DOI: 10.2478/auom-2021-0010
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 29(1),2021, 143–164

Filters of strong Sheffer stroke non-associative MV-algebras

Tahsin Oner, Tugce Katican, Arsham Borumand Saeid* and Mehmet Terziler

Abstract

In this paper, at first we study strong Sheffer stroke NMV-algebra. For getting more results and some classification, the notions of filters and subalgebras are introduced and studied. Finally, by a congruence relation, we construct a quotient strong Sheffer stroke NMV-algebra and isomorphism theorems are proved.

1 Introduction

MV-algebras were introduced and developed by C. C. Chang as a model of the Lukasiewicz many-valued logic ([11], [12]). Also, a current and abbreviated axiomatization of MV-algebras is found in [13]. The concept of non-associative MV-algebra (for short, NMV-algebra) is introduced and studied by Chajda et al. ([9], [10]) since associativity of the binary operation of MV-algebras leads to serious problems in expert systems in artificial intelligence ([5], [10]). Chajda and Khr generalized MV-algebras in such a way that the binary relation defined on these algebras derived from associativity property is still a partial order. However, the transitivity of $\leq$ cannot be shown without associativity of the operation on MV-algebras, called the identity $(MV1)$. Hence, $(MV1)$ is replaced by two other axioms holding in MV-algebras and enabling $\leq$ to be transitive [9].

Key Words: Sheffer stroke, (Sheffer stroke) NMV-algebra, filter, congruence, homomorphism.
2010 Mathematics Subject Classification: Primary 06F05, 03G25; Secondary 03G10.
Received: 01.04.2020
Accepted: 25.05.2020

Besides, filter theory plays a crucial role in the studies on the algebraic structures such as Hilbert algebras, BCI/BCK-algebras, BZ-algebras, implication algebras, residuated lattices, etc. which have a distinguished element and satisfy some common identities. Especially, certain types of filters in the MV-algebras have been investigated in the studies [2].

On the other hand, it is well known that all Boolean operations on Boolean algebras can be defined by a single binary operation introduced by H. M. Sheffer, called the Sheffer stroke operation [15]. The most important application is to have all diods on the chip forming processor in a computer, and so it is simpler and cheaper than to produce different diods for other Boolean operations. Besides, it has many algebraic applications in algebraic structures of classic and non-classic logics such as Boolean algebras [14], [15], ortholattices [6] and orthoimplication algebras [1]. Recently, Chajda et al. introduced strong Sheffer stroke NMV-algebras which are equivalent to NMV-algebras with the Sheffer stroke operation [8].

In this study, it is first presented Sheffer operation, a strong Sheffer stroke NMV-algebra and a partial order on this algebraic structure. Then a filter and a congruence relation on a strong Sheffer stroke NMV-algebra are defined and the relationship between them is analyzed. Consequently, it is proved that the family of all its filters is isomorphic to the family of all its congruence relations. By means of the congruence relation, a quotient strong Sheffer stroke NMV-algebra is built from a strong Sheffer stroke NMV-algebra. Besides, some properties of the quotient strong Sheffer stroke NMV-algebras are investigated. A homomorphism between strong Sheffer stroke NMV-algebras is described and its relationships with the mentioned concepts are examined. The isomorphism theorems are proved by the kernel of a homomorphism.

2 Preliminaries

In this section, basic definitions and notions about Sheffer stroke operation and strong Sheffer stroke NMV-algebras are given.

Definition 2.1. *[14] Let $\mathcal{A} = (A, |)$ be a groupoid. The operation $|$ on A is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$(S1)$ $x|y = y|x$,
$(S2)$ $(x|x)|(x|y) = x$,
$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2. *[8] A non-associative MV-algebra (NMV-algebra, for short) is an algebra $\boldsymbol{A} = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities*

(1) $x \oplus y \approx y \oplus x$,
(2) $x \oplus 0 \approx x$,
(3) $x \oplus 1 \approx 1$,
(4) $\neg(\neg x) \approx x$,
(5) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$,
(6) $\neg x \oplus (x \oplus y) \approx 1$,
(7) $x \oplus (\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus z) \approx 1$,
where 1 *denotes the algebraic constant* $\neg 0$.

Lemma 2.1. *[9] Let $\boldsymbol{A} = (A, \oplus, \neg, 0)$ be an NMV-algebra. Then the binary relation $\leq$ defined by*

$$x \leq y \text{ if and only if } \neg x \oplus y \approx 1$$

is a partial order on A. Thus, $(A, \leq)$ is a poset with the least element 0 and the greatest element 1 which we call the poset induced by $\boldsymbol{A}$.

Theorem 2.1. *[8] Let $\boldsymbol{A} = (A, \oplus, \neg, 0)$ be an NMV-algebra and put*

$$x|y := \neg x \oplus \neg y$$

for all $x, y \in A$. Then $S(\boldsymbol{A}) = (A, |, 1)$ is a strong Sheffer stroke NMV-algebra.

Theorem 2.2. *[8] Let $\boldsymbol{S} = (A, |, 1)$ be a strong Sheffer stroke NMV-algebra and put*

$$x \oplus y := (x|1)|(y|1),$$
$$\neg x := x|1,$$
$$0 := 1|1,$$

for all $x, y \in A$. Then $A(\boldsymbol{S}) = (A, \oplus, \neg, 0)$ is an NMV-algebra.

Definition 2.3. *[8] An implication NMV-algebra is a non-empty groupoid $\boldsymbol{A} = (A, \rightarrow)$ satisfying the identities*

(a) $x \rightarrow x \approx y \rightarrow y$,
(b) $x \rightarrow 1 \approx 1$,
(c) $1 \rightarrow x \approx x$,
(d) $(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$,
(e) $x \rightarrow (y \rightarrow x) \approx 1$,
(f) $x \rightarrow ((((x \rightarrow y) \rightarrow y) \rightarrow z) \rightarrow z) \approx 1$,
where 1 *denotes the algebraic constant* $x \rightarrow x$.

An implication NMV-algebra with 0 is an algebra $\boldsymbol{A} = (A, \rightarrow, 0)$ of type $(2, 0)$ satisfying the identities $(a) - (f)$ as well as the identity

$$0 \rightarrow x \approx 1.$$

Theorem 2.3. *[8]* $\boldsymbol{A} = (A, \oplus, \neg, 0)$ *be an NMV-algebra and define*

$$x \to y := \neg x \oplus y$$

for all $x, y \in A$. *Then* $(A, \to, 0)$ *is an implication NMV-algebra with* 0.

3 Strong Sheffer stroke NMV-algebras

In this section, we introduce some properties of a strong Sheffer stroke NMV-algebra.

Definition 3.1. *[8] A strong Sheffer stroke NMV-algebra is an algebra* $(A, |, 1)$ *of type* $(2, 0)$ *satisfying the identities for all* $x, y, z \in A$:

(n1) $x|y \approx y|x$,
(n2) $x|0 \approx 1$,
(n3) $(x|1)|1 \approx x$,
(n4) $((x|1)|y)|y \approx ((y|1)|x)|x$,
(n5) $(x|1)|((x|y)|1) \approx 1$,
(n6) $x|(((((x|y)|y)|z)|z)|1) \approx 1$,
where 0 *denotes the algebraic constant* $1|1$.

Proposition 3.1. *Let* $(A, |, 1)$ *be a strong Sheffer stroke NMV-algebra. Then the binary relation* $\leq$ *defined by*

$$x \leq y \text{ if and only if } x|(y|1) \approx 1$$

is a partial order on A. *Hence,* $(A, \leq)$ *is a poset with the least element* 0 *and the greatest element* 1.

Example 3.1. *Consider a structure* $(A, |, 1)$ *with the following Hasse diagram where a set* $A = \{0, a, b, c, d, e, f, 1\}$:

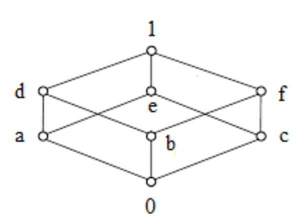

Figure 1:

The binary operation $|$ defined on A has the following Cayley table:

$\vert$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

Table 1: The Cayley table of Sheffer stroke $|$

Then this structure is a strong Sheffer stroke NMV-algebra.

Lemma 3.1. *In a strong Sheffer stroke NMV-algebra $(A, |, 1)$, the following properties hold for all $x, y, z \in A$:*

(*i*) $x|(x|1) \approx 1$,
(*ii*) $x \leq y \Leftrightarrow y|1 \leq x|1$,
(*iii*) $y \leq x|(y|1)$,
(*iv*) $y|1 \leq x|y$,
(*v*) $x \leq (x|y)|y$,
(*vi*) $x \leq (((x|y)|y)|z)|z$,
(*vii*) $((x|y)|y)|y \approx x|y$,
(*viii*) $x|1 \approx x|x$,
(*ix*) $x|(x|x) \approx 1$,
(*x*) $1|(x|x) \approx x$,
(*xi*) $x \leq y \Rightarrow y|z \leq x|z$,
(*xii*) $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$,
(*xiii*) $x|(y|1) \leq (z|(x|1))|((z|(y|1))|1)$,
(*xiv*) $x \leq y$ *and* $z \leq t$ *imply* $y|t \leq x|z$.

Proof. (*i*) It is obtained by (n1), (n3) and (n5).

(*ii*) It follows from Proposition 3.1, (n1) and (n3).

(*iii*) It follows from (n1), (n3) and (n5) that $y|((x|(y|1))|1) \approx ((y|1)|1)|(((y|1)|x)|1) \approx 1$. Then we have from Proposition 3.1 that $y \leq x|(y|1)$.

(iv) By putting $[y := y|1]$ in (iii), we get $y|1 \leq x|((y|1)|1) \approx x|y$ from (n3).

(v) It follows from (n3) and (n6) that $x|(((x|y)|y)|1) \approx x|(((((x|y)|y)|1)|1)|1) \approx 1$. Then $x \leq (x|y)|y$ from Proposition 3.1.

(vi) It is proved from (n6) and Proposition 3.1.

(vii) It is known from (v) that $x|y \leq ((x|y)|y)|y$. Since

$$\begin{array}{rcl}(((x|y)|y)|y)|((x|y)|1) & \approx & ((x|y)|1)|(((x|y)|y)|y) \\ & \approx & ((x|y)|1)|(((((x|y)|1)|1)|y)|y) \\ & \approx & ((x|y)|1)|(((y|1)|((x|y)|1))|((x|y)|1)) \\ & \approx & ((x|y)|1)|(((x|y)|1)|((y|1)|((y|x)|1))) \\ & \approx & 1\end{array}$$

from (n1), (n3)-(n5) and (i), we get $((x|y)|y)|y \leq x|y$. Hence, $((x|y)|y)|y \approx x|y$.

$(viii)$ It is obtained from (iii) that $x|1 \leq x|x$. Since

$$\begin{array}{rcl}(x|x)|((x|1)|1) & \approx & x|(((x|x)|1)|1) \\ & \approx & x|(((x|x)|(x|0))|1) \\ & \approx & x|(x|1) \\ & \approx & 1\end{array}$$

from (n1)-(n3), $(S2)$ and (i), we have $x|x \leq x|1$. Then $x|1 \approx x|x$.

(ix) $x|(x|x) \approx x|(x|1) \approx 1$ from $(viii)$ and (i), respectively.

(x) $1|(x|x) \approx 1|(x|1) \approx x$ by $(viii)$, (n1) and (n3), respectively.

(xi) Let $x \leq y$, i. e., $x|(y|1) \approx 1$. Since

$$\begin{array}{rcl}(y|z)|((x|z)|1) & \approx & (y|z)|((z|x)|(z|x)) \\ & \approx & (((y|z)|z)|((y|z)|z))|x \\ & \approx & ((((z|1)|(y|1))|(y|1))|(((z|1)|(y|1))|(y|1)))|x \\ & \approx & ((z|1)|(y|1))|((x|(y|1))|(x|(y|1))) \\ & \approx & ((z|1)|(y|1))|(1|1) \\ & \approx & 1\end{array}$$

from $(viii)$, (n1)-(n4) and $(S3)$, we have $y|z \leq x|z$.

(xii) Since

$$\begin{aligned}
&(x|(y|1))|(((y|(z|1))|((x|(z|1))|1))|1)\\
&\approx (x|(y|1))|(((y|(z|1))|((x|(z|1))|(x|(z|\\
&\quad 1))))|((y|(z|1))|((x|(z|1))|(x|(z|1)))))\\
&\approx (x|(y|1))|(((((y|(z|1))|(z|1))|((y|(z|1))|(z|1)))\\
&\quad |x)|((((y|(z|1))|(z|1))|((y|(z|1))|(z|1)))|x))\\
&\approx (x|(y|1))|(((((z|(y|1))|(y|1))|((z|(y|1))|(y|1)))\\
&\quad |x)|((((z|(y|1))|(y|1))|((z|(y|1))|(y|1)))|x))\\
&\approx (x|(y|1))|(((z|(y|1))|((x|(y|1))|(x|(y|\\
&\quad 1))))|((z|(y|1))|((x|(y|1))|(x|(y|1)))))\\
&\approx (((x|(y|1))|((x|(y|1))|(x|(y|1))))|((x|\\
&\quad (y|1))|((x|(y|1))|(x|(y|1)))))|(z|(y|1))\\
&\approx (1|1)|(z|(y|1))\\
&\approx 1
\end{aligned}$$

from $(viii)$, (n1)-(n4), $(S3)$ and (ix), it follows $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$.

$(xiii)$ Because

$$\begin{aligned}
&(x|(y|1))|(((z|(x|1))|((z|(y|1))|1))|1)\\
&\approx (x|(y|1))|(((z|(x|1))|((z|(y|1))|(z|(y|\\
&\quad 1))))|((z|(x|1))|((z|(y|1))|(z|(y|1)))))\\
&\approx (x|(y|1))|((((((x|1)|z)|z)|(((x|1)|z)|z))|(y\\
&\quad |1))|(((((x|1)|z)|z)|(((x|1)|z)|z))|(y|1)))\\
&\approx (x|(y|1))|((((((z|1)|x)|x)|(((z|1)|x)|x))|(y\\
&\quad |1))|(((((z|1)|x)|x)|(((z|1)|x)|x))|(y|1)))\\
&\approx (x|(y|1))|((((z|1)|x)|((x|(y|1))|(x|(y|\\
&\quad 1))))|(((z|1)|x)|((x|(y|1))|(x|(y|1)))))\\
&\approx (((x|(y|1))|((x|(y|1))|(x|(y|1))))|((x|\\
&\quad (y|1))|((x|(y|1))|(x|(y|1)))))|((z|1)|x)\\
&\quad \approx (1|1)|((z|1)|x)\\
&\quad \approx 1
\end{aligned}$$

from $(viii)$, (n1)-(n2), (n4), $(S3)$ and (ix) that it is obtained $x|(y|1) \leq (z|(x|1))|((z|(y|1))|1)$.

(xiv) Let $x \leq y$ and $z \leq t$. Then $y|t \leq x|t$ and $x|t \approx t|x \leq z|x \approx x|z$ from (xi) and (n1). Since $\leq$ is transitive, $y|t \leq x|z$. □

Theorem 3.1. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras. Then $(A \times B, |_{A\times B}, 1_{A\times B})$ is a strong Sheffer stroke NMV-algebra* *where the Sheffer stroke* $|_{A\times B}$ *on* $A \times B$ *is defined by* $(x_1, y_1)|_{A\times B}(x_2, y_2) \approx (x_1|_A x_2, y_1|_B y_2)$*, and the constant is* $1_{A\times B} \approx (1_A, 1_B)$*.*

Proof. Let $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) be arbitrary elements in $A \times B$. Since $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ are strong Sheffer stroke NMV-algebras, we have

(1) :

$$\begin{aligned}(x_1, y_1)|_{A\times B}(x_2, y_2) &\approx (x_1|_A x_2, y_1|_B y_2)\\ &\approx (x_2|_A x_1, y_2|_B y_1)\\ &\approx (x_2, y_2)|_{A\times B}(x_1, y_1).\end{aligned}$$

(2) : $(x_1, y_1)|_{A\times B}(0_A, 0_B) = (x_1|_A 0_A, y_1|_B 0_B) \approx (1_A, 1_B) \approx 1_{A\times B}$.

(3) :

$$\begin{aligned}((x_1, y_1)|_{A\times B}1_{A\times B})|_{A\times B}1_{A\times B} &\approx ((x_1, y_1)|_{A\times B}(1_A, 1_B))|_{A\times B}(1_A, 1_B)\\ &\approx ((x_1|_A 1_A)|_A 1_A, (y_1|_B 1_B)|_B 1_B)\\ &\approx (x_1, y_1).\end{aligned}$$

(4) :

$$\begin{aligned}&(((x_1, y_1)|_{A\times B}1_{A\times B})|_{A\times B}(x_2, y_2))|_{A\times B}(x_2, y_2)\\ &\approx (((x_1, y_1)|_{A\times B}(1_A, 1_B))|_{A\times B}(x_2, y_2))|_{A\times B}(x_2, y_2)\\ &\approx (((x_1|_A 1_A)|_A x_2)|_A x_2, ((y_1|_B 1_B)|_B y_2)|_B y_2)\\ &\approx (((x_2|_A 1_A)|_A x_1)|_A x_1, ((y_2|_B 1_B)|_B y_1)|_B y_1)\\ &\approx (((x_2, y_2)|_{A\times B}(1_A, 1_B))|_{A\times B}(x_1, y_1))|_{A\times B}(x_1, y_1)\\ &\approx (((x_2, y_2)|_{A\times B}1_{A\times B})|_{A\times B}(x_1, y_1))|_{A\times B}(x_1, y_1).\end{aligned}$$

(5) :

$$\begin{aligned}&((x_1, y_1)|_{A\times B}1_{A\times B})|_{A\times B}(((x_1, y_1)|_{A\times B}(x_2, y_2))|_{A\times B}1_{A\times B})\\ &\approx ((x_1, y_1)|_{A\times B}(1_A, 1_B))|_{A\times B}(((x_1, y_1)|_{A\times B}(x_2, y_2))|_{A\times B}(1_A, 1_B))\\ &\approx ((x_1|_A 1_A)|_A((x_1|_A x_2)|_A 1_A), (y_1|_B 1_B)|_B((y_1|_B y_2)|_B 1_B))\\ &\approx (1_A, 1_B)\\ &\approx 1_{A\times B}.\end{aligned}$$

(6) :

$$\begin{aligned}&(x_1, y_1)|_{A\times B}((((((x_1, y_1)|_{A\times B}(x_2, y_2))|_{A\times B}\\ &(x_2, y_2))|_{A\times B}(x_3, y_3))|_{A\times B}(x_3, y_3))|_{A\times B}1_{A\times B})\\ &\approx (x_1, y_1)|_{A\times B}((((((x_1, y_1)|_{A\times B}(x_2, y_2))|_{A\times B}\\ &\quad (x_2, y_2))|_{A\times B}(x_3, y_3))|_{A\times B}(x_3, y_3))|_{A\times B}(1_A, 1_B))\end{aligned}$$

$$\begin{aligned}
&\approx (x_1|_A(((((x_1|_A x_2)|_A x_2)|_A x_3)|_A x_3)|_A 1_A),\\
&\quad y_1|_B(((((y_1|_B y_2)|_B y_2)|_B y_3)|_B y_3)|_B 1_B))\\
&\approx (1_A, 1_B)\\
&\approx 1_{A\times B}.
\end{aligned}$$

□

4 Filters of strong Sheffer stroke NMV-algebras

In this section, we introduce filters of a strong Sheffer stroke NMV-algebra. Here, $(A, |, 1)$ represents a strong Sheffer stroke NMV-algebra.

Definition 4.1. *A nonempty subset $F \subseteq A$ is called a filter of A if it satisfies the following properties:*

$(S_f - 1)$ $1 \in F$,
$(S_f - 2)$ *For all $x, y \in A$, $x|(y|1) \in F$ and $x \in F$ imply $y \in F$.*

Example 4.1. *Consider the strong Sheffer stroke NMV-algebra A in Example 3.1. Then clearly A itself, $\{1\}$, $\{1, d\}$, $\{1, f\}$, $\{1, a, d, e\}$ and $\{1, c, e, f\}$ are filters of A.*

Lemma 4.1. *A nonempty subset $F \subseteq A$ is a filter of A if and only if $1 \in F$ and $x \leq y$ and $x \in F$ imply $y \in F$.*

Theorem 4.1. *The family Π_A of all filters of A forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of filters of A. Since $1 \in F_i$ for all $i \in I$, it follows that $1 \in \bigcup_{i\in I} F_i$ and $1 \in \bigcap_{i\in I} F_i$.

(i) Suppose that $x|(y|1) \in \bigcap_{i\in I} F_i$ and $x \in \bigcap_{i\in I} F_i$ hold for any $x, y \in A$. Then $x|(y|1) \in F_i$ and $x \in F_i$ hold for all $i \in I$. Because every F_i is a filter of A, $y \in F_i$ for all $i \in I$. So, $y \in \bigcap_{i\in J} I_i$.

(ii) Let W be the family of all filters of A containing the union $\bigcup_{i\in I} F_i$. Then $\bigcap W$ is a filter of S from (i). If $\bigwedge_{i\in I} F_i = \bigcap_{i\in I} F_i$ and $\bigvee_{i\in I} F_i = \bigcap W$, then $(\Pi_A, \bigwedge, \bigvee)$ is a complete lattice. □

Corollary 4.1. *Let V be a subset of A. Then there is the minimal filter $\langle V\rangle$ of A containing the subset V.*

Proof. Let $E = \{F : F \textit{ is a filter of } A \textit{ containing } V \subseteq A\}$. Then $\langle V\rangle = \{x \in A : x \in \bigcap_{F\in E} F\}$ is the minimal filter of A containing $V \subseteq A$. □

Definition 4.2. *A subset B of A is called a strong Sheffer stroke NMV-subalgebra of A if 1 of A is in B and $(B, |, 1)$ forms a strong Sheffer stroke*

NMV-algebra. Clearly, A itself and $\{1\}$ are strong Sheffer stroke NMV-subalgebras of A.

Lemma 4.2. *Any filter F of a strong Sheffer stroke NMV-algebra A is a strong Sheffer stroke NMV-subalgebra of A.*

Proof. Let F be a filter of A. Then $1 \in F$ from $(S_f - 1)$, and F satisfies (n1)-(n6) for all $x, y, z \in F$ since $F \subseteq A$ and A is a strong Sheffer stroke NMV-algebra. Hence, $(F, |, 1)$ is also a strong Sheffer stroke NMV-algebra. □

However, the converse of Lemma 4.2 is not true.

Example 4.2. *Consider the strong Sheffer stroke NMV-algebra A in Example 3.1. Then a subset $B = \{0, a, f, 1\} \subseteq A$ is a strong Sheffer stroke NMV-subalgebra of A but it is not a filter of A since $d \notin B$ when $a|(d|d) = 1 \in B$ and $a \in B$.*

Definition 4.3. *Let F be a filter of A. Define the binary relation $\propto_F$ on A as below: for all $x, y \in A$*

$$x \propto_F y \ if \ and \ only \ if \ x|(y|1) \in F \ and \ y|(x|1) \in F. \tag{1}$$

Example 4.3. *Given the strong Sheffer stroke NMV-algebra A in Example 3.1. For the filter $F_1 = \{c, e, f, 1\}$ of A, $\propto_{F_1} = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (1,1), (c,e), (e,c), (c,f), (f,c), (c,1), (1,c), (e,f), (f,e), (e,1), (1,e), (f,1), (1,f), (a,0), (0,a), (a,b), (b,a), (a,d), (d,a), (b,d), (d,b), (b,0), (0,b), (d,0), (0,d)\}$ is a binary relation on A. It can be seen easily that $\propto_{F_1}$ is an equivalence relation on A.*

Definition 4.4. *If $x\xi y$ implies $x|k\xi y|k$ for all $x, y, k \in A$, then the equivalence relation ξ is called a congruence relation on A.*

Example 4.4. *Consider the strong Sheffer stroke NMV-algebra A in Example 3.1. Then the equivalence relation $\xi = \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (1,1), (f,1), (1,f), (0,a), (a,0), (e,c), (c,e), (b,d), (d,b)\}$ is a congruence relation on A.*

Lemma 4.3. *An equivalence relation ξ is a congruence relation on A if and only if $x\xi y$ and $k_1 \xi k_2$ imply $x|k_1 \xi y|k_2$.*

Lemma 4.4. *Let F be a filter of A and the binary relation $\propto_F$ be defined as (1). Then $\propto_F$ is a congruence relation on A.*

Proof. It is first shown that the binary relation $\propto_F$ is an equivalence relation on A.

• Reflexive: It follows from Lemma 3.1 (i).

• Symmetric: Let x, y be arbitrary elements in A such that $x \propto_F y$, i.e., $x|(y|1) \in F$ and $y|(x|1) \in F$. Then $y|(x|1) \in F$ and $x|(y|1) \in F$, i.e., $y \propto_F x$.

• Transitive: Let x, y, z be any elements in A such that $x \propto_F y$ and $y \propto_F z$, i.e., $x|(y|1), y|(x|1) \in F$ and $y|(z|1), z|(y|1) \in F$. Since $x|(y|1) \leq (y|(z|1))|((x|(z|1))|1)$ from Lemma 3.1 (xii), it follows from Lemma 4.1 that $(y|(z|1))|((x|(z|1))|1) \in F$. Then $x|(z|1) \in F$ by $(S_f - 2)$. In a similar way, $(z|(y|1))|((z|(x|1))|1) \in F$ by Lemma 4.1 because it is known from Lemma 3.1 $(xiii)$ that $y|(x|1) \leq (z|(y|1))|((z|(x|1))|1)$. Thus, $z|(x|1) \in F$ from $(S_f - 2)$. Hence, $x|(z|1), z|(x|1) \in F$, i. e., $x \propto_F z$.

Now, it is demonstrated that the equivalence relation $\propto_F$ is a congruence relation on A. Let $x, y, m, n \in A$ such that $x \propto_F m$ and $y \propto_F n$, i.e., $x|(m|1), m|(x|1) \in F$ and $y|(n|1), n|(y|1) \in F$.

(a) It is obtained from (n1) and (n3) that $(m|1)|((x|1)|1) \in F$ and $(x|1)|((m|1)|1) \in F$. Then it follows from Lemma 3.1 $(xiii)$, (n1) and (n3) that

$$\begin{aligned}(x|1)|((m|1)|1) &\leq (y|((x|1)|1))|((y|((m|1)|1))|1) \\ &\approx (x|y)|((m|y)|1),\end{aligned}$$

and so, $(x|y)|((m|y)|1) \in F$ by Lemma 4.1. Similarly, we have $(m|y)|((x|y)|1) \in F$ by substituting, simultaneously, $[x := m]$ and $[m := x]$ in above statement. Hence, $x|y \propto_F m|y$.

(b) It follows from (n1) that $m|y \propto_F m|n$ by putting, simultaneously, $[x := y]$, $[y := m]$ and $[m := n]$ in (a). Hence, $x|y \propto_F m|y$.

Thus, $x|y \propto_F m|n$ from transitivity of $\propto_F$. □

Lemma 4.5. *Let ξ be a congruence relation on A. Then $\boldsymbol{F}(\xi) = \{x \in A : x\xi 1\}$ is a filter of A.*

Proof. • Since ξ is reflexive, $1\xi 1$. Then $1 \in \boldsymbol{F}(\xi)$.

• Let $x, x|(y|1) \in \boldsymbol{F}(\xi)$. Then $x\xi 1$ and $x|(y|1)\xi 1$. So, $x\xi x|(y|1)$. Since $y\xi y$, $x|y\xi(x|(y|1))|y \approx y|((y|1)|x) \approx ((y|y)|(y|y))|((y|1)|x) \approx ((y|1)|(y|1))|((y|1)|x) \approx y|1$ from Lemma 4.3, (n1), Lemma 3.1 $(viii)$ and $(S2)$. Then $x|(y|1)\xi(y|1)|1 \approx y$ from (n3). Since ξ is transitive, $y\xi 1$ which means $y \in \boldsymbol{F}(\xi)$. □

Lemma 4.6. *Let F be a filter of A and ξ be a congruence relation on A. Then*

(a) $F = \boldsymbol{F}(\propto_F)$

(b) $\xi = \propto_{\boldsymbol{F}(\xi)}$.

Proof. (a)

$$\begin{aligned}\boldsymbol{F}(\propto_F) &= \{x \in A : x \propto_F 1\}\\ &= \{x \in A : x|(1|1) \approx x|0 \approx 1 \in F \textit{ and } 1|(x|x) \approx x \in F\}\\ &= \{x \in A : x \in F\}\\ &= F\end{aligned}$$

from (n2) and Lemma 3.1 (x).

(b) Let $x, y \in A$ be such that $x \propto_{\boldsymbol{F}(\xi)}$.

$$\begin{aligned}x \propto_{\boldsymbol{F}(\xi)} &\Leftrightarrow x|(y|1), y|(x|1) \in \boldsymbol{F}(\xi)\\ &\Leftrightarrow x|(y|1)\xi 1 \textit{ and } y|(x|1)\xi 1\\ &\Leftrightarrow x|(y|1)\xi y|(x|1)\\ &\Leftrightarrow (x|(y|1))|(y|1)\xi(y|(x|1))|(y|1) \approx y,\end{aligned}$$

and similarly, $(y|(x|1))|(x|1)\xi x$. Thus, $x\xi(y|(x|1))|(x|1) \approx (x|(y|1))|(y|1)\xi y$ from the symmetry of ξ, (n3) and (n4). Therefore, $\xi = \propto_{\boldsymbol{F}(\xi)}$. □

Corollary 4.2. *Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra, $Fil(A)$ be the class of all filters of A and $Con(A)$ be the class of all congruence relations on A. Then*

$$\begin{aligned}f : Fil(A) &\longrightarrow Con(A)\\ F &\longmapsto f(F) = \propto_F\end{aligned}$$

and

$$\begin{aligned}g : Con(A) &\longrightarrow Fil(A)\\ \xi &\longmapsto g(\xi) = \boldsymbol{F}(\xi)\end{aligned}$$

are isomorphisms, i.e,

$$Fil(A) \cong Con(A).$$

Theorem 4.2. *Let F be a filter of A and $\propto$ be a congruence relation on A defined by F. Then $A/F \equiv A/\propto = \{[x]_\propto : x \in A\}$ is also a strong Sheffer stroke NMV-algebra with the strong Sheffer stroke operation $|_\propto$ on A/F which is defined by $[x]_\propto|_\propto[y]_\propto \approx [x|y]_\propto$, for all $x, y \in A$ and $F \approx [1]_\propto$.*

Proof. Assume that F is a filter of A and $\propto$ is the congruence relation on A defined by F. Let $A/F \equiv A/\propto = \{[x]_\propto : x \in A\}$ be a structure with the strong Sheffer stroke operation $|_\propto$ described by $[x]_\propto|_\propto[y]_\propto \approx [x|y]_\propto$ for all $x, y \in A$.

•It is first shown that $F \approx [1]_{\propto}$.

For any $x \in F$, $x \in [1]_{\propto}$, i. e., $x \propto 1$, since $x|(1|1) \approx x|0 \approx 1 \in F$ and $1|(x|1) \approx (x|1)|1 \approx x \in F$ from (n1)-(n3) and $(S_f - 1)$. Then $F \subseteq [1]_{\propto}$.

For any $x \in [1]_{\propto}$, $x \propto 1$, i. e., $1 \approx x|0 \approx x|(1|1) \in F$ and $x \approx (x|1)|1 \approx 1|(x|1) \in F$ from (n1)-(n3) and $(S_f - 1)$. Then $[1]_{\propto} \subseteq F$.

Thus, $F \approx [1]_{\propto}$ and so, $0_{A/F} \approx [0]_{\propto} \approx [1|1]_{\propto} \approx [1]_{\propto}|_{\propto}[1]_{\propto}$.

•Now, it is demonstrated that the structure $(A/F, |_{\propto}, F)$ is a strong Sheffer stroke NMV-algebra. By the definition of $|_{\propto}$ and the fact that $(A, |, 1)$ is a strong Sheffer stroke NMV-algebra, it is obtained

(1) : $[x]_{\propto}|_{\propto}[y]_{\propto} \approx [x|y]_{\propto} \approx [y|x]_{\propto} \approx [y]_{\propto}|_{\propto}[x]_{\propto}$.

(2) : $[x]_{\propto}|_{\propto}0_{A/F} \approx [x]_{\propto}|_{\propto}[0]_{\propto} \approx [x|0]_{\propto} \approx [1]_{\propto} \approx F$.

(3) : $([x]_{\propto}|_{\propto}F)|_{\propto}F \approx ([x]_{\propto}|_{\propto}[1]_{\propto})|_{\propto}[1]_{\propto} \approx [(x|1)|1]_{\propto} \approx [x]_{\propto}$.

(4) :

$$\begin{aligned}(([x]_{\propto}|_{\propto}F)|_{\propto}[y]_{\propto})|_{\propto}[y]_{\propto} &\approx ([x]_{\propto}|_{\propto}[1]_{\propto})|_{\propto}[y]_{\propto})|_{\propto}[y]_{\propto}\\ &\approx [((x|1)|y)|y]_{\propto} \approx [((y|1)|x)|x]_{\propto}\\ &\approx ([y]_{\propto}|_{\propto}[1]_{\propto})|_{\propto}[x]_{\propto})|_{\propto}[x]_{\propto}\\ &\approx ([y]_{\propto}|_{\propto}F)|_{\propto}[x]_{\propto})|_{\propto}[x]_{\propto}.\end{aligned}$$

(5) :

$$\begin{aligned}([x]_{\propto}|_{\propto}F)|_{\propto}(([x]_{\propto}|_{\propto}[y]_{\propto})|_{\propto}F) &\approx ([x]_{\propto}|_{\propto}[1]_{\propto})|_{\propto}(([x]_{\propto}|_{\propto}[y]_{\propto})|_{\propto}[1]_{\propto})\\ &\approx [(x|1)|((x|y)|1)]_{\propto}\\ &\approx [1]_{\propto}\\ &\approx F.\end{aligned}$$

(6) :

$$\begin{aligned}&[x]_{\propto}|_{\propto}((((([x]_{\propto}|_{\propto}[y]_{\propto})|_{\propto}|_{\propto}[y]_{\propto})|_{\propto}[z]_{\propto})|_{\propto}[z]_{\propto})|_{\propto}F)\\ &\approx [x]_{\propto}|_{\propto}((((([x]_{\propto}|_{\propto}[y]_{\propto})|_{\propto}|_{\propto}[y]_{\propto})|_{\propto}[z]_{\propto})|_{\propto}[z]_{\propto})|_{\propto}[1]_{\propto})\\ &\approx [x|(((((x|y)|y)|z)|z)|1)]_{\propto}\\ &\approx [1]_{\propto}\\ &\approx F.\end{aligned}$$

□

Example 4.5. *Consider the strong Sheffer stroke NMV-algebra A in Example 3.1. For the filter $F = \{1, f\}$ of A, $\propto_F = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(a,0),(0,a),(1,f),(f,1),(b,d),(d,b),(c,e),(e,c)\}$ is a congruence* *relation on* A *defined by this filter. Then* $(A/F, |_{\propto}, F)$ *is a strong Sheffer stroke NMV-algebra in which the quotient set* $A/F = \{[0]_{\propto_F}, [b]_{\propto_F}, [c]_{\propto_F}, [1]_{\propto_F}\}$*,* $F \approx [1]_{\propto_F} = \{1, f\}$*, and the strong Sheffer stroke operation* $|_{\propto_F}$ *on* A/F *has the following Cayley table:*

$\|_{\propto_F}$	$[0]_{\propto_F}$	$[b]_{\propto_F}$	$[c]_{\propto_F}$	$[1]_{\propto_F}$
$[0]_{\propto_F}$	$[1]_{\propto_F}$	$[1]_{\propto_F}$	$[1]_{\propto_F}$	$[1]_{\propto_F}$
$[b]_{\propto_F}$	$[1]_{\propto_F}$	$[c]_{\propto_F}$	$[1]_{\propto_F}$	$[c]_{\propto_F}$
$[c]_{\propto_F}$	$[1]_{\propto_F}$	$[1]_{\propto_F}$	$[b]_{\propto_F}$	$[b]_{\propto_F}$
$[1]_{\propto_F}$	$[1]_{\propto_F}$	$[c]_{\propto_F}$	$[b]_{\propto_F}$	$[0]_{\propto_F}$

Table 2: The Cayley table of Sheffer stroke $|_{\propto_F}$

Corollary 4.3. *If* A/F *is associative, then it is an MV-algebra.*

Corollary 4.4. A/F *is an implication NMV-algebra with* 0.

5 Homomorphism of strong Sheffer stroke NMV-algebras

In this section, we introduce a homomorphism of strong Sheffer stroke NMV-algebras.

Definition 5.1. *Let* $(A, |_A, 1_A)$ *and* $(B, |_B, 1_B)$ *be strong Sheffer stroke NMV-algebras. A mapping* $h : A \longrightarrow B$ *is called a homomorphism if*

$$h(x|_A y) = h(x)|_B h(y)$$

for all $x, y \in A$.

Lemma 5.1. *Let* $(A, |_A, 1_A)$ *and* $(B, |_B, 1_B)$ *be strong Sheffer stroke NMV-algebras and* $h : A \longrightarrow B$ *be a homomorphism. Then* $h(A)$ *is a filter of* B.

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras and $h : A \longrightarrow B$ be a homomorphism. It is obtained from Lemma 3.1 (ix) that

$$\begin{aligned} 1_B &\approx h(1_A)|_B(h(1_A)|_B h(1_A)) \\ &\approx h(1_A|_A(1_A|_A 1_A)) \\ &\approx h(1_A) \in h(A). \end{aligned}$$

Let $h(x) \in h(A)$ and $h(x)|_B(h(y)|_B 1_B) \in h(A)$, i. e., $h(x|_A(y|_A 1_A)) \approx h(x)|_B(h(y)|_B h(1_A)) \approx h(x)|_B(h(y)|_B 1_B) \in h(A)$. Then $x \in A$ and $x|_A(y|_A 1_A) \in A$. Since A itself is a filter of A, $y \in A$, and so, $h(y) \in h(A)$. $\square$

Remark 5.1. *The class of strong Sheffer stroke NMV-algebras forms a variety.*

Theorem 5.1. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras and $h : A \longrightarrow B$ be a homomorphism. Then*

(1) *If F is a filter of A, then $h(F)$ is a filter of B.*

(2) *If G is a filter of B, then $h^{-1}(G)$ is a filter of A.*

Proof. Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, and let $h : A \longrightarrow B$ be a homomorphism.

(1) Assume that F is a filter of A. Since $1_A \in F$, $1_B \approx h(1_A) \in h(F)$. Let $h(x) \in h(F)$ and $h(x)|_B(h(y)|_B 1_B) \in h(F)$, i. e., $h(x|_A(y|_A 1_A)) \approx h(x)|_B(h(y)|_B h(1_A)) \approx h(x)|_B(h(y)|_B 1_B) \in h(F)$. Then $x \in F$ and $x|_A(y|_A 1_A) \in F$. By the assumption, $y \in F$, i. e., $h(y) \in h(F)$.

(2) Suppose that G is a filter of B. Because $h(1_A) \approx 1_B \in G$, $1_A \approx h^{-1}h(1_A) \approx h^{-1}(1_B) \in h^{-1}(G)$. Let $x \in h^{-1}(G)$ and $x|_A(y|_A 1_A) \in h^{-1}(G)$, i. e., $h(x) \in hh^{-1}(G) \subseteq G$ and $h(x)|_B(h(y)|_B 1_B) \approx h(x)|_B$ $(h(y)|_B h(1_A)) \approx h(x|_A(y|_A 1_A)) \in ff^{-1}(G) \subseteq G$. By the assumption, $h(y) \in G$, i. e., $y \in h^{-1}(G)$.

$\square$

Theorem 5.2. *Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra and F and G be two filters of A. Then $A/(F \cap G) \cong A/F \times A/G$.*

Proof. Since $F \cap G$ is a filter of A, $A/(F \cap G)$ is a strong Sheffer stroke NMV-algebra by Theorem 4.2. Also, A/F and A/G are strong Sheffer stroke NMV-algebras, and so $A/F \times A/G$ is a strong Sheffer stroke NMV-algebra from Theorem 3.1. Define

$$\begin{aligned} f : A/(F \cap G) &\longrightarrow A/F \times A/G \\ [x]_{\propto_{F\cap G}} &\longmapsto ([x]_{\propto_F}, [x]_{\propto_G}). \end{aligned}$$

Then f is a homomorphism.

- f is injective: Let $([x]_{\propto_F}, [x]_{\propto_G}), ([y]_{\propto_F}, [y]_{\propto_G}) \in A/F \times A/G$ such that $([x]_{\propto_F}, [x]_{\propto_G}) \approx ([y]_{\propto_F}, [y]_{\propto_G})$. Then $[x]_{\propto_F} \approx [y]_{\propto_F}$ and $[x]_{\propto_G} \approx [y]_{\propto_G}$. So, $x \propto_F y$ and $x \propto_G y$, i. e., $x|(y|y), y|(x|x) \in F$ and $x|(y|y), y|(x|x) \in G$. Thus, $x|(y|y), y|(x|x) \in F \cap G$, i. e., $x \propto_{F\cap G} y$. Hence, $[x]_{\propto_{F\cap G}} \approx [y]_{\propto_{F\cap G}}$.

• f is surjective: Let $([a]_{\propto_F}, [a]_{\propto_G}) \in A/F \times A/G$ be such that $f([x]_{\propto_{(F\cap G)}}) \approx ([a]_{\propto_F}, [a]_{\propto_G})$. Then $([x]_{\propto_F}, [x]_{\propto_G}) \approx ([a]_{\propto_F}, [a]_{\propto_G})$ which is $[x]_{\propto_F} \approx [a]_{\propto_F}$ and $[x]_{\propto_G} \approx [a]_{\propto_G}$. So, $x \propto_F a$ and $x \propto_G a$, i.e., $x|(a|a), a|(x|x) \in F$ and $x|(a|a), a|(x|x) \in G$. Hence, $x|(a|a), a|(x|x) \in F \cap G$ which means $x \propto_{F\cap G} a$. Thus, $[x]_{\propto_{F\cap G}} \approx [a]_{\propto_{F\cap G}}$.

Therefore, f is an isomorphism, i.e., $A/(F \cap G) \cong A/F \times A/G$. □

Theorem 5.3. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras, and let F and G be two filters of A and B, resp. Then $(A\times B)/(F \times G) \cong A/F \times B/G$.*

Proof. A/F and B/G are strong Sheffer stroke NMV-algebras by Theorem 4.2. Also, $A/F \times B/G$ and $A \times B$ are strong Sheffer stroke NMV-algebras from Theorem 3.1.

Now, we show that $F \times G$ is a filter of $A \times B$. Since F and G are filters, $1_{A\times B} \approx (1_A, 1_B) \in F \times G$. Let $(x_1, y_1), (x_1, y_1)|_{A\times B}((x_2, y_2)|_{A\times B}(x_2, y_2)) \in F \times G$. Then $(x_1|_A(x_2|_Ax_2), y_1|_B(y_2|_By_2)) \in F \times G$. So, $x_1, x_1|_A(x_2|_Ax_2) \in F$ and $y_1, y_1|_B(y_2|_By_2) \in G$. Since F and G are filters, $x_2 \in F$ and $y_2 \in G$, i.e., $(x_2, y_2) \in F \times G$. Thus, $A \times B/F \times G$ is a strong Sheffer stroke NMV-algebra by Theorem 4.2. Define

$$\begin{aligned} g : (A \times B)/(F \times G) &\longrightarrow A/F \times B/G \\ [(x, y)]_{\propto_{F\times G}} &\longmapsto ([x]_{\propto_F}, [y]_{\propto_G}). \end{aligned}$$

Then g is a surjective homomorphism.

g is injective: Let $([x_1]_{\propto_F}, [y_1]_{\propto_G})$ and $([x_2]_{\propto_F}, [y_2]_{\propto_G})$ such that $([x_1]_{\propto_F}, [y_1]_{\propto_G}) \approx ([x_2]_{\propto_F}, [y_2]_{\propto_G})$. Then $[x_1]_{\propto_F} \approx [x_2]_{\propto_F}$ and $[y_1]_{\propto_G} \approx [y_2]_{\propto_G}$, i.e., $x_1 \propto_F x_2$ and $y_1 \propto_G y_2$. Thus, $x_1|_A(x_2|_Ax_2), x_2|_A(x_1|_Ax_1) \in F$ and $y_1|_B(y_2|_By_2), y_2|_B(y_1|_By_1) \in G$. So, $(x_1, y_1)|_{A\times B}((x_2, y_2)|_{A\times B}(x_2, y_2)) \approx (x_1|_A (x_2|_Ax_2), y_1|_B(y_2|_By_2)) \in F \times G$ and $(x_2, y_2)|_{A\times B}((x_1, y_1)|_{A\times B}(x_1, y_1)) \approx (x_2|_A (x_1|_Ax_1), y_2|_B(y_1|_By_1)) \in F \times G$. Hence, $(x_1, y_1) \propto_{F\times G} (x_2, y_2)$, i.e.,

$$[(x_1, y_1)]_{\propto_{F\times G}} \approx [(x_2, y_2)]_{\propto_{F\times G}}.$$

Thereby, g is an isomorphism. □

Corollary 5.1. *Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra, and F and G be two filters of A. Then $A/(F \cap G) \cong (A \times A)/(F \times G)$.*

Theorem 5.4. *Let F be a filter of A. Then G is a filter of A such that $F \subseteq G$ if and only if the set $G/F = \{[x]_F \in A/F : x \in G\}$ is a filter of A/F.*

Proof. ($\Rightarrow$) $(A/F, |_F, F)$ is a strong Sheffer stroke NMV-algebra and $F \approx [1]_F$ from Theorem 4.2. Then $F \approx [1]_F \in G/F$ from the definition of G/F. Let $[x]_F \in G/F$ and $[x]_F|_F([y]_F|_F F) \in G/F$, i.e., $[x|(y|1)]_F \approx [x]_F|_F([y]_F|_F[1]_F) \approx [x]_F|_F([y]_F|_F F) \in G/F$. So, $x \in G$ and $x|(y|1) \in G$. Since G is a filter of A, then $y \in G$, i.e., $[y]_F \in G/F$.

($\Leftarrow$) It is obtained from the definition of G/F. □

Theorem 5.5. *Let $(A, |_A, 1_A)$ and $(B, |_B, 1_B)$ be strong Sheffer stroke NMV-algebras and $h : A \longrightarrow B$ be a homomorphism. Then*

$$Ker h = \{x \in A : h(x) \approx 1_B\}$$

is a filter of A.

Proof. Since $h(1_A) \approx 1_B$, $1_B \in Ker h$. Let $x \in Ker h$ and $x|_A(y|_A y) \in Ker h$. Then $h(x) \approx 1_B$ and $h(x|_A(y|_A y)) \approx 1_B$. Since

$$\begin{aligned} h(y) &\approx 1_B|_B(h(y)|_B h(y)) \\ &\approx h(x)|_B(h(y)|_B h(y)) \\ &\approx h(x|_A(y|_A y)) \\ &\approx 1_B, \end{aligned}$$

it follows $y \in Ker h$. Thus, $Ker h$ is a filter of A. □

Theorem 5.6. *Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra and F be a filter of A. Then there exists the natural homomorphism $g : A \longrightarrow A/F$ defined by $x \mapsto [x]_F$.*

Proof. By Theorem 4.2, $g : A \longrightarrow A/F$ is a homomorphism, and it is called the natural homomorphism. □

Theorem 5.7. *Let $(A, |, 1)$ be a strong Sheffer stroke NMV-algebra and F be a filter of a. Then there exists a canonical surjective homomorphism $f : A \longrightarrow A/F$ defined by $f(x) \approx [x]_F$ and $Ker f = F$.*

Proof. By Theorem 5.6, $f : A \longrightarrow A/F,\ x \mapsto [x]_F$ is a homomorphism and is surjective from its definition. Then it follows from (n2) and Lemma 3.1 (x) that

$$\begin{aligned} Ker f &= \{x \in A : f(x) \approx [1]_F\} \\ &= \{x \in A : [x]_F \approx [1]_F\} \\ &= \{x \in A : x \propto_F 1\} \\ &= \{x \in A : x|(1|1) \approx x|0 \approx 1 \in F \text{ and } 1|(x|x) \approx x \in F\} \\ &= \{x \in A : x \in F\} \\ &= F. \end{aligned}$$

□

Theorem 5.8. *Let* $(A, |_A, 1_A)$ *and* $(B, |_B, 1_B)$ *be strong Sheffer stroke NMV-algebras and* $h : A \longrightarrow B$ *be a homomorphism. Then there exists a unique homomorphism* $f : A/Kerh \longrightarrow B$ *such that* $h \approx f \circ g$ *where* g *is the natural homomorphism* $A \longrightarrow A/Kerh$*. Moreover,* g *is surjective and* f *is injective.*

Proof. Since $Kerh$ is a filter of A from Theorem 5.5, $A/Kerh$ is a strong Sheffer stroke NMV-algebra by Theorem 4.2. We define $f : A/Kerh \longrightarrow B$, $[x]_{Kerh} \longmapsto h(x)$.

Let $[x]_{Kerh}, [y]_{Kerh} \in A/Kerh$ such that $[x]_{Kerh} \approx [y]_{Kerh}$. Then $x \propto_{Kerh} y$, i. e., $x|_A(y|_A y)$, $y|_A(x|_A x) \in Kerh$. So, $h(x)|_B(h(y)|_B h(y)) \approx h(x|_A(y|_A y)) \approx 1_B$ and $h(y)|_B(h(x)|_B h(x)) \approx h(y|_A(x|_A x)) \approx 1_B$. Hence, $h(x) \leq h(y)$ and $h(y) \leq h(x)$ from Proposition 3.1, and so $f([x]_{Kerh}) \approx h(x) \approx h(y) \approx f([y]_{Kerh})$. Thus, f is well-defined.

Let $[x]_{Kerh}, [y]_{Kerh} \in A/Kerh$. Then

$$\begin{aligned} f([x]_{Kerh}|_{Kerh}[y]_{Kerh}) &\approx f([x|_A y]_{Kerh}) \\ &\approx h(x|_A y) \\ &\approx h(x)|_B h(y) \\ &\approx f([x]_{Kerh})|_B f([y]_{Kerh}). \end{aligned}$$

Also, $f(Kerh) \approx f([1_A]_{Kerh}) \approx h(1_A) \approx 1_B$. Hence, f is a homomorphism.

For the natural homomorphism $g : A \longrightarrow A/Kerh$, $g(x) \approx [x]_{Kerh}$ and for any $x \in A$,

$$f \circ g(x) \approx f(g(x)) \approx f([x]_{Kerh}) \approx h(x).$$

Then $h \approx f \circ g$.

Now, we show the uniqueness of f. Let $\lambda : A/Kerf \longrightarrow B$ be a homomorphism such that $h \approx \lambda \circ g$. For $[x]_{Kerh} \in A/Kerh$, $\lambda([x]_{Kerh}) \approx \lambda(g(x)) \approx \lambda \circ g(x) \approx h(x) \approx f \circ g(x) \approx f(g(x)) \approx f([x]_{Kerh})$. Thus, $\lambda \approx f$.

Moreover, it is obvious that g is surjective.

Let $[x]_{Kerh}, [y]_{Kerh} \in A/Kerh$ such that $f([x]_{Kerh}) \approx f([y]_{Kerh})$. Then $h(x) \approx h(y)$. Hence, $h(x|_A(y|_A y)) \approx h(x)|_B(h(y)|_B h(y)) \approx h(x)|_B(h(x)|_B h(x)) \approx 1_B$ and similarly $h(y|_A(x|_A x)) \approx 1_B$ from Lemma 3.1 (ix). So, $x|_A(y|_A y), y|_A (x|_A x) \in Kerh$ which means $x \propto_{Kerh} y$, i. e., $[x]_{Kerh} \approx [y]_{Kerh}$. Therefore, f is injective. □

Corollary 5.2. *(1st Isomorphism Theorem) Let* $(A, |_A, 1_A)$ *and* $(B, |_B, 1_B)$ *be strong Sheffer stroke NMV-algebras and* $h : A \longrightarrow B$ *be a homomorphism. Then* $A/Kerh \cong h(A)$*. If* h *is surjective, then* $A/Kerh \cong B$*.*

Theorem 5.9. *(2nd Isomorphism Theorem) Let B be a strong Sheffer stroke NMV-subalgebra of a strong Sheffer stroke NMV-algebra $(A, |, 1)$ and F be a filter of A. Then $BF/F \cong B/B \cap F$ where $BF = \bigcup [x]_F$, for $x \in B$ and $BF/F = \{[x]_F \in A/F : x \in BF\}$.*

Proof. We first show that BF is a strong Sheffer stroke NMV-subalgebra of A. Since B is a strong Sheffer stroke NMV-subalgebra of A, $1 \in B$. Then $1 \in F = [1]_F \subseteq \bigcup [x]_F = BF$. Also, (n1-n6) are satisfied for all $x, y, z \in BF \subseteq A$, that is, $(BF, |, 1)$ forms a strong Sheffer stroke NMV-algebra. Thus, BF/F is a strong Sheffer stroke NMV-subalgebra of A/F. Since BF/F is a strong Sheffer stroke NMV-algebra and $B \subseteq BF$, then $h : B \longrightarrow BF/F,\ x \mapsto [x]_F$ is a homomorphism. So,

$$
\begin{aligned}
Kerh &= \{x \in B : h(x) \approx [1]_F\} \\
&= \{x \in B : [x]_F \approx [1]_F\} \\
&= \{x \in B : x \propto_F 1\} \\
&= \{x \in B : x|(1|1) \approx x|0 \approx 1 \in F \text{ and } 1|(x|x) \approx x \in F\} \\
&= \{x \in B : x \in F\} \\
&= B \cap F.
\end{aligned}
$$

from (n2) and Lemma 3.1 (x). □

For any $[a]_F \in BF/F$, $a \in BF = \bigcup [x]_F$. Then there exists $x \in B$ such that $a \in [x]_F$, and so $[a]_F \approx [x]_F$. Hence, $h(x) \approx [x]_F \approx [a]_F$, i.e., h is onto. By Corollary 5.2, $B/B \cap F = B/Kerh \cong h(B) = BF/F$.

Theorem 5.10. *(3rd Isomorphism Theorem) Let F and G be filters of A such that $F \subseteq G$. Then $(A/F)/(G/F) \cong A/G$.*

Proof. It is known from Theorem 5.4 that $G/F = \{[x]_F \in A/F : x \in G\}$ is a filter of a strong Sheffer stroke NMV-algebra A/F. Then $(A/F)/(G/F) = \{[[x]_F]_{G/F} : [x]_F \in A/F\}$ and also $A/G = \{[x]_G : x \in A\}$ are strong Sheffer stroke NMV-algebras by Theorem 4.2. Define $\mu : (A/F)/(G/F) \longrightarrow A/G$, $[[x]_F]_{G/F} \longmapsto [x]_G$. Since

$$
\begin{aligned}
\mu([[x]_F]_{G/F}|_{G/F}[[y]_F]_{G/F}) &\approx \mu([[x]_F|_F[y]_F]_{G/F}) \\
&\approx \mu([[x|y]_F]_{G/F}) \\
&\approx [x|y]_G \\
&\approx [x]_G|_G[y]_G \\
&\approx \mu([[x]_F]_{G/F})|_G\mu([[y]_F]_{G/F})
\end{aligned}
$$

for any elements $[[x]_F]_{G/F}, [[y]_F]_{G/F} \in (A/F)/(G/F)$ and $G \approx [1]_G \approx \mu([[1]_F]_{G/F})$, it is obtained that μ is a homomorphism.

• μ is one-to-one: Let $[[x]_F]_{G/F} \in Ker\mu$, i. e., $[0]_G \approx \mu([[x]_F]_{G/F}) \approx [x]_G$. Then $x \propto_G 0$, i. e., $x|(0|1), 0|(x|1) \in G$, and so, $[x]_F|_F([0]_F|_F[1]_F) \approx [x|(0|1)]_F$,
$[0]_F|_F([x]_F|_F[1]_F) \approx [0|(x|1)]_F \in G/F$. Thus, $[x]_F \propto_{G/F} [0]_F$, i. e., $[[x]_F]_{G/F} \approx [[0]_F]_{G/F}$. Hence, $Ker\mu = \{[[0]_F]_{G/F}\}$.

• μ is onto: it is obvious from the definition of μ.

Therefore, μ is an isomorphism. □

6 Conclusion

In this work, a partially order on a strong Sheffer stroke NMV-algebra is given. The Cartesian product of two strong Sheffer stroke NMV-algebras, filters, strong Sheffer stroke NMV-subalgebras, congruence relation and homomorphisms are analysed. It is shown that cartesian product of two strong Sheffer stroke NMV-algebras is a strong Sheffer stroke NMV-algebra. It is proved that any filter of a strong Sheffer stroke NMV-algebra is its strong Sheffer stroke NMV-subalgebra but the converse is not true. It is also shown that the family of all filters of a strong Sheffer stroke NMV-algebra forms a complete lattice, and thereby for a subset of strong Sheffer stroke NMV-algebra there exists a minimal filter containing this subset. Besides, it is introduced a congruence relation on a strong Sheffer stroke NMV-algebra defined by a filter, and its filter defined by a congruence relation on this algebraic structure. Thereby, it is demonstrated that the class of all filters of a strong Sheffer stroke NMV-algebra is isomorphic to the class of all its congruences. A quotient strong Sheffer stroke NMV-algebra is built from a strong Sheffer stroke NMV-algebra by its congruence. Also, it is proved the quotient strong Sheffer stroke NMV-algebra is a MV-algebra if it is associative, and that it is implication NMV-algebra with 0. The relationships between the quotient strong Sheffer stroke algebras are stated. By determining a homomorphism between strong Sheffer stroke NMV-algebras and its kernel, the isomorphism theorems on this algebraic structure are proved.

In future works, the relationships between strong Sheffer stroke NMV-algebras and other algebraic structures, their fuzzy concepts, and different types of their filters can be examined, and so it can be studied in details for some applications in computer science and artificial intelligence.

Acknowledgment

We wish to thank the reviewers for excellent suggestions that have been incorporated into the paper. This study is supported by Ege University Scientific Research Projects Directorate with the project number 20772.

References

[1] Abbott, J. C., *Implicational algebras*, Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, 11(1), 3-23, 1967.

[2] Bailey, C. G., *Prime filters in MV-Algebra II*, arXiv preprint, 2009. Available at arXiv:0907.3332v1[math.RA].

[3] H. Banivaheb and A. Borumand Saeid, *MV -modules of fractions*, J. Algebra and its Applications, (2020) 2050131 (19 pages).

[4] Birkhoff, G., *Lattice Theory*, Proceedings of the American Mathematical Society, 1967.

[5] Botur, M., Halaš, R., *Commutative basic algebras and non-assocative fuzzy logics*, Arch. Math. Log. 48, 243-255, 2009.

[6] Chajda, I., *Sheffer operation in ortholattices*, Acta Universitatis Palackianae Olomucensis Facultas Rerum Naturalium Mathematica, 44(1), 19-23, 2008.

[7] Chajda, I., *Orthomodular semilattices*, Discrete Mathematics, 307(1), 115-118, 2007.

[8] Chajda, I., Halaš, R., Länger, H., *Operations and structures derived from non-associative MV-algebras*, Soft Computing, 23(12), 3935-3944, 2019.

[9] Chajda, I., Khr, J., *A non-associative generalization of MV-algebras*, Math. Slov., 57, 301-312, 2007.

[10] Chajda, I., Länger, H., *Properties of non-associative MV-algebras*, Math. Slov., 67, 1095-1104, 2017.

[11] Chang, C. C., *Algebraic analysis of many-valued logics*, Transactions of the American Mathematical Society, 88, 467-490, 1958.

[12] Chang, C. C., *A new proof of the completeness of the ukasiewicz axioms*, Transactions of the American Mathematical Society, 93, 7480, 1959.

[13] Cignoli, R. L., d'Ottaviano, I. M. L., Mundici, G., *Algebraic foundations of many-valued reasoning*, Springer Science and Business Media, Dordrecht, Holland, 2000.

[14] McCune, W., Veroff, R. Fitelson, B. Harris, K. Feist, A., Wos, L., *Short single axioms for Boolean algebra*, Journal of Automated Reasoning, 29(1), 1-16, 2002.

[15] Sheffer, H. M., *A set of five independent postulates for Boolean algebras, with application to logical constants*, Transactions of the American Mathematical Society, 14(4), 481-488, 1913.

Tahsin ONER,
Department of Mathematics,
Ege University,
35100 Izmir, Turkey.
Email: tahsin.oner@ege.edu.tr

Tugce KATICAN,
Department of Mathematics,
Ege University,
35100 Izmir, Turkey.
Email: tugcektcn@gmail.com

Arsham Borumand SAEID,
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran.
Email: arsham@uk.ac.ir

Mehmet TERZILER,
Department of Mathematics,
Yasar University,
35100 Izmir, Turkey.
Email: mehmet.terziler@yasar.edu.tr