation (6), we get our sought result. Equation (5) can be derived using the same reasoning. In this case one just has to extract a multiplier $(-1)^n$ from the power series form of the exponential function. □

It is worth noting that Bernoulli exchanged the order of the integration and the summation in equation (6) without a worry, and he also does not mention why the terms disappear at zero in expression (7), as it is noted by Dunham in [5].

Proof of proposition 1, second variant. Another approach to calculate the integrals on the right hand side of equation (6) is to change the variable via the substitution

$$x = \exp\left(-\frac{u}{n+1}\right)$$

obtaining

$$\lim_{\omega\to 0^+}\int_\omega^1 x^n \ln^n x\,dx = (-1)^n \frac{1}{(n+1)^{n+1}} \lim_{\omega\to 0^+}\int_0^{-(n+1)\ln\omega} u^n e^{-u}\,du$$

where the integral on the right hand side - by Euler's integral identity, see equation 6.1.1 in [1] - is the integral representation of $\Gamma(n+1)$. □

The results of Galidakis, see [6], concerning the integration of the so called "power tower" should be also mentioned. This can be regarded as another generalization of integral (2).

1.2 New results

We assume from the start that $a \neq 0$ and $c \neq 0$ in expression (1), otherwise the solution is trivial. We are going to apply the identity $x^x = \exp(x \ln x)$ like in the first proof of proposition 1, so we require that $ax + b > 0$ when $x \in [0, 1]$, which means that $b > 0$ and $a > -b$ should hold. By using continuity arguments we can extend these requirements to $b \geq 0$ and $a \geq -b$.

Another requirement is that $ax+b$ shouldn't become zero when $cx+d < 0$ holds. Because $ax + b$ is linear, this can happen at most at one point in the interval $[0, 1]$. So if $-b/a \in [0, 1]$, then $d - cb/a \geq 0$ should hold.

Proposition 2. *Let $a, b, c, d \in \mathbb{R}$, with $a \geq -b$, $a \neq 0$, $b \geq 0$ and $c \neq 0$. When $-b/a \in [0, 1]$, then $d - cb/a \geq 0$ should hold, otherwise we require that $d - cb/a \neq -1, -2, \ldots$ holds. With these requirements, integral (1) is convergent and finite; furthermore when $b \neq 0$, then it is equal to the series*

$$\frac{1}{a}\sum_{n=0}^{\infty} \frac{(-c/a)^n}{n!(n+d-cb/a+1)^{n+1}} \left[\Gamma(n+1, -(n+d-cb/a+1)\ln t)\right]_b^{a+b}$$

otherwise when $b = 0$, it is equal to the series

$$\frac{1}{a}\sum_{n=0}^{\infty} \frac{(-c/a)^n}{n!(n+d+1)^{n+1}}\Gamma(n+1, -(n+d+1)\ln a).$$

The function $\Gamma(s, z)$ in the proposition is the upper incomplete gamma function defined as

$$\Gamma(s, z) := \int_z^\infty t^{s-1} e^{-t}\,dt \tag{9}$$

where $s, z \in \mathbb{C}$, see section 9.1 of [3], section 6.5 of [1] or section 1.1 of [7].

Applying our proposition with $a = c = 1$ and $b = d = 0$ we get back equation (4), while with $a = 1$, $c = -1$ and $b = d = 0$ we get back equation (5), because $\Gamma(s, 0) = \Gamma(s)$ in both cases. For the properties of the gamma function $\Gamma(s)$, see chapter 1 of [2] or chapter 6 of [1].

2 Proof of the new results

When s is a positive integer, then the equality

$$\Gamma(s,z) = (s-1)!e^{-z}\sum_{k=0}^{s-1}\frac{z^k}{k!} \tag{10}$$

holds for all $z \in \mathbb{C}$, see section 9.2.1 of [3].

Lemma 3. *Let s be a positive integer and $z \in \mathbb{R}$. Then*

$$|\Gamma(s,z)| \leq (s-1)!e^{2|z|}.$$

Proof of lemma 3. When z is zero, then $\Gamma(s,z) = \Gamma(s) = (s-1)!$, so the inequality holds in this case. Let's assume that $z \neq 0$. Because s is a positive integer, we can use equation (10) to get

$$|\Gamma(s,z)| = (s-1)!e^{-z}\left|\sum_{k=0}^{s-1}\frac{z^k}{k!}\right| \leq (s-1)!e^{-z}\sum_{k=0}^{s-1}\frac{|z|^k}{k!} \leq (s-1)!e^{-z}e^{|z|}$$

where we could apply the triangle inequality due to the finiteness of the sum. When z is a positive real number, then the right hand side is equal to $(s-1)!$, otherwise when z is negative it it equal to $(s-1)!e^{2|z|}$. □

Lemma 4. *The series*

$$\sum_{n=0}^{\infty}\frac{a^n}{n!(n+d+1)^{n+1}}\Gamma(n+1,(n+b)c)$$

converges absolutely for every $a, b, c \in \mathbb{R}$ and $d \in \mathbb{R} \setminus \mathbb{Z}^-$.

Proof of lemma 4. We are going to show that the series

$$\sum_{n=0}^{\infty}\left|\frac{a^n}{n!(n+d+1)^{n+1}}\Gamma(n+1,(n+b)c)\right|$$

is convergent for every $a, b, c \in \mathbb{R}$ and $d \in \mathbb{R} \setminus \mathbb{Z}^-$. By applying lemma 3, we get that this expression is smaller than or equal to

$$\sum_{n=0}^{\infty}\frac{|a|^n e^{2|(n+b)c|}}{|n+d+1|^{n+1}} \leq e^{2|bc|}\sum_{n=0}^{\infty}\frac{1}{|n+d+1|}\left(\frac{|a|e^{2|c|}}{|n+d+1|}\right)^n$$

where the right hand side is convergent by the root test. □

Lemma 5. *Let $\alpha \neq -1$, $0 \leq \lambda_1 < \lambda_2 < \infty$ be real numbers, and n be a non-negative integer. Then*

$$\int_{\lambda_1}^{\lambda_2} x^\alpha \ln^n x\, dx = \frac{(-1)^n}{(\alpha+1)^{n+1}} \lim_{\omega\to\lambda_1^+} \left[\Gamma(n+1, -(\alpha+1)\ln x)\right]_\omega^{\lambda_2}.$$

Proof of lemma 5. When $n = 0$, then the equality follows from definition (9). Otherwise when $n > 0$, then we apply integration by parts as in section 1.1, to get that the integral is equal to

$$\lim_{\omega\to\lambda_1^+} \left[\frac{x^{\alpha+1}}{\alpha+1} \ln^n x\right]_\omega^{\lambda_2} - \frac{n}{\alpha+1} \lim_{\omega\to\lambda_1^+} \int_\omega^{\lambda_2} x^\alpha \ln^{n-1} x\, dx$$

which application we can continue iteratively, to find that the initial integral is equal to the sum

$$\sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \lim_{\omega\to\lambda_1^+} \left[\frac{x^{\alpha+1}}{(\alpha+1)^{k+1}} \ln^{n-k} x\right]_\omega^{\lambda_2}.$$

Here we reverse the order of summation as

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \lim_{\omega\to\lambda_1^+} \left[\frac{x^{\alpha+1}}{(\alpha+1)^{n-k+1}} \ln^{k} x\right]_\omega^{\lambda_2}$$

and then rearrange the terms to get the expression

$$(-1)^n \frac{n!}{(\alpha+1)^{n+1}} \lim_{\omega\to\lambda_1^+} \left[x^{\alpha+1} \sum_{k=0}^{n} \frac{(-(\alpha+1)\ln x)^k}{k!}\right]_\omega^{\lambda_2}.$$

Now one applies equation (10) and performs the cancellations, to get the sought result. □

Lemma 6. *Let $\lambda > -1$ be a real number, and n be a non-negative integer. Then*

$$\lim_{\omega\to 0^+} \Gamma(n+1, -(n+\lambda+1)\ln\omega) = 0.$$

Proof of lemma 6. By using equation (10), we have the equality

$$\lim_{\omega\to 0^+} \Gamma(n+1, -(n+\lambda+1)\ln\omega) = n! \lim_{\omega\to 0^+} e^{(n+\lambda+1)\ln\omega} \sum_{k=0}^{n} \frac{(-(n+\lambda+1)\ln\omega)^k}{k!}$$

where both $n + \lambda + 1$ and ω are positive, so the right hand side is equal to

$$n! \sum_{k=0}^{n} (-1)^k \frac{(n+\lambda+1)^k}{k!} \lim_{\omega \to 0^+} \omega^{n+\lambda+1} \ln^k \omega$$

where the summand is zero when $k = 0$, and it is also zero for every $k > 0$ based on limit (8). $\square$

Proof of proposition 2. First, we start by removing the terms from the parentheses using the linear substitution $t = ax + b$ as

$$\int_0^1 (ax+b)^{cx+d}\, dx = \frac{1}{a} \int_b^{a+b} t^{c(t-b)/a+d}\, dt = \frac{1}{a} \int_b^{a+b} t^{Ct+D}\, dt$$

where $C = c/a \neq 0$ and $D = d - cb/a$. Now we can rewrite the integrand on the right hand side as in the case of the proof of proposition 1 as

$$\frac{1}{a} \int_b^{a+b} e^{\ln t^{Ct}} t^D\, dt = \frac{1}{a} \int_b^{a+b} e^{Ct \ln t} t^D\, dt.$$

Substituting the series representation of the exponential function, we get

$$\frac{1}{a} \int_b^{a+b} \left(\sum_{n=0}^{\infty} \frac{C^n}{n!} t^n \ln^n t \right) t^D\, dt = \frac{1}{a} \sum_{n=0}^{\infty} \frac{C^n}{n!} \int_b^{a+b} t^{n+D} \ln^n t\, dt$$

where the exchange of the order of the integration and the summation is allowed because of the uniform convergence of the power series. Based on our requirements $n + D \neq -1$ and $0 \leq b, a+b < \infty$, so we can apply lemma 5 on the integral to get

$$\frac{1}{a} \sum_{n=0}^{\infty} \frac{(-C)^n}{n!(n+D+1)^{n+1}} \lim_{\omega \to b^+} \left[\Gamma(n+1, -(n+D+1)\ln t)\right]_{\omega}^{a+b}.$$

When $b \neq 0$, then this is equal to the series

$$\frac{1}{a} \sum_{n=0}^{\infty} \frac{(-c/a)^n}{n!(n+d-cb/a+1)^{n+1}} \left[\Gamma(n+1, -(n+d-cb/a+1)\ln t)\right]_b^{a+b}$$

which converges based on lemma 4. Otherwise when $b = 0$, then based on lemma 6, it is equal to the series

$$\frac{1}{a} \sum_{n=0}^{\infty} \frac{(-c/a)^n}{n!(n+d+1)^{n+1}} \Gamma(n+1, -(n+d+1)\ln a).$$

which converges too, based on lemma 4 again. $\square$

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