DOI: 10.2478/auom-2021-0012
An. Şt. Univ. Ovidius Constanţa **Vol. 29(1),2021, 183–200**

Gődel filters in residuated lattices

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Abstract

In this paper, in the spirit of [4], we study a new type of filters in residuated lattices : *Gődel filters.* So, we characterize the filters for which the quotient algebra that is constructed via these filters is a *Gődel algebra* and we establish the connections between these filters and other types of filters. Using Gődel filters we characterize the residuated lattices which are Gődel algebras. Also, we prove that a residuated lattice is a Gődel algebra (divisible residuated lattice, MTL algebra, BL algebra) if and only if every filter is a Gődel filter (divisible filter, MTL filter, BL filter). Finally, we present some results about injective Gődel algebras showing that complete Boolean algebras are injective objects in the category of Gődel algebras.

1 Introduction

Residuated lattices have been studied in [1], [7], [19], [20], etc. Residuation is a fundamental concept of ordered structures. The commutative residuated lattices were introduced by Ward and Dilworth in [20] as generalization of ideal lattices of rings; non-commutative residuated lattices are algebraic counterpart of substructural logics. Filters are important in defining congruence relations in such lattices. Filters correspond to subsets closed with respect to Modus Ponens and they are sometimes called *deductive systems.* At present, the filter theory of residuated lattices has been studied, and some important results have been published. There are a lot of classes of filters: Boolean filters

Key Words: Residuated lattice, Gődel algebra, Gődel filter, injective object.
2010 Mathematics Subject Classification: Primary 03B50, 06A06, 03G05; Secondary 03G25, 08A72, 06D35, 06E05, 06B20, 03C05, 18C05.
Received: 15.01.2020
Accepted: 13.03.2020

(or Boolean deductive systems or implicative deductive systems or positive implicative filters), Heyting filters (or implicative filters or G(RL) filters), BL filters, MV filters (or fantastic filters), MTL filters, divisible filters, involution filters, etc. (see [3], [4], [5], [9], [15], [21], [22]).

In this paper we present a new type of filters in a residuated lattice : *Gődel filters.* So, we characterize the filters for which the quotient algebra that is constructed via these filters is a *Gődel algebra* (i.e., a BL algebra L in which $x^2 = x$, for every $x \in L$) and, using these filters, the residuated lattices which are Gődel algebras. Gődel filters have been studied in [12] and [14] in the particular case of BL algebras. In this paper we generalize these results in the context of residuated lattices. We state and prove some results which establish the relationships between Gődel filters and other filters of a residuated lattice: Boolean, Heyting, BL, MV, MTL, divisible filters and by some examples we show that these filters are different. We prove that a residuated lattice is a Gődel algebra (divisible residuated lattice, MTL algebra, BL algebra) if and only if every filter is a Gődel filter (divisible filter, MTL filter, BL filter). Also, using Gődel filters, we obtain a new characterization for Gődel algebras: A residuated lattice L is a Gődel algebra if and only if $[x \to (x \odot y)] \vee [y \to (x \odot y)] = 1$, for every $x, y \in L$, see Corollary 59. Finally, we show that a complete Boolean algebra is an injective object in the category of Gődel algebras.

2 Preliminaries

We begin by reminding definitions and properties of residuated lattices used in this paper.

Definition 1. *([2], [10], [11], [18], [19], [20]) A* residuated lattice *is an algebra* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *with an order* $\leq$ *such that:*

(LR_1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice relative to the order* $\leq$*;*

(LR_2) $(L, \odot, 1)$ *is a commutative monoid;*

(LR_3) $\odot$ *and* $\to$ *form an adjoint pair :* $a \odot x \leq y$ *iff* $a \leq x \to y$ *for every* $a, x, y \in L$.

We denote by $\mathcal{RL}$ the class of residuated lattices and by L a residuated lattice (unless otherwise mentioned).

For $x \in L$ and a natural number n we define $x^* = x \to 0, x^{**} = (x^*)^*, x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

In our paper we use the following rules of calculus in residuated lattices (see [3], [8], [11], [12], [16], [17], [20]):

(c_1) $1 \to x = x, x \to x = 1, x \to 1 = 1$;

(c_2) $x \leq y$ iff $x \to y = 1$;

(c_3) If $x \leq y$, then $x \odot z \leq y \odot z, z \to x \leq z \to y$ and $y \to z \leq x \to z$;

(c_4) $x \leq y \to x, x \leq (x \to y) \to y, ((x \to y) \to y) \to y = x \to y$;

(c_5) $y \to z \leq (x \to y) \to (x \to z) \leq x \to (y \to z)$;

(c_6) $x \to y \leq (y \to z) \to (x \to z)$;

(c_7) $x \odot (x \to y) \leq y$;

(c_8) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(c_9) $x \to (y \wedge z) = (x \to y) \wedge (x \to z), (y \vee z) \to x = (y \to x) \wedge (z \to x)$;

(c_{10}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $(y \wedge z) \to x \geq (y \to x) \vee (z \to x)$;

(c_{11}) $x \odot x^* = 0, x \odot y = 0$ iff $x \leq y^*$;

(c_{12}) $x \leq x^{**}, x \leq x^* \to y, (x \vee y)^* = x^* \wedge y^*$.

A Boolean algebra $(L, \vee, \wedge, ', 0, 1)$ becomes a residuated lattice $(L, \vee, \wedge, \odot, \to, 0, 1)$ defining for every $x, y \in L, x \odot y = x \wedge y$, $x \to y = x' \vee y$. A Boolean algebra is a residuated lattice L in which $x \vee x^* = 1$ for every $x \in L$ (see [9]).

Definition 2. *([9], [13], [16], [22]) A residuated lattice L in which $x^2 = x$ for all $x \in L$ (or, equivalently, $x \odot y = x \wedge y$ for all $x, y \in L$) is called a Heyting algebra (or G(RL) algebra or pseudo Boolean algebra).*

In a residuated lattice L we consider the identities:

(BL_1) $x \odot (x \to y) = x \wedge y$ (*divisibility*);

(BL_2) $(x \to y) \vee (y \to x) = 1$ (*prelinearity*);

(MV) $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$.

Definition 3. *([11], [18]) We say that the residuated lattice L is*

(i) divisible *if L verifies* (BL_1);

(ii) *a* MTL algebra *if L verifies* (BL_2);

(iii) *a* BL algebra *if L verifies* (BL_1) *and* (BL_2), *that is, L is a divisible MTL algebra;*

(iv) a MV algebra *if L verifies (MV).*

It is not hard to see that an equivalent presentation of MV algebras can be given as BL algebras plus condition $x^{**} = x$, for every $x \in L$ (see [6]).

Definition 4. *([11]) A BL algebra L in which $x^2 = x$ for every $x \in L$ is called a* Gődel algebra.

The standard Gődel algebra is the BL algebra $([0,1], \max, \min, \odot_G, \rightarrow_G, 0, 1)$ determined by the Gődel t-norm. (see [11])

Remark 5. *Heyting algebras are divisible residuated lattices. Indeed, if L is a Heyting algebra, then $x \odot y = x \wedge y$ for every $x, y \in L$. But $x \odot y \leq x \odot (x \rightarrow y) \leq x \wedge y$, so $x \odot (x \rightarrow y) = x \wedge y$, for every $x, y \in L$.*

In a residuated lattice we can define both the notion of deductive system (Definition 6) and the notion of filter (Definition 7) and they are equivalent:

Definition 6. *([18]) Let L be a residuated lattice. A subset $F \subseteq L$ is called* deductive system *if :*

(F_1) $1 \in F$;

(F_2) *If $x, x \rightarrow y \in F$, then $y \in F$.*

Definition 7. *([18]) Let L be a residuated lattice. A subset $F \subseteq L$ is called a* filter*:*

(F_1') *If $x \leq y$ and $x \in F$, then $y \in F$;*

(F_2') *If x, $y \in F$, then $x \odot y \in F$.*

In this paper, we shall work with the notion of filter.

We denote by $\mathbf{F}(\mathbf{L})$ the set of all filters of L.

For $F \in \mathbf{F}(\mathbf{L})$, the relation $\sim_F$ defined on L by $(x, y) \in \sim_F$ iff $x \rightarrow y, y \rightarrow x \in F$ iff $(x \rightarrow y) \odot (y \rightarrow x) \in F$ is a congruence relation on L (see [18]). The quotient algebra $L/\sim_F$ denoted by L/F becomes a residuated lattice.

For $x \in L$ we denote by x/F the congruence class of x modulo $\sim_F$. So, the order relation on L/F is given by $x/F \leq y/F$ iff $x \rightarrow y \in F$. Clearly, $x/F = 1/F$ iff $x \in F$.

3 Types of filters in residuated lattices

Let $\mathcal{V}$ be a subvariety of the variety $\mathcal{RL}$ of residuated lattices.

Definition 8. *([4]) A filter $F \in \mathbf{F}(\mathbf{L})$ will be called a $\mathcal{V}$* filter *if $L/F \in \mathcal{V}$.*

We denote by $\mathcal{V}\mathbf{F}(\mathbf{L})$ the set of all $\mathcal{V}$ filters of L.

For different subvarieties of residuated lattices we obtain a classification of filters.

3.1 The class of Boolean filters

Definition 9. *([3], [4], [15], [22]) A filter F of a residuated lattice L is called* Boolean filter *if L/F is a Boolean algebra.*

Let $\mathbf{BF(L)}$ be the set of all Boolean filters of L.

Theorem 10. *([3], [12], [14], [22]) For a filter F of a residuated lattice L the following conditions are equivalent:*

(*i*) $F \in \mathbf{BF(L)}$;

(*ii*) $x \vee x^* \in F$ *for every* $x \in L$.

Theorem 11. *([22]) A residuated lattice is a Boolean algebra if and only if any filter is a Boolean filter.*

Corollary 12. *([22]) A residuated lattice L is a Boolean algebra if and only if $\{1\}$ is a Boolean filter of L.*

3.2 The class of Heyting filters

Definition 13. *([4], [22]) A filter F of a residuated lattice L is called* Heyting filter *if L/F is a Heyting algebra.*

Let $\mathbf{HF(L)}$ be the set of all Heyting filters of L.

Theorem 14. *([3], [9], [14], [22]) For a filter F of a residuated lattice L the following statements are equivalent:*

(*i*) $F \in \mathbf{HF(L)}$;

(*ii*) $x \to x^2 \in F$ *for every* $x \in L$;

(*iii*) $(x \wedge y) \to (x \odot y) \in F$, *for every* $x, y \in L$;

(*iv*) *If* $x \to (y \to z) \in F$, *then* $(x \to y) \to (x \to z) \in F$.

Theorem 15. *([22]) A residuated lattice is a Heyting algebra if and only if any filter is a Heyting filter.*

Corollary 16. *([22]) A residuated lattice L is a Heyting algebra if and only if $\{1\}$ is a Heyting filter of L.*

3.3 The class of MTL filters

Definition 17. *([4], [5], [21]) A filter F of a residuated lattice L is called* MTL filter *if L/F is a MTL algebra.*

Let $\mathbf{MTLF(L)}$ be the set of all MTL filters of L.

Theorem 18. *([21]) For a filter F of a residuated lattice L the following are equivalent:*

(*i*) $F \in \mathbf{MTLF(L)}$;

(*ii*) $(x \to y) \vee (y \to x) \in F$ *for every* $x, y \in L$.

Theorem 19. *A residuated lattice is a MTL algebra if and only if any filter is a MTL filter.*

Proof. Let L be a MTL algebra, so for every $x, y \in L$ we have $(x \to y) \vee (y \to x) = 1$. If F is a filter of L, then $1 \in F$, so $(x \to y) \vee (y \to x) \in F$, thus $F \in \mathbf{MTLF(L)}$, by Theorem 18.

Conversely, if any filter is a MTL filter, then $F = \{1\}$ is a MTL filter, so $(x \to y) \vee (y \to x) \in F = \{1\}$, for every $x, y \in L$. We conclude that L is a MTL algebra. $\square$

Corollary 20. *A residuated lattice L is a MTL algebra if and only if $\{1\}$ is a MTL filter of L.*

3.4 The class of divisible filters

Definition 21. *([4], [5]) A filter F of a residuated lattice L is called* divisible filter *if L/F is a divisible residuated lattice.*

Let $\mathbf{DivF(L)}$ be the set of all divisible filters of L.

Theorem 22. *([5]) For a filter F of a residuated lattice L the following conditions are equivalent:*

(*i*) $F \in \mathbf{DivF(L)}$;

(*ii*) $(x \wedge y) \to [x \odot (x \to y)] \in F$, *for every* $x, y \in L$.

Theorem 23. *Let L be a residuated lattice. L is a divisible residuated lattice if and only if any filter of L is a divisible filter.*

Proof. If we suppose that L is a divisible residuated lattice then $x \wedge y = x \odot (x \to y)$, for every $x, y \in L$. So, $(x \wedge y) \to [x \odot (x \to y)] = 1 \in F$, for any filter F of L. From Theorem 22 we deduce that $F \in \mathbf{DivF}(\mathbf{L})$.

Conversely, if any filter of L is a divisible filter, then $F = \{1\}$ is a divisible filter, so $(x \wedge y) \to [x \odot (x \to y)] = 1$, for every $x, y \in L$. We deduce that $x \wedge y = x \odot (x \to y)$, so, L is a divisible residuated lattice. □

Corollary 24. *A residuated lattice L is divisible if and only if $\{1\}$ is a divisible filter of L.*

3.5 The class of BL filters

Definition 25. *([4], [5]) A filter F of a residuated lattice L is called* BL filter *if L/F is a BL algebra.*

Let $\mathbf{BLF}(\mathbf{L})$ be the set of all BL filters of L.

In [5] it is proved the following result : $\mathbf{BLF}(\mathbf{L}) = \mathbf{MTLF}(\mathbf{L}) \cap \mathbf{DivF}(\mathbf{L})$.

Theorem 26. *([5]) For a filter F of a residuated lattice L the following three conditions are equivalent:*

(*i*) $F \in \mathbf{BLF}(\mathbf{L})$;

(*ii*) *If* $(x \to y) \to (x \to z) \in F$, *then* $(x \to z) \vee (y \to z) \in F$;

(*iii*) $((x \to y) \to (x \to z)) \to ((x \to z) \vee (y \to z)) \in F$, *for every* $x, y, z \in L$.

Theorem 27. *A residuated lattice is a BL algebra if and only if any filter is a BL filter.*

Proof. We consider L a BL algebra. Then L is a divisible residuated lattice and a MTL algebra. So, since $\mathbf{BLF}(\mathbf{L}) = \mathbf{MTLF}(\mathbf{L}) \cap \mathbf{DivF}(\mathbf{L})$, by Theorems 19 and 23 any filter of L is a BL filter.

Conversely, if any filter of L is a BL filter, then $\{1\}$ is a BL filter, so $\{1\} \in \mathbf{MTLF}(\mathbf{L}) \cap \mathbf{DivF}(\mathbf{L})$. From Corollaries 20 and 24, L is a BL algebra. □

Corollary 28. *A residuated lattice L is a BL algebra if and only if $\{1\}$ is a BL filter of L.*

3.6 The class of MV filters

Definition 29. *([4], [22]) A filter F of a residuated lattice L is called MV filter of L if L/F is a MV algebra.*

Let $\mathbf{MVF}(\mathbf{L})$ be the set of all MV filters of L.

Theorem 30. *([3], [22]) For a filter F of a residuated lattice L the following statements are equivalent:*

(*i*) $F \in \mathbf{MVF(L)}$;

(*ii*) $((x \to y) \to y) \to ((y \to x) \to x) \in F$ *for every* $x, y \in L$.

Theorem 31. *([22]) A residuated lattice is a MV algebra if and only if any filter is a MV filter.*

Corollary 32. *([22]) A residuated lattice L is a MV algebra if and only if $\{1\}$ is a MV filter.*

4 Gődel filters in a residuated lattice

In this section we introduce a new type of filters in a residuated lattice (taking [4] as guideline): *Gődel filters.* So, we characterize the filters for which the quotient algebra that is constructed via these filters is a *Gődel algebra*.

We deduce that a residuated lattice is a Gődel algebra if and only if every filter is a Gődel filter. Also, using these filters, we obtain a new characterization for Gődel algebras: A residuated lattice L is a Gődel algebra if and only if $[x \to (x \odot y)] \vee [y \to (x \odot y)] = 1$, for every $x, y \in L$, see Corollary 59.

In [12] and [14], Gődel filters were studied in the particular case of BL algebras, so in this section we generalize these results for residuated lattices.

As in [4] we say:

Definition 33. *A filter F of a residuated lattice L is called* Gődel filter *if L/F is a Gődel algebra.*

Let $\mathbf{GoF(L)}$ be the set of all Gődel filters of L.

Remark 34. *In [12] and [14], for a Gődel filter in a BL algebra is used the name of implicative filter.*

We recall some equivalent conditions for Gődel filters in a BL algebra:

Proposition 35. *([12], [14]) For a filter F of a BL algebra L the following conditions are equivalent:*

(*i*) *F is a Gődel filter of L;*

(*ii*) $x \to x^2 \in F$ *for every* $x \in L$;

(*iii*) *If* $y \to (y \to x) \in F$, *then* $y \to x \in F$;

(*iv*) *If* $x \to (y \to z) \in F$, *then* $(x \to y) \to (x \to z) \in F$, *for every* $x, y, z \in L$.

Theorem 36. *In any residuated lattice* L,

$$\mathbf{GoF}(\mathbf{L}) = \mathbf{BLF}(\mathbf{L}) \cap \mathbf{HF}(\mathbf{L}).$$

Proof. If we consider $F \in \mathbf{BLF}(\mathbf{L}) \cap \mathbf{HF}(\mathbf{L})$, then L/F is a BL algebra and since $F \in \mathbf{HF}(\mathbf{L})$, then $x \to x^2 \in F$, for every $x \in L$ (by Theorem 14, (ii)). We deduce that $x/F \to x^2/F = 1/F$, so $x/F \leq x^2/F$. Since for every $x \in L$, $x^2/F \leq x/F$, we obtain $x/F = x^2/F$, so L/F is a Gődel algebra and F is a Gődel filter.

Conversely, we consider $F \in \mathbf{GoF}(\mathbf{L})$. Then L/F is a Gődel algebra, i.e., L/F is a BL algebra and $x/F = (x/F)^2$, for every $x \in L$. We conclude that F is a BL filter and $x \to x^2, x^2 \to x \in F$, hence also F is a Heyting filter (see Theorem 14, (ii)). So, $F \in \mathbf{BLF}(\mathbf{L}) \cap \mathbf{HF}(\mathbf{L})$. $\square$

Theorem 37. *For a filter F of a residuated lattice L, the following are equivalent:*

(i) *$F \in \mathbf{GoF}(\mathbf{L})$;*

(ii) *For $x, y, z \in L$, if $x \to (y \to z) \in F$, then $(x \to z) \vee (y \to z) \in F$.*

Proof. $(i) \Rightarrow (ii)$. We presume that $F \in \mathbf{GoF}(\mathbf{L})$ and let $x, y, z \in L$ such that $x \to (y \to z) \in F$. Since every Gődel filter is a Heyting filter (see Theorem 36), we deduce from Theorem 14, (iv), that $(x \to y) \to (x \to z) \in F$. Since every Gődel filter is also a BL filter (see Theorem 36), we deduce from Theorem 26, (ii), that $(x \to z) \vee (y \to z) \in F$.

$(ii) \Rightarrow (i)$. To show that F is a Gődel filter it is suffice to prove that F is a Heyting filter and a BL filter. To demonstrate that F is a Heyting filter it is enough to justify that F verifies condition (ii) from Theorem 14. Since $x \to (x \to x^2) = x^2 \to x^2 = 1 \in F$, using (ii), we deduce that $(x \to x^2) \vee (x \to x^2) = x \to x^2 \in F$, that is, F is a Heyting filter of L. Following Theorem 26, to prove that F is a BL filter it is suffice to confirm that F verifies condition (ii) from this Theorem. So, let $x, y, z \in L$ such that $(x \to y) \to (x \to z) \in F$. Since $(x \to y) \to (x \to z) \overset{c_5}{\leq} x \to (y \to z)$ and F is a filter, we deduce that $x \to (y \to z) \in F$. Now, using the condition (ii) from this Theorem, we obtain that $(x \to z) \vee (y \to z) \in F$, that is, F is a BL filter. $\square$

Proposition 38. *Let F be a filter of a residuated lattice L. If F is a Gődel filter of L, then $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, for every $x, y \in L$.*

Proof. Let $F \in \mathbf{GoF}(\mathbf{L})$. Using Theorem 36, $F \in \mathbf{BLF}(\mathbf{L}) \cap \mathbf{HF}(\mathbf{L})$.

Since $F \in \mathbf{HF}(\mathbf{L})$, following Theorem 14, (iii), $(x \wedge y) \to (x \odot y) \in F$, for every $x, y \in L$. Since $x \odot (x \to y) \leq x \wedge y$ we deduce that $(x \wedge y) \to (x \odot y) \leq$

$[x \odot (x \to y)] \to (x \odot y) \overset{c_8}{=} (x \to y) \to [x \to (x \odot y)]$. By hypothesis, F is a filter, so $(x \to y) \to [x \to (x \odot y)] \in F$. But $F \in \mathbf{BLF(L)}$, so using Theorem 26, (ii), we get $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, for every $x, y \in L$. □

Proposition 39. *Let F be a filter of a residuated lattice L. If $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, for every $x, y \in L$, then $F \in \mathbf{MTLF(L)} \cap \mathbf{HF(L)}$.*

Proof. Since $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, for every $x, y \in L$, then for $x = y$ we obtain that $(x \to x^2) \vee (x \to x^2) = x \to x^2 \in F$, so $F \in \mathbf{HF(L)}$, see Theorem 14, (ii). Since $x \odot y \leq x, y$ we obtain $x \to (x \odot y) \leq x \to y$ and $y \to (x \odot y) \leq y \to x$, so $[x \to (x \odot y)] \vee [y \to (x \odot y)] \leq (x \to y) \vee (y \to x)$, for every $x, y \in L$. By hypothesis, F is a filter, so $(x \to y) \vee (y \to x) \in F$. Using Theorem 18, (ii), $F \in \mathbf{MTLF(L)}$, so, $F \in \mathbf{MTLF(L)} \cap \mathbf{HF(L)}$. □

In [3] it is proved that $\mathbf{HF(L)} \cap \mathbf{MVF(L)} = \mathbf{BF(L)}$.

Remark 40. *In any residuated lattice L,*

$$\mathbf{HF(L)} \subsetneqq \mathbf{DivF(L)}.$$

Indeed, if we suppose that $F \in \mathbf{HF(L)}$, then L/F is a Heyting algebra. By Remark 5, L/F is a divisible residuated lattice, so $F \in \mathbf{DivF(L)}$. Obviously, $\mathbf{HF(L)} \neq \mathbf{DivF(L)}$. Indeed, since every MV algebra is a BL algebra, thus a divisible residuated lattice, then $F = \{1, d\}$ from Example by Remark 51 is a divisible filter but $b \to b^2 = b \to 0 = c \notin F$, so by Theorem 14, it is not a Heyting filter of L.

Theorem 41. *In any residuated lattice L,*

$$\mathbf{GoF(L)} = \mathbf{MTLF(L)} \cap \mathbf{HF(L)}.$$

Proof. Let $F \in \mathbf{MTLF(L)} \cap \mathbf{HF(L)}$. Since $\mathbf{HF(L)} \subseteq \mathbf{DivF(L)}$, see Remark 40, we deduce that $F \in \mathbf{MTLF(L)} \cap \mathbf{DivF(L)} = \mathbf{BLF(L)}$, see [5]. Thus, $F \in \mathbf{BLF(L)} \cap \mathbf{HF(L)} = \mathbf{GoF(L)}$, see Theorem 36.

Conversely, if $F \in \mathbf{GoF(L)}$, then by Theorem 36, $F \in \mathbf{BLF(L)} \cap \mathbf{HF(L)}$. Since every BL algebra is a MTL algebra, $F \in \mathbf{MTLF(L)} \cap \mathbf{HF(L)}$. □

Using Propositions 38, 39 and Theorem 41, we deduce:

Corollary 42. *For a filter F of a residuated lattice L, the following conditions are equivalent:*

(i) $F \in \mathbf{GoF(L)}$;

(ii) $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, *for every* $x, y \in L$.

Using Corollary 42 and Theorem 37, we obtain:

Corollary 43. *Let $F \in \mathbf{F}(\mathbf{L})$. The following are equivalent:*

(i) $F \in \mathbf{GoF}(\mathbf{L})$;

(ii) $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, *for every* $x, y \in L$;

(iii) *For* $x, y, z \in L$, *if* $x \to (y \to z) \in F$, *then* $(x \to z) \vee (y \to z) \in F$.

Remark 44. *It is possible to give a direct proof for the equivalence conditions* (ii) *and* (iii) *from Corollary 43. Indeed, let* F *be a filter of a residuated lattice* L : $(ii) \Rightarrow (iii)$. *Let* $x, y, z \in L$ *such that* $x \to (y \to z) \in F$. *Using* (ii), $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$. *Since* F *is a filter* $a = [x \to (y \to z)] \odot ([x \to (x \odot y)] \vee [y \to (x \odot y)]) \in F$. *From* c_{10}, $a = ([x \to (y \to z)] \odot [x \to (x \odot y)]) \vee ([x \to (y \to z)] \odot [y \to (x \odot y)])$. *By* c_5 *and* c_8, $x \to (y \to z) = (x \odot y) \to z \leq [x \to (x \odot y)] \to (x \to z)$. *Hence,* $[x \to (y \to z)] \odot ([x \to (x \odot y)] \leq x \to z$. *In a similar way,* $[x \to (y \to z)] \odot [y \to (x \odot y)] = [y \to (x \to z)] \odot [y \to (x \odot y)] \leq y \to z$. *So,* $a \leq (x \to z) \vee (y \to z)$. *Since* F *is a filter and* $a \in F$, *we deduce that* $(x \to z) \vee (y \to z) \in F$. *Now we prove* $(iii) \Rightarrow (ii)$. *Let* $x, y \in L$. *Since* $x \to [y \to (x \odot y)] = (x \odot y) \to (x \odot y) = 1 \in F$, *from* (iii) *we deduce that* $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$.

Remark 45. *In Corollary 43, for* L *a BL algebra, the condition* (iii) *becomes: If* $x \to (y \to z) \in F$, *then* $(x \to z) \vee (y \to z) = (x \wedge y) \to z = [x \odot (x \to y)] \to z = (x \to y) \to (x \to z) \in F$. *So, in this case, the condition* (iii) *from Corollary 43 coincides with the condition* (iv) *from Proposition 35. Thus, if the residuated lattice is a BL algebra we get the results from [12] and [14].*

Remark 46. *Using the equivalent conditions for Gődel and Heyting filters from Theorem 14 and Corollary 43 it results directly that:* $\mathbf{GoF}(\mathbf{L}) \subseteq \mathbf{HF}(\mathbf{L})$. *Indeed, if* F *is a Gődel filter, then* $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$, *for every* $x, y \in L$. *But from* c_{10}, $[x \to (x \odot y)] \vee [y \to (x \odot y)] \leq (x \wedge y) \to (x \odot y)$. *Since* F *is a filter,* $(x \wedge y) \to (x \odot y) \in F$, *so* F *is a Heyting filter.*

Theorem 47. *In any residuated lattice* L,

$$\mathbf{GoF}(\mathbf{L}) \cap \mathbf{MVF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L}).$$

Proof. Since $\mathbf{GoF}(\mathbf{L}) \subseteq \mathbf{HF}(\mathbf{L})$ and $\mathbf{HF}(\mathbf{L}) \cap \mathbf{MVF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L})$, see [3], we have $\mathbf{GoF}(\mathbf{L}) \cap \mathbf{MVF}(\mathbf{L}) \subseteq \mathbf{BF}(\mathbf{L})$.

Conversely, we consider $F \in \mathbf{BF}(\mathbf{L})$. Then L/F is a Boolean algebra, so is a Gődel algebra and a MV algebra (see [6], [16], [18]). We conclude that $F \in \mathbf{GoF}(\mathbf{L}) \cap \mathbf{MVF}(\mathbf{L})$, so $\mathbf{GoF}(\mathbf{L}) \cap \mathbf{MVF}(\mathbf{L}) = \mathbf{BF}(\mathbf{L})$. □

Remark 48. *In a residuated lattice L, we have*

$$\mathbf{BF}(\mathbf{L}) \subsetneqq \mathbf{GoF}(\mathbf{L}).$$

Indeed, from Theorem 47, $\mathbf{BF}(\mathbf{L}) \subseteq \mathbf{GoF}(\mathbf{L})$. *We show that* $\mathbf{GoF}(\mathbf{L}) \neq \mathbf{BF}(\mathbf{L})$. *We consider the following example ([13]) of a BL-algebra (it is not a MV algebra): $L = \{0, a, b, c, 1\}$, with $0 < c < a, b < 1$, a, b incomparable and the operations:*

$$\begin{array}{c|ccccc} \rightarrow & 0 & c & a & b & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 \\ c & 0 & 1 & 1 & 1 & 1 \\ a & 0 & b & 1 & b & 1 \\ b & 0 & a & a & 1 & 1 \\ 1 & 0 & c & a & b & 1 \end{array}, \qquad \begin{array}{c|ccccc} \odot & 0 & c & a & b & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & c & c & c & c \\ a & 0 & c & a & c & a \\ b & 0 & c & c & b & b \\ 1 & 0 & c & a & b & 1 \end{array}.$$

It is easy to see that $F = \{1, a\}$ is a Gődel filter, since $[x \rightarrow (x \odot y)] \vee [y \rightarrow (x \odot y)] = 1$, for every $x, y \in L$. Also $b \vee b^* = b \vee 0 = b \notin F$, so, F is not a Boolean filter.

Example 49. *In [13] it is proved that $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and a, b incomparable is a residuated lattice with the operations:*

$$\begin{array}{c|ccccc} \rightarrow & 0 & a & b & c & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 \\ a & b & 1 & b & 1 & 1 \\ b & a & a & 1 & 1 & 1 \\ c & 0 & a & b & 1 & 1 \\ 1 & 0 & a & b & c & 1 \end{array}, \qquad \begin{array}{c|ccccc} \odot & 0 & a & b & c & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & 0 & a & a \\ b & 0 & 0 & b & b & b \\ c & 0 & a & b & c & c \\ 1 & 0 & a & b & c & 1 \end{array}.$$

In fact, L is a Gődel algebra. Obviously, $F = \{a, c, 1\} \in$ $\mathbf{BF}(\mathbf{L})$, see [3]. Using Theorem 47 we obtain that $F \in$ $\mathbf{GoF}(\mathbf{L})$.

Remark 50. *There are Gődel filters which are not MV filters, i.e.,* $\mathbf{GoF}(\mathbf{L}) \setminus \mathbf{MVF}(\mathbf{L}) \neq \emptyset$. *Indeed, in [3] is proved that $F = \{1, a\}$ from Example by Remark 48 is not a MV filter but it is a Gődel filter.*

Remark 51. *There are MV filters which are not Gődel filters, i.e.,* $\mathbf{MVF}(\mathbf{L}) \setminus \mathbf{GoF}(\mathbf{L}) \neq \emptyset$. *We consider $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1, 0 < b < d < 1$, a, b and, respective c, d incomparable (see [13]) a finite residuated lattice which is a MV algebra. We have in L the following operations:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

It is easy to see that $F = \{1, d\}$ is a filter and F is not a Gődel filter because $[b \to (b \odot b)] \vee [b \to (b \odot b)] = (b \to 0) \vee (b \to 0) = c \vee c = c \notin F$. Since L is an MV algebra, every filter of L is a MV filter. In particular $\{1, d\}$ is a MV filter.

Remark 52. *In any residuated lattice L, we have*

$$\mathbf{GoF}(\mathbf{L}) \subsetneqq \mathbf{BLF}(\mathbf{L}).$$

Indeed, from Theorem 36, $\mathbf{GoF}(\mathbf{L}) \subseteq \mathbf{BLF}(\mathbf{L})$. We show that $\mathbf{GoF}(\mathbf{L}) \neq \mathbf{BLF}(\mathbf{L})$. Since every MV algebra is a BL algebra, then the filter $F = \{1, d\}$ from Example by Remark 51 is a BL filter which is not a Gődel filter.

Remark 53. *In any residuated lattice L, we have*

$$\mathbf{GoF}(\mathbf{L}) \subsetneqq \mathbf{HF}(\mathbf{L}).$$

Indeed, from Theorem 36, $\mathbf{GoF}(\mathbf{L}) \subseteq \mathbf{HF}(\mathbf{L})$. To show that $\mathbf{GoF}(\mathbf{L}) \neq \mathbf{HF}(\mathbf{L})$, we consider the following example ([13]) of a finite residuated lattice which is a Heyting algebra : Let $L = \{0, a, b, p, n, c, d, 1\}$, with $0 < a, b < p < n < c, d < 1$, a, b and c, d incomparable, and the following operations:

$\to$	0	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
p	0	a	b	1	1	1	1	1
n	0	a	b	p	1	1	1	1
c	0	a	b	p	d	1	d	1
d	0	a	b	p	c	c	1	1
1	0	a	b	p	n	c	d	1

,

$\odot$	0	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
p	0	a	b	p	p	p	p	p
n	0	a	b	p	n	n	n	n
c	0	a	b	p	n	c	n	c
d	0	a	b	p	n	n	d	d
1	0	a	b	p	n	c	d	1

.

It is easy to see that $F = \{1, d\}$ is a Heyting filter and is not a Gődel filter, since $[a \to (a \odot b)] \vee [b \to (a \odot b)] = (a \to 0) \vee (b \to 0) = b \vee a = p \notin F$.

Remark 54. *In any residuated lattice L, we have*

$$\mathbf{GoF(L)} \subsetneqq \mathbf{MTLF(L)}.$$

Indeed, from Theorem 41, $\mathbf{GoF(L)} \subseteq \mathbf{MTLF(L)}$. *We show that* $\mathbf{GoF(L)} \neq \mathbf{MTLF(L)}$. *Since every MV algebra is a MTL algebra, then the filter $F = \{1, d\}$ from Example by Remark 51 is a MTL filter which is not a Gődel filter.*

Proposition 55. *Let L be a residuated lattice and $F, G \in \mathbf{GoF(L)}$. Then $F \cap G \in \mathbf{GoF(L)}$.*

Proof. Since $F, G \in \mathbf{GoF(L)}$, for $x, y \in L$, by Corollary 43, (ii), $[x \to (x \odot y)] \vee [y \to (x \odot y)]$ is in F and also in G, so $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F \cap G$, that is, $F \cap G \in \mathbf{GoF(L)}$. $\square$

Proposition 56. *Suppose that F and G are two filters of a residuated lattice L and $F \subseteq G$. If $F \in \mathbf{GoF(L)}$, then $G \in \mathbf{GoF(L)}$.*

Proof. Since $F \in \mathbf{GoF(L)}$, for $x, y \in L$, from Corollary 43, (ii), we have that $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in F$. By hypothesis, $F \subseteq G$, so $[x \to (x \odot y)] \vee [y \to (x \odot y)] \in G$, for each $x, y \in L$. $\square$

Theorem 57. *A residuated lattice is a Gődel algebra if and only if any filter is a Gődel filter.*

Proof. Let L be a Gődel algebra, i.e., a BL algebra with $x^2 = x$, for every $x \in L$ (or equivalent, $x \wedge y = x \odot y$, for every $x, y \in L$). If F is a filter of L, then $1 \in F$ and $[x \to (x \odot y)] \vee [y \to (x \odot y)] = [x \to (x \wedge y)] \vee [y \to (x \wedge y)] = (x \to y) \vee (y \to x) = 1 \in F$, so by Corollary 43 we deduce that $F \in \mathbf{GoF(L)}$.

Conversely, if any filter of L is a Gődel filter, then $F = \{1\}$ is a Gődel filter. Using Theorem 36 we deduce that $\{1\}$ is a BL filter of L and from Corollary 28, L is a BL algebra. Now, using Proposition 35, $x \to x^2 \in F = \{1\}$, for every $x \in L$, so $x = x^2$, thus L is a Gődel algebra. $\square$

Corollary 58. *A residuated lattice L is a Gődel algebra if and only if $\{1\}$ is a Gődel filter of L.*

Using Corollary 58 we obtain:

Corollary 59. *A residuated lattice L is a Gődel algebra if and only if $[x \to (x \odot y)] \vee [y \to (x \odot y)] = 1$, for every $x, y \in L$.*

So, we obtain an equivalent definition of Gődel algebras:

In a residuated lattice L we consider the condition:

(Go) $[x \to (x \odot y)] \vee [y \to (x \odot y)] = 1$, for every $x, y \in L$.

Definition 60. *The residuated lattice L is called* Gődel algebra *if it satisfies* (Go) *condition.*

Finally, we remark that the connections studied in this paper, between Gődel filters and other filters are in concordance with the relations between the corresponding classes of algebras, mentioned in Section 2.

5 Injective objects in the Gődel algebras category

In this section we present some results for injective Gődel algebras. So, we show that a complete Boolean algebra is an injective object in the category of Gődel algebras.

In what follows, we consider G a Gődel algebra and $B(G)$ the Boolean algebra of all complemented elements of G.

Proposition 61. *For $g \in G$, the following are equivalent:*

(i) $g \in B(G)$;

(ii) $g = g^{**}$;

(iii) $g \vee g^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Let $g \in B(G)$. There exists $x \in G$ such that $g \vee x = 1$ and $g \wedge x = 0$. Thus, $x \leq g^*$. But $g^* = 1 \odot g^* = (g \vee x) \odot g^* = (g \odot g^*) \vee (x \odot g^*) = 0 \vee (x \odot g^*) = x \odot g^* \leq x$. It results $x = g^*$. So $g^* \in B(G)$. Analogously g^{**} is the complement of g^*. Because G is a distributive lattice, we get $g = g^{**}$.

$(ii) \Rightarrow (iii)$. $(g \to g^*) \to g^* = [(g \odot g) \to 0] \to g^* = g^* \to g^* = 1$. Since $g^* \odot (g^* \to g) = g^* \wedge g = g^* \odot g = 0$ we obtain that $g^* \to g \leq g^{**} = g$. But, $g \leq g^* \to g$, so we have that $g^* \to g = g$. Since G is a BL algebra, $g \vee g^* = [(g \to g^*) \to g^*] \wedge [(g^* \to g) \to g] = 1$.

$(iii) \Rightarrow (i)$. From $g \vee g^* = 1$ we obtain $g \wedge g^* = 0$, so $g \in B(G)$. ☐

Remark 62. *From Proposition 61, $B(G) = \{g \in G : g^{**} = g\} = \{g^* : g \in G\}$.*

Definition 63. *For G_1, G_2 two Gődel algebras, $f : G_1 \to G_2$ is an morphism of Gődel algebras if for every $x, y \in G_1$: $f(0_{G_1}) = 0_{G_2}$, $f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$, for every $x, y \in G_1$.*

Remark 64. *A morphism of Gődel algebras $f : G_1 \to G_2$ verifies $f(x^*) = [f(x)]^*$, $f(1_{G_1}) = 1_{G_2}$, $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee y) = f(x) \vee f(y)$, $(f(x))^2 = f(x)$ for every $x, y \in G_1$. Hence every morphism of Gődel algebras is also a morphism of Boolean algebras.*

Let $\mathcal{GO}$ be the category of Gődel algebras. Obviously, $\mathcal{B} \subseteq \mathcal{GO}$, where $\mathcal{B}$ is the category of Boolean algebras.

Remark 65. *Since $\mathcal{GO}$ and $\mathcal{B}$ are equational categories, then the monomorphisms and the injective morphisms are the same.*

Theorem 66. *$\mathcal{B}$ is a reflective subcategory of $\mathcal{GO}$. The reflector $\mathcal{R} : \mathcal{GO} \to \mathcal{B}$ preserves monomorphisms.*

Proof. Let $G \in Ob(\mathcal{GO})$ and we define, using Proposition 61,

$$\mathcal{R}(G) = B(G) = \{g^* : g \in G\} = \{g \in G : g^{**} = g\},$$

(which is a Boolean subalgebra of G) and $\Phi_{\mathcal{R}}(G) : G \to \mathcal{R}(G)$ by $\Phi_{\mathcal{R}}(G)(x) = x^{**}$, for every $x \in G$. Since G is a Gődel algebra, $(x \wedge y)^{**} = x^{**} \wedge y^{**}$, $(x \vee y)^{**} = x^{**} \vee y^{**}, (x \odot y)^{**} = x^{**} \odot y^{**}$, $(x \to y)^{**} = x^{**} \to y^{**}$ for every $x, y \in G$, so $\Phi_{\mathcal{R}}(G)$ is a morphism in $\mathcal{GO}$. For $G_1, G_2 \in Ob(\mathcal{GO})$ and $f \in Hom_{\mathcal{GO}}(G_1, G_2)$, let $\mathcal{R}(f) : \mathcal{R}(G_1) \to \mathcal{R}(G_2)$, $\mathcal{R}(f)(x^*) = f(x^*) = (f(x))^*$ for each $x \in G_1$. We prove that $\mathcal{R}(f)$ is a morphism in $\mathcal{B}$. For $x, y \in G_1$, $(x \wedge y)^* = x^* \vee y^*$, $(x \vee y)^* = x^* \wedge y^*$, so $\mathcal{R}(f)(x^* \wedge y^*) = \mathcal{R}(f)((x \vee y)^*) = (f(x \vee y))^* = f(x)^* \wedge f(y)^* = \mathcal{R}(f)(x^*) \wedge \mathcal{R}(f)(y^*)$, $\mathcal{R}(f)(x^* \vee y^*) = \mathcal{R}(f)((x \wedge y)^*) = (f(x \wedge y))^* = f(x)^* \vee f(y)^* = \mathcal{R}(f)(x^*) \vee \mathcal{R}(f)(y^*)$, $\mathcal{R}(f)(0) = \mathcal{R}(f)(1^*) = (f(1))^* = 1^* = 0$. Analogously, $\mathcal{R}(f)(1) = 1$ and $\mathcal{R}(f)((x^*)^*) = (\mathcal{R}(f)(x^*))^*$. In this way, $\mathcal{R} : \mathcal{GO} \to \mathcal{B}$ becomes a covariant functor.

For $G_1, G_2 \in Ob(\mathcal{GO})$, let us consider the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{f} & G_2 \\ \downarrow_{\Phi_{\mathcal{R}}(G_1)} & & \downarrow_{\Phi_{\mathcal{R}}(G_2)} \\ \mathcal{R}(G_1) & \xrightarrow{\mathcal{R}(f)} & \mathcal{R}(G_2) \end{array}$$

For $x \in G_1$, $(\Phi_{\mathcal{R}}(G_2) \circ f)(x) = (f(x))^{**}$ and $(\mathcal{R}(f) \circ \Phi_{\mathcal{R}}(G_1))(x) = \mathcal{R}(f)(x^{**}) = (f(x))^{**}$, hence the diagram is a commutative one. For $G \in Ob(\mathcal{GO})$, $B \in Ob(\mathcal{B})$ and $f : G \to B$ a morphism in $\mathcal{GO}$ we consider $f' = f_{|\mathcal{R}(G)} : \mathcal{R}(G) \to B$, such that, for $x \in G$, $f'(x^*) = f(x)^*$. For $x, y \in G$, we have $f'(x^* \wedge y^*) = f'((x \vee y)^*) = (f(x \vee y))^* = f(x)^* \wedge f(y)^* = f'(x^*) \wedge f'(y^*)$. Analogously, $f'(x^* \vee y^*) = f'(x^*) \vee f'(y^*)$. Also, $f'((x^*)^*) = f((x^*)^*) = f(x)^{**} = f(x^*)^* = (f'(x^*))^*$, and similarly $f'(0) = 0$ and $f'(1) = 1$. We conclude that f' is an morphism in $\mathcal{B}$. From $(f' \circ \Phi_{\mathcal{R}}(G))(x) = f'(\Phi_{\mathcal{R}}(G)(x)) = f'(x^{**}) = f(x)^{**} = f(x)$, we get $f' \circ \Phi_{\mathcal{R}}(G) = f$.

To prove the uniqueness of f' we consider $f'' : \mathcal{R}(G) \to B$ a morphism in $\mathcal{B}$ with $f'' \circ \Phi_{\mathcal{R}}(G) = f$. Then $f'(x^*) = f(x^*) = (f'' \circ \Phi_{\mathcal{R}}(G))(x^*) = f''(x^*)$, for each $x \in G$ and so $f'' = f'$.

Finally, to verify that $\mathcal{R}$ preserves monomorphisms, let us consider $f : G_1 \to G_2$ a monomorphism in $\mathcal{GO}$ and $x, y \in G_1$ such that $\mathcal{R}(f)(x^*) = \mathcal{R}(f)(y^*)$.

From $f(x^*) = f(y^*)$, we obtain $x^* = y^*$, so $\mathcal{R}(f)$ is a monomorphism in $\mathcal{B}$, using Remark 65. □

Theorem 67. *Each complete Boolean algebra is an injective object in $\mathcal{GO}$.*

Proof. We know that the injective objects in $\mathcal{B}$ are exactly the complete Boolean algebras. From Theorem 66, $\mathcal{B}$ is a reflective subcategory of $\mathcal{GO}$ and the reflector $\mathcal{R} : \mathcal{GO} \to \mathcal{B}$ preserves monomorphisms. Using [1], these facts imply that each complete Boolean algebra is an injective object in $\mathcal{GO}$. □

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