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Topological Transversality Coincidence Theory for Multivalued Maps with Selections in a Given Class

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Abstract

This paper presents the topological transversality coincidence theorem for general multivalued maps who have selections in a given class of maps.

1. Introduction.

Fix a map Φ. To consider a coincidence problem between a complicated map F and Φ (i.e. to find an x with $F(x) \cap \Phi(x) \neq \emptyset$) the idea in this paper is to try and relate it to a simpler and solvable coincidence problem between a map G and Φ where G is homotopic (in an appropriate way) to F and from this hope to deduce a coincidence between F and Φ. In this paper we consider multivalued maps F and G with selections in a given class of maps and $F \cong G$ in this setting. The topological transversality theorem will state that F is Φ–essential if and only if G is Φ–essential (essential maps were introduced by Granas [2] and extended by many authors [1, 3, 4, 5]). By introducing our notion of a Φ–essential (or d–Φ–essential) map we establish a simple result which will immediately yield the topological transversality theorem in this setting.

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2. Topological transversality theorems.

Let E be a completely regular topological space and U an open subset of E.

We will consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 2.1. We say $F \in D(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the boundary of U in E.

In this paper we use bold face only to indicate properties of our maps and usually $D = \mathbf{D}$ etc. Examples of $F \in \mathbf{D}(\overline{U}, E)$ might be that $F : \overline{U} \to K(E)$ is a upper semicontinuous compact map with convex (or acyclic) values or it might be that $F : \overline{U} \to E$ is a single valued continuous compact map; here $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2.2. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$ and now we present the notion of coincidence free on the boundary and the notion of homotopy.

Definition 2.3. We say $F \in A_{\partial U}(\overline{U}, E)$ (respectively $F \in D_{\partial U}(\overline{U}, E)$) if $F \in A(\overline{U}, E)$ (respectively $F \in D(\overline{U}, E)$) with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. Let E be a completely regular (respectively, normal) topological space and let $\Psi, \Lambda \in D_{\partial U}(\overline{U}, E)$. We say $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\left\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (resp., closed), for any continuous map $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have $\left\{x \in \overline{U} : \ \Phi(x) \cap H(x, t\,\eta(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is closed, $H_0 = \Psi$ and $H_1 = \Lambda$. In addition here we assume for any map $\Theta \in \mathbf{D}(\overline{U} \times [0,1], E)$ and any maps $g \in \mathbf{C}(\overline{U}, \overline{U} \times [0,1])$ and $f \in \mathbf{C}(\overline{U} \times [0,1], \overline{U} \times [0,1])$ then $\Theta \circ g \in \mathbf{D}(\overline{U}, E)$ and $\Theta \circ f \in \mathbf{D}(\overline{U} \times [0,1], E)$; here $\mathbf{C}$ denotes the class of single valued continuous functions.

Remark 2.5. (a). In our results below alternatively we could use the following definition for $\cong$ in $D_{\partial U}(\overline{U}, E)$: $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in \mathbf{D}(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\left\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (respectively, closed), for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have that $\left\{x \in \overline{U} : \ \Phi(x) \cap H(x, t\,\eta(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is closed,

$H_0 = \Psi$ and $H_1 = \Lambda$ (here $H_t(x) = H(x,t)$).

(b). Throughout we assume $\cong$ in $D_{\partial U}(\overline{U},E)$ is a reflexive, symmetric relation.

Definition 2.6. Let $F, G \in A_{\partial U}(\overline{U},E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U},E)$ if for any selection $\Psi \in D_{\partial U}(\overline{U},E)$ (respectively, $\Lambda \in D_{\partial U}(\overline{U},E)$) of F (respectively, of G) we have $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U},E)$.

Next we present the notion of Φ–essentiality which is more general than notions considered in the literature.

Definition 2.7. We say $F \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$ if for any selection $\Psi \in D(\overline{U},E)$ of F and any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U},E)$ there exists a $x \in U$ with $J\,(x) \cap \Phi\,(x) \neq \emptyset$.

Remark 2.8. If $F \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$ and if $\Psi \in D(\overline{U},E)$ is any selection of F then there exists an $x \in U$ with $\Psi\,(x) \cap \Phi\,(x) \neq \emptyset$ (take $J = \Psi$ in Definition 2.7), and $\emptyset \neq \Psi\,(x) \cap \Phi\,(x) \subseteq F\,(x) \cap \Phi\,(x)$.

Theorem 2.9. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U},E)$ and $G \in A_{\partial U}(\overline{U},E)$ is Φ–essential in $A_{\partial U}(\overline{U},E)$. Also suppose*

(2.1)
$$\begin{cases} \textit{for any selection } \ \Psi \in D_{\partial U}(\overline{U},E) \ \ \textit{(respectively, } \ \Lambda \in D_{\partial U}(\overline{U},E)\textit{)} \\ \textit{of } \ F \ \ \textit{(respectively, of } \ G\textit{) and any map } \ J \in D_{\partial U}(\overline{U},E) \ \ \textit{with} \\ J|_{\partial U} = \Psi|_{\partial U} \ \ \textit{and } \ J \cong \Psi \ \ \textit{in } \ D_{\partial U}(\overline{U},E) \ \ \textit{we have} \\ \Lambda \cong J \ \ \textit{in } \ D_{\partial U}(\overline{U},E). \end{cases}$$

Then F is Φ–essential in $A_{\partial U}(\overline{U},E)$.

Proof: Let $\Psi \in D_{\partial U}(\overline{U},E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U},E)$. We must show there exists an $x \in U$ with $J\,(x) \cap \Phi\,(x) \neq \emptyset$. Let $\Lambda \in D_{\partial U}(\overline{U},E)$ be any selection of G. Now (2.1) guarantees that there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$ is compact (resp., closed), for any continuous map $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have that $\{x \in \overline{U} : \ \Phi(x) \cap H(x, t\,\eta(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\}$ is closed, $H_0 = \Lambda$ and $H_1 = J$. Let

$$\Omega = \left\{x \in \overline{U} : \ \Phi(x) \cap H(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Now since G is Φ–essential in $A_{\partial U}(\overline{U},E)$ then Remark 2.8 (note $H_0 = \Lambda$) guarantees that $\Omega \neq \emptyset$. Also Ω is compact (respectively, closed) if E is a completely regular (respectively, normal) topological space. Next note $\Omega \cap$

$\partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = H(x, \mu(x)) = H \circ g(x)$ for $x \in \overline{U}$; here $g : \overline{U} \to \overline{U} \times [0,1]$ is given by $g(x) = (x, \mu(x))$. Note $R \in \mathbf{D}(\overline{U}, E)$ and in fact $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = H_0|_{\partial U} = \Lambda|_{\partial U}$. We now show $\Lambda \cong R$ in $D_{\partial U}(\overline{U}, E)$. To see this let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\,\mu(x)) = H \circ f(x,t)$ where $f : \overline{U} \times [0,1] \to \overline{U} \times [0,1]$ is given by $f(x,t) = (x, t\,\mu(x))$. Note $Q \in \mathbf{D}(\overline{U} \times [0,1], E)$, $Q_0 = \Lambda$, $Q_1 = R$ and $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (note if $x \in \partial U$ and $t \in (0,1)$ and $\emptyset \neq \Phi(x) \cap H(x, t\,\mu(x)) = \Phi(x) \cap H_{t\,\mu(x)}(x)$, then $x \in \Omega$ so $\mu(x) = 1$ i.e. $\emptyset \neq \Phi(x) \cap H(x, t\,\mu(x)) = \Phi(x) \cap H_t(x)$), also

$$\begin{aligned}&\big\{x \in \overline{U} :\ \Phi(x) \cap Q(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}\\ &\qquad = \big\{x \in \overline{U} :\ \Phi(x) \cap H(x, t\,\mu(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}\end{aligned}$$

is closed so $\big\{x \in \overline{U} :\ \Phi(x) \cap Q(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}$ is compact (respectively, closed) (note

$$\begin{aligned}&\big\{x \in \overline{U} :\ \Phi(x) \cap H(x, t\,\mu(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}\\ &\qquad \subseteq \big\{x \in \overline{U} :\ \Phi(x) \cap H(x, s) \neq \emptyset \ \text{ for some } \ s \in [0,1]\big\}),\end{aligned}$$

and finally for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, note if $w(x) = \eta(x)\,\mu(x)$ (so $w : \overline{U} \to [0,1]$ with $w(\partial U) = 0$) then

$$\begin{aligned}&\big\{x \in \overline{U} :\ \Phi(x) \cap Q(x, t\,\eta(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}\\ &\qquad = \big\{x \in \overline{U} :\ \Phi(x) \cap H(x, t\,w(x)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}\end{aligned}$$

is closed. Thus $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = \Lambda|_{\partial U}$ and $\Lambda \cong R$ in $D_{\partial U}(\overline{U}, E)$. Since G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $R(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$). Thus $x \in \Omega$, $\mu(x) = 1$ so $\emptyset \neq H_1(x) \cap \Phi(x) = J(x) \cap \Phi(x)$, and we are finished. $\square$

Now with this simple result we present the topological transversality theorem. Assume

$$\cong \ \text{ in } \ D_{\partial U}(\overline{U}, E) \ \text{ is an equivalence relation} \tag{2.2}$$

Theorem 2.10. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, and assume* (2.2) *holds. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now F is Φ–essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is Φ–essential in $A_{\partial U}(\overline{U}, E)$.*

Proof: Assume G is Φ–essential in $A_{\partial U}(\overline{U}, E)$. We will use Theorem 2.9. Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of F, $\Lambda \in D_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (so $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$) then (2.2) guarantees that $\Lambda \cong J$ in $D_{\partial U}(\overline{U}, E)$. Thus (2.1) holds so Theorem 2.9 guarantees that F is Φ–essential in $A_{\partial U}(\overline{U}, E)$. A similar argument shows if F is Φ–essential in $A_{\partial U}(\overline{U}, E)$ then G is Φ–essential in $A_{\partial U}(\overline{U}, E)$. □

Now we consider a generalization of Φ–essential maps, namely the d–Φ–essential maps (motivated from the notion of the degree of a map). Let E be a completely regular topological space and U an open subset of E. For any map $\Psi \in D(\overline{U}, E)$ let $\Psi^\star = I \times \Psi : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ \left(\Psi^\star\right)^{-1} (B) \right\} \cup \{\emptyset\} \to K \tag{2.3}$$

be any map with values in the nonempty set K; here $B = \left\{ (x, \Phi(x)) :\ x \in \overline{U} \right\}$.

Definition 2.11. Let E be a completely regular (respectively, normal) topological space and let $\Psi, \Lambda \in D_{\partial U}(\overline{U}, E)$. We say $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\left\{ x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1] \right\}$ is compact (resp., closed), for any continuous map $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have

$$\left\{ x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,\eta(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1] \right\}$$

is closed, $H_0 = \Psi$ and $H_1 = \Lambda$. In addition here we assume for any map $\Theta \in \mathbf{D}(\overline{U} \times [0,1], E)$ and any maps $g \in \mathbf{C}(\overline{U}, \overline{U} \times [0,1])$ and $f \in \mathbf{C}(\overline{U} \times [0,1], \overline{U} \times [0,1])$ then $\Theta \circ g \in \mathbf{D}(\overline{U}, E)$ and $\Theta \circ f \in \mathbf{D}(\overline{U} \times [0,1], E)$; here $\mathbf{C}$ denotes the class of single valued continuous functions.

Remark 2.12. There is an analogue Remark 2.5 in this situation also.

Definition 2.13. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if for any selection $\Psi \in D_{\partial U}(\overline{U}, E)$ (respectively, $\Lambda \in D_{\partial U}(\overline{U}, E)$) of F (respectively, of G) we have $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11).

Definition 2.14. Let $F \in A_{\partial U}(\overline{U}, E)$ and write $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–Φ–essential if for any selection $\Psi \in D(\overline{U}, E)$ of F and any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11) we have that $d\left(\left(\Psi^\star\right)^{-1} (B) \right) = d\left(\left(J^\star\right)^{-1} (B) \right) \neq d(\emptyset)$; here $\Psi^\star = I \times \Psi$ and $J^\star = I \times J$.

Remark 2.15. If $F^\star$ is d–Φ–essential then for any selection $\Psi \in D(\overline{U}, E)$ of F (with $\Psi^\star = I \times \Psi$) we have

$$\emptyset \neq \left(\Psi^\star\right)^{-1} (B) = \{ x \in \overline{U} :\ (x, \Psi(x)) \cap (x, \Phi(x)) \neq \emptyset \},$$

so there exists a $x \in U$ with $(x, \Psi(x)) \cap (x, \Phi(x)) \neq \emptyset$ (i.e. $\Phi(x) \cap \Psi(x) \neq \emptyset$ so in particular $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 2.16. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d is defined in (2.3), $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$ and $G^\star = I \times G$. Suppose $G^\star$ is d–Φ–essential and*

(2.4)
$$\begin{cases} \textit{for any selection } \Psi \in D_{\partial U}(\overline{U}, E) \ \textit{(respectively, } \Lambda \in D_{\partial U}(\overline{U}, E)) \\ \textit{of } F \ \textit{(respectively, of } G) \textit{ and any map } J \in D_{\partial U}(\overline{U}, E) \\ \textit{with } J|_{\partial U} = \Psi|_{\partial U} \ \textit{and } J \cong \Psi \ \textit{in } D_{\partial U}(\overline{U}, E) \\ \textit{we have } \Lambda \cong J \ \textit{in } D_{\partial U}(\overline{U}, E) \textit{ (Definition 2.11) and} \\ d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right); \textit{ here } \Psi^\star = I \times \Psi \textit{ and } \Lambda^\star = I \times \Lambda. \end{cases}$$

Then $F^\star$ is d–Φ–essential.

Proof: Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11). We must show $d\left((\Psi^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$; here $\Psi^\star = I \times \Psi$ and $J^\star = I \times J$. Let $\Lambda \in D_{\partial U}(\overline{U}, E)$ be any selection of G and let $\Lambda^\star = I \times \Lambda$. Now (2.4) guarantees that there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (resp., closed), for any continuous map $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have

$$\left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\,\eta(x))) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is closed, $H_0 = \Lambda$ and $H_1 = J$ and $d\left((\Psi^\star)^{-1}(B)\right) = d\left((\Lambda^\star)^{-1}(B)\right)$. Let

$$\Omega = \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ since $G^\star$ is d–Φ–essential (and $H_0 = \Lambda$). Also Ω is compact (respectively, closed) if E is a completely regular (respectively, normal) topological space. Next note $\Omega \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R by $R(x) = H(x, \mu(x))$ for $x \in \overline{U}$ and write $R^\star = I \times R$. Note $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = H_0|_{\partial U} = \Lambda|_{\partial U}$. Next we show $\Lambda \cong R$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11). To see this let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\,\mu(x))$. Now as in Theorem 2.9 note $Q \in \mathbf{D}(\overline{U} \times [0,1], E)$, $Q_0 = \Lambda$, $Q_1 = R$ and

$\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, also

$$\begin{aligned} &\big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, Q(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\} \\ &\qquad = \big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,\mu(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\} \end{aligned}$$

is closed so $\big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, Q(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\}$ is compact (respectively, closed) (note

$$\begin{aligned} &\big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,\mu(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\} \\ &\qquad \subseteq \big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, s)) \neq \emptyset \ \text{ for some } \ s \in [0,1]\big\}), \end{aligned}$$

and finally for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, note if $w(x) = \eta(x)\,\mu(x)$ (so $w : \overline{U} \to [0,1]$ with $w(\partial U) = 0$) then

$$\begin{aligned} &\big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, Q(x, t\,\eta(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\} \\ &\qquad = \big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,w(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1]\big\} \end{aligned}$$

is closed. Thus $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = \Lambda|_{\partial U}$ and $\Lambda \cong R$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11). Since $G^\star$ is d–Φ–essential then

$$d\left((\Lambda^\star)^{-1}\,(B)\right) = d\left((R^\star)^{-1}\,(B)\right) \neq d(\emptyset). \tag{2.5}$$

Now since $\mu(\Omega) = 1$ we have

$$\begin{aligned} (R^\star)^{-1}\,(B) &= \big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\big\} \\ &= \big\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\big\} = (J^\star)^{-1}\,(B), \end{aligned}$$

so from (2.5) we have $d\left((\Lambda^\star)^{-1}\,(B)\right) = d\left((J^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. Now combine with the above and we have $d\left((\Psi^\star)^{-1}\,(B)\right) = d\left((J^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. $\square$

Now assume

$$\cong \ \text{ in } \ D_{\partial U}(\overline{U}, E) \text{ (Definition 2.11) is an equivalence relation.} \tag{2.6}$$

Theorem 2.17. *Let E be a completely regular (respectively, normal) topological space, U an open subset of E, $B = \big\{(x, \Phi(x)) :\ x \in \overline{U}\big\}$, d is defined in* (2.3)*, and assume* (2.6) *holds. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$, $G^\star = I \times G$ and $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.13). Then $F^\star$ is d–Φ–essential if and only if $G^\star$ is d–Φ–essential.*

Proof: Assume $G^\star$ is d–Φ–essential. Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of F, $\Lambda \in D_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $J \in D_{\partial U}(\overline{U}, E)$

with $J|_{\partial U} = \Psi|_{\partial U}$ and $J \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11). If we show (2.4) then $F^\star$ is d–Φ–essential from Theorem 2.16. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.13) (so $\Psi \cong \Lambda$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11)) then (2.6) guarantees that $\Lambda \cong J$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11). To complete (2.4) it remains to show $d\left((\Psi^\star)^{-1}\,(B)\right) = d\left((\Lambda^\star)^{-1}\,(B)\right)$; here $\Psi^\star = I \times \Psi$ and $\Lambda^\star = I \times \Lambda$. We will show this by following the argument in Theorem 2.16. Note $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.13) so let $H : \overline{U} \times [0,1] \to 2^E$ with $H \in \mathbf{D}(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $\left\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (resp., closed), for any continuous map $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ we have

$$\left\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,\eta(x))) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is closed, $H_0 = \Lambda$ and $H_1 = \Psi$. Let

$$\Omega = \left\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ and there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H(x, \mu(x))$ and write $R^\star = I \times R$. Note (as in Theorem 2.16) $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = \Lambda|_{\partial U}$ and $\Lambda \cong R$ in $D_{\partial U}(\overline{U}, E)$ (Definition 2.11) so since $G^\star$ is d–Φ–essential then $d\left((\Lambda^\star)^{-1}\,(B)\right) = d\left((R^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. Now since $\mu(\Omega) = 1$ we have (see the argument in Theorem 2.16) $(R^\star)^{-1}\,(B) = (\Psi^\star)^{-1}\,(B)$ and as a result we have $d\left((\Psi^\star)^{-1}\,(B)\right) = d\left((\Lambda^\star)^{-1}\,(B)\right)$. $\square$

Remark 2.18. It is also easy to extend the above ideas to other natural situations [3, 4]. Let E be a (Hausdorff) topological vector space (so automatically completely regular), Y a topological vector space, and U an open subset of E. Also let $L : dom\,L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\,L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L+T : dom\,L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$. We say $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ and we could discuss Φ–essential and d–Φ–essential in this situation.

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