Steiner Degree Distance of Two Graph Products

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Abstract

The degree distance $DD(G)$ of a connected graph G was invented by Dobrynin and Kochetova in 1994. Recently, one of the present authors introduced the concept of k-center Steiner degree distance defined as

$$SDD_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left[\sum_{v \in S} deg_G(v) \right] d_G(S),$$

where $d_G(S)$ is the Steiner k-distance of S and $deg_G(v)$ is the degree of the vertex v in G. In this paper, we investigate the Steiner degree distance of complete and Cartesian product graphs.

1 Introduction

All graphs in this paper are assumed to be undirected, finite, and simple. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Then the order and size of G are $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$. In

other words, G has n vertices and m edges. The degree $deg_G(u)$ of the vertex $u \in V(G)$ is number of first neighbors of this vertex.

Distance is one of the basic concepts of graph theory [7]. If G is connected and $u, v \in V(G)$, then the *distance* $d(u, v) = d_G(u, v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $\varepsilon(v)$ of v is defined as $\varepsilon(v) = \max\{d(u, v) \,|\, u \in V(G)\}$. Furthermore, the *radius* $rad(G)$ and *diameter* $diam(G)$ of G are $rad(G) = \min\{\varepsilon(v) \,|\, v \in V(G)\}$ and $diam(G) = \max\{\varepsilon(v) \,|\, v \in V(G)\}$. These latter two quantities are related by the inequalities $rad(G) \leq diam(G) \leq 2\,rad(G)$. More details on this subject can be found in [14]. We refer to [5] for graph theoretical notation and terminology not specified here.

For a graph G with vertex set $V(G)$, the *degree distance* is defined as [13]

$$DD = DD(G) = \sum_{\{u,v\} \in V(G)} [deg_G(u) + deg_G(v)]\, d_G(u, v). \qquad (1)$$

For more details on degree–and–distance–based graph invariant, we refer to [2, 3, 4, 6, 17, 18, 30, 32, 35].

The Wiener index $W(G)$ of the graph G is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)\,. \qquad (2)$$

Details on this oldest distance–based topological index can be found in numerous surveys, e.g., in [11, 12, 19, 20, 33, 34, 36].

The Steiner distance of a graph, introduced by Chartrand et al. [9] in 1989, is a natural and nice generalization of the concept of classical graph distance. For a subset S of the vertex set $V(G)$, consisting of at least two vertices, the *Steiner distance* $d(S)$ (or simply the distance of S) is the minimum size (number of edges) of a connected subgraphs whose vertex set contains S. This connected subgraph is necessarily a tree and is referred to as a *Steiner tree*.

Note that if $S = \{u, v\}$, then $d(S) = d(u, v)$ is nothing new, but the classical distance between u and v. Clearly, if $|S| = k$, then $d(S) \geq k - 1$.

Let n and k be integers such that $2 \leq k \leq n$. The *Steiner k-eccentricity* $\varepsilon_k(v)$ of a vertex v of G is $\varepsilon_k(v) = \max\{d(S) \,|\, S \subseteq V(G), \ |S| = k, \text{and } v \in S\}$. The *Steiner k-radius* of G is $srad_k(G) = \min\{\varepsilon_k(v) \,|\, v \in V(G)\}$, whereas the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{\varepsilon_k(v) \,|\, v \in V(G)\}$. Note that for every connected graph G, $\varepsilon_2(v) = \varepsilon(v)$ for all vertices v of G, $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. For more details on Steiner distance, we refer to [1, 8, 9, 10, 14, 31].

The following result is immediate.

Observation 1.1. *If H is a connected spanning subgraph of G, then*

$$sdiam_k(G) \leq sdiam_k(H)$$

holds for all k, $2 \leq k \leq n$.

Li et al. [21] put forward a Steiner–distance–based generalization of the Wiener index concept. According to [21], the *k-center Steiner Wiener index* $SW_k(G)$ of the graph G is defined by

$$SW_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S| = k}} d(S). \tag{3}$$

For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index, Eq. (2). It is usual to consider SW_k for $2 \leq k \leq n - 1$, but the above definition would be applicable also in the cases $k = 1$ and $k = n$, implying $SW_1(G) = 0$ and $SW_n(G) = n - 1$. For more details on Steiner Wiener index, we refer to [21, 22, 26, 27, 28, 29].

Recently, Gutman [16] offered an analogous generalization of the concept of degree distance, Eq. (1). Thus, the *k-center Steiner degree distance* $SDD_k(G)$

of G is defined as

$$SDD_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left[\sum_{v \in S} deg_G(v) \right] d_G(S) \,. \tag{4}$$

Mao et al. [28] reported expressions for the k-center Steiner degree distance of the star, path, as well as the complete and complete bipartite graphs, and got general expressions for $SDD_k(G)$ for $k = n, n-1$. They also obtained sharp lower and upper bounds for SDD_k. Very recently, Mao and Das [25] generalize the concept of Gutman index by Steiner distance. The Steiner Gutman k-index $SGut_k(G)$ of graph G is defined by

$$SGut_k(G) = \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left[\prod_{v \in S} deg_G(v) \right] d_G(S) \,.$$

The join and Cartesian products are defined as follows:

The *join* or *complete product* $G \vee H$ of two disjoint graphs G and H, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

The *Cartesian product* $G \square H$ of two graphs G and H, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

2 Complete product

In this section, we give the exact expression of $SDD_k(G \vee H)$.

Theorem 2.1. *Let G, H be two connected graphs of order a, b $(a \leq b)$, respectively. Let k be an integer with $3 \leq k \leq a + b$.*

(1) *If $1 \le k \le a$, then*

$$
SDD_k(G \vee H) = k\binom{a}{k} - x - 2e(G)\binom{a-1}{k-1} - \sum_{i=1}^{x}\left(\sum_{v \in S_i} deg_G(v)\right)
$$

$$
+k\binom{b}{k} - y - 2e(H)\binom{b-1}{k-1} - \sum_{i=1}^{y}\left(\sum_{v \in S_i'} deg_H(v)\right)
$$

$$
+2(k-1)[e(G) + e(H) + ab]\binom{a+b-1}{k-1}
$$

$$
-(k-1)ab\binom{a-1}{k-1} - (k-1)ab\binom{b-1}{k-1},
$$

where $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(G)$ such that $G[S_i]$ $(1 \le i \le x)$ is connected, and $S_1', S_2', \ldots, S_y'$ are all the k-subsets of $V(H)$ such that $H[S_i']$ $(1 \le i \le y)$ is connected.

(2) *If $a < k \le b$, then*

$$
SDD_k(G \vee H) = ak^2\binom{b}{k} - akx + 2e(H)\binom{b-1}{k-1} - \sum_{i=1}^{x}\left(\sum_{v \in S_i} deg_H(v)\right)
$$

$$
+2(k-1)[e(G) + e(H) + ab]\binom{a+b-1}{k-1} - (k-1)ab\binom{a-1}{k-1}
$$

$$
-(k-1)ab\binom{b-1}{k-1} - (k-1) \cdot 2e(G)\binom{a-1}{k-1},
$$

where $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(H)$ such that $H[S_i]$ $(1 \le i \le x)$ is connected.

(3) *If $b < k \le a + b$, then*

$$
SDD_k(G \vee H) = 2(k-1)e(G)\left[\binom{a+b-1}{k-1} - \binom{a-1}{k-1}\right] - (k-1)ab\left[\binom{a-1}{k-1}\right.
$$

$$
\left. +\binom{b-1}{k-1}\right] + 2(k-1)e(H)\left[\binom{a+b-1}{k-1} - \binom{b-1}{k-1}\right].
$$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_a\}$ and $V(H) = \{v_1, v_2, \ldots, v_b\}$. Without loss of generality, we can assume that $a \leq b$. We distinguish the following three cases to show this theorem.

Case 1. $1 \leq k \leq a$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap V(G) = \varnothing$, or $S \cap V(H) = \varnothing$, or $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.

If $S \cap V(H) = \varnothing$, then $S \subseteq V(G)$. Recall that $S_1, S_2, \ldots, S_{\binom{a}{k}}$ be all the k-subsets of $V(G)$, and $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(G)$ such that $G[S_i]$ $(1 \leq i \leq x)$ is connected. Then $d_G(S_i) = d_{G \vee H}(S_i) = k - 1$ for each i $(1 \leq i \leq x)$. For each i $(x + 1 \leq i \leq \binom{a}{k})$, we have $d_G(S_i) \geq k$ and $d_{G \vee H}(S_i) = k$. For $v \in V(G \vee H)$, if $v \in V(G)$, then $deg_{G \vee H}(v) = deg_G(v) + b$. This case contributes to $SDD_k(G \vee H)$ by

$$\sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_{G \vee H}(v) \right) d_{G \vee H}(S)$$

$$= \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} (deg_G(v) + b) \right) d_{G \vee H}(S)$$

$$= \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_G(v) \right) d_{G \vee H}(S) + bk \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_{G \vee H}(S)$$

$$= (k-1) \sum_{\substack{S \subseteq V(G),\ |S|=k \\ d_G(S)=k-1}} \left(\sum_{v \in S} deg_G(v) \right) + k \sum_{\substack{S \subseteq V(G),\ |S|=k \\ d_G(S) \geq k}} \left(\sum_{v \in S} deg_G(v) \right) + bk \sum_{\substack{S \subseteq V(G) \\ |S|=k}} d_{G \vee H}(S)$$

$$= k \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_G(v) \right) - \sum_{\substack{S \subseteq V(G),\ |S|=k \\ d_G(S)=k-1}} \left(\sum_{v \in S} deg_G(v) \right) + bk \left[k \binom{a}{k} - x \right]$$

$$= 2ke(G) \binom{a-1}{k-1} - \sum_{i=1}^{x} \left(\sum_{v \in S_i} deg_G(v) \right) + bk^2 \binom{a}{k} - bkx.$$

If $S \cap V(G) = \varnothing$, then $S \subseteq V(H)$. Similarly, this case contributes to $SDD_k(G \vee H)$ by

$$2ke(H)\binom{b-1}{k-1} - \sum_{i=1}^{y}\left(\sum_{v \in S_i'} deg_H(v)\right) + ak^2\binom{b}{k} - aky.$$

Suppose $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. In this case, $d_{G \vee H}(S) = k - 1$. This case contributes to $SDD_k(G \vee H)$ by

$$\sum_{\substack{S \subseteq V(G), |S|=k \\ S \cap V(G) \neq \varnothing, S \cap V(H) \neq \varnothing}} \left(\sum_{v \in S} deg_{G \vee H}(v)\right) d_{G \vee H}(S)$$

$$= \quad (k-1) \sum_{\substack{S \subseteq V(G), |S|=k \\ S \cap V(G) \neq \varnothing, S \cap V(H) \neq \varnothing}} \left(\sum_{v \in S} deg_{G \vee H}(v)\right)$$

$$= \quad (k-1) \cdot 2e(G \vee H)\binom{a+b-1}{k-1} - (k-1) \cdot 2e(G)\binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1) \cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}$$

$$= \quad (k-1)(2e(G) + 2e(H) + 2ab)\binom{a+b-1}{k-1} - (k-1) \cdot 2e(G)\binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1) \cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}.$$

From the above arguments, we conclude that

$$SDD_k(G \vee H) \quad = \quad bk^2\binom{a}{k} - bkx + 2e(G)\binom{a-1}{k-1} - \sum_{i=1}^{x}\left(\sum_{v \in S_i} deg_G(v)\right)$$

$$+ak^2\binom{b}{k} - aky + 2e(H)\binom{b-1}{k-1} - \sum_{i=1}^{y}\left(\sum_{v \in S_i'} deg_H(v)\right)$$

$$+2(k-1)[e(G) + e(H) + ab]\binom{a+b-1}{k-1} - (k-1)ab\binom{a-1}{k-1} - (k-1)ab\binom{b-1}{k-1},$$

where $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(G)$ such that $G[S_i]$ $(1 \leq i \leq x)$ is connected, and $S'_1, S'_2, \ldots, S'_y$ are all the k-subsets of $V(H)$ such that $H[S'_i]$ $(1 \leq i \leq y)$ is connected.

Case 2. $a < k \leq b$**.** For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap V(G) = \varnothing$, or $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.

If $S \cap V(G) = \varnothing$, then $S \subseteq V(H)$. Recall that $S_1, S_2, \ldots, S_{\binom{b}{k}}$ be all the k-subsets of $V(H)$, and $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(H)$ such that $G[S_i]$ $(1 \leq i \leq x)$ is connected. Then $d_H(S_i) = d_{G \vee H}(S_i) = k - 1$ for each i $(1 \leq i \leq x)$. For each i $(x + 1 \leq i \leq \binom{b}{k})$, we have $d_H(S_i) \geq k$ and $d_{G \vee H}(S_i) = k$. For $v \in V(G \vee H)$, if $v \in V(H)$, then $deg_{G \vee H}(v) = deg_H(v) + a$. This case contributes to $SDD_k(G \vee H)$ by

$$\sum_{\substack{S \subseteq V(H) \\ |S|=k}} \left(\sum_{v \in S} deg_{G \vee H}(v) \right) d_{G \vee H}(S)$$

$$= \sum_{\substack{S \subseteq V(H) \\ |S|=k}} \left(\sum_{v \in S} (deg_H(v) + a) \right) d_{G \vee H}(S)$$

$$= \sum_{\substack{S \subseteq V(H) \\ |S|=k}} \left(\sum_{v \in S} deg_H(v) \right) d_{G \vee H}(S) + ak \sum_{\substack{S \subseteq V(H) \\ |S|=k}} d_{G \vee H}(S)$$

$$= (k-1) \sum_{\substack{S \subseteq V(H),\ |S|=k \\ d_H(S)=k-1}} \left(\sum_{v \in S} deg_H(v) \right) + k \sum_{\substack{S \subseteq V(H),\ |S|=k \\ d_H(S) \geq k}} \left(\sum_{v \in S} deg_H(v) \right)$$
$$+ ak \sum_{\substack{S \subseteq V(H) \\ |S|=k}} d_{G \vee H}(S)$$

$$= k \sum_{\substack{S \subseteq V(H) \\ |S|=k}} \left(\sum_{v \in S} deg_H(v) \right) - \sum_{\substack{S \subseteq V(G),\ |S|=k \\ d_H(S)=k-1}} \left(\sum_{v \in S} deg_H(v) \right) + ak \left[k \binom{b}{k} - x \right]$$

$$= \quad 2ke(H)\binom{b-1}{k-1} - \sum_{i=1}^{x}\left(\sum_{v\in S_i} deg_H(v)\right) + ak^2\binom{b}{k} - akx\,.$$

Suppose $S\cap V(G)\neq\varnothing$ and $S\cap V(H)\neq\varnothing$. In this case, $d_{G\vee H}(S)=k-1$. Then this case contributes to $SDD_k(G\vee H)$ by

$$\sum_{\substack{S\subseteq V(G),|S|=k \\ S\cap V(G)\neq\varnothing, S\cap V(H)\neq\varnothing}}\left(\sum_{v\in S} deg_{G\vee H}(v)\right)d_{G\vee H}(S)$$

$$= \quad (k-1)\sum_{\substack{S\subseteq V(G),|S|=k \\ S\cap V(G)\neq\varnothing, S\cap V(H)\neq\varnothing}}\left(\sum_{v\in S} deg_{G\vee H}(v)\right)$$

$$= \quad (k-1)\cdot 2e(G\vee H)\binom{a+b-1}{k-1} - (k-1)\cdot 2e(G)\binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1)\cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}$$

$$= \quad (k-1)(2e(G)+2e(H)+2ab)\binom{a+b-1}{k-1} - (k-1)\cdot 2e(G)\binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1)\cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}\,.$$

From the above arguments, we conclude that

$$SDD_k(G\vee H) \quad = \quad ak^2\binom{b}{k} - akx + 2e(H)\binom{b-1}{k-1} - \sum_{i=1}^{x}\left(\sum_{v\in S_i} deg_H(v)\right)$$

$$+2(k-1)[e(G)+e(H)+ab]\binom{a+b-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1)ab\binom{b-1}{k-1} - (k-1)\cdot 2e(G)\binom{a-1}{k-1}\,,$$

where $S_1, S_2, \ldots, S_x$ are all the k-subsets of $V(H)$ such that $H[S_i]$ $(1\leq i\leq x)$ is connected.

Case 3. $b < k \leq a+b$. For any $S \subseteq V(G)$ and $|S| = k$, we have $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. In this case, $d_{G \vee H}(S) = k - 1$. Then

$$SDD_k(G \vee H)$$

$$= \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_{G \vee H}(v) \right) d_{G \vee H}(S)$$

$$= (k-1) \sum_{\substack{S \subseteq V(G) \\ |S|=k}} \left(\sum_{v \in S} deg_{G \vee H}(v) \right)$$

$$= (k-1) \cdot 2e(G \vee H) \binom{a+b-1}{k-1} - (k-1) \cdot 2e(G) \binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1) \cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}$$

$$= (k-1)[2e(G) + 2e(H) + 2ab]\binom{a+b-1}{k-1} - (k-1) \cdot 2e(G)\binom{a-1}{k-1}$$

$$-(k-1)ab\binom{a-1}{k-1} - (k-1) \cdot 2e(H)\binom{b-1}{k-1} - (k-1)ab\binom{b-1}{k-1}$$

$$=2(k-1)e(G)\left[\binom{a+b-1}{k-1} - \binom{a-1}{k-1}\right] - (k-1)ab\left[\binom{a-1}{k-1} + \binom{b-1}{k-1}\right]$$

$$+2(k-1)e(H)\left[\binom{a+b-1}{k-1} - \binom{b-1}{k-1}\right].$$

This completes the proof of the theorem.

2.1 Cartesian product

In [15], Gologranc obtained the following lower bound for Steienr distance.

Lemma 2.1. [15] *Let* $k \geq 2$ *be an integer, and let* G, H *be two connected graphs. Let* $S = \{(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)\}$ *be a set of distinct vertices of*

$G \square H$. Let $S_G = \{g_1, g_2, \ldots, g_k\}$ and $S_H = \{h_1, h_2, \ldots, h_k\}$. Then

$$d_{G \square H}(S) \geq d_G(S_G) + d_H(S_H).$$

Mao et al. derived the following results for Steiner distance.

Lemma 2.2. *Let G, H be two connected graphs of order n, m, respectively. Let k be an integer with $3 \leq k \leq mn$. Let $S = \{(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)\}$ be a set of distinct vertices of $G \square H$. Let $S_G = \{g_1, g_2, \ldots, g_k\}$ and $S_H = \{h_1, h_2, \ldots, h_k\}$. Then*

$$
\begin{aligned}
d_G(S_G) + d_H(S_H) &\leq d_{G \square H}(S) \\
&\leq \min\{d_G(S_G) + (k-2)d_H(S_H), d_H(S_H) + (k-2)d_G(S_G)\}.
\end{aligned}
$$

In this section, we give the upper and lower bounds of $SDD_k(G \square H)$.

Theorem 2.2. *Let G, H be two connected graphs of order a, b ($a \leq b$), respectively. Let k be an integer with $3 \leq k \leq a + b$. Then*

$$X \leq SDD_k(G \square H) \leq \left\lceil \frac{k-1}{2} \right\rceil X,$$

where

$$
X = \sum_{i=1}^{k} b \binom{b-1}{r_1 - 1} \binom{b}{r_2} \cdots \binom{b}{r_i} SDD_i(G) + \sum_{j=1}^{k} a \binom{a-1}{s_1 - 1} \binom{a}{s_2} \cdots \binom{a}{s_j} SDD_j(H)
$$

$$
+ 2e(H) \sum_{i=1}^{k} b \binom{b-1}{r_1 - 1} \binom{b}{r_2} \cdots \binom{b}{r_i} iSW_i(G)
$$

$$
+ 2e(G) \sum_{j=1}^{k} a \binom{a-1}{s_1 - 1} \binom{a}{s_2} \cdots \binom{a}{s_j} jSW_j(H),
$$

$\sum_{p=1}^{i} r_p = k$, $\sum_{q=1}^{j} s_q = k$, $r_p \geq 1$ $(1 \leq p \leq i)$, $s_q \geq 1$ $(1 \leq q \leq j)$.

Proof. From the definition of Steiner degree distance, we have

$$
SDD_k(G \square H) = \sum_{\substack{S \subseteq V(G \square H) \\ |S| = k}} \left(\sum_{(u,v) \in S} deg_{G \square H}(u, v) \right) d_{G \square H}(S).
$$

Let $S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$. Then

$$\sum_{(u,v) \in S} deg_{G\square H}(u,v) = \sum_{i=1}^{k} deg_G(u_i) + \sum_{i=1}^{k} deg_H(v_i).$$

From Lemma 2.1, we have $d_{G\square H}(S) \geq d_G(S_G) + d_H(S_H)$, and hence

$$SDD_k(G\square H)$$

$$\geq \sum_{\substack{S \subseteq V(G\square H) \\ |S|=k}} \left(\sum_{i=1}^{k} deg_G(u_i) + \sum_{i=1}^{k} deg_H(v_i) \right) [d_G(S_G) + d_H(S_H)]$$

$$= \sum_{\substack{S \subseteq V(G\square H) \\ |S|=k}} \left(\sum_{i=1}^{k} deg_G(u_i) \right) d_G(S_G) + \sum_{\substack{S \subseteq V(G\square H) \\ |S|=k}} \left(\sum_{i=1}^{k} deg_H(v_i) \right) d_H(S_H)$$

$$+ \sum_{\substack{S \subseteq V(G\square H) \\ |S|=k}} \left(\sum_{i=1}^{k} deg_H(v_i) \right) d_G(S_G) + \sum_{\substack{S \subseteq V(G\square H) \\ |S|=k}} \left(\sum_{i=1}^{k} deg_G(u_i) \right) d_H(S_H)$$

$$= \sum_{i=1}^{k} b \binom{b-1}{r_1-1} \binom{b}{r_2} \cdots \binom{b}{r_i} SDD_i(G) + \sum_{j=1}^{k} a \binom{a-1}{s_1-1} \binom{a}{s_2} \cdots \binom{a}{s_j} SDD_j(H)$$

$$+ 2e(H) \sum_{i=1}^{k} b \binom{b-1}{r_1-1} \binom{b}{r_2} \cdots \binom{b}{r_i} iSW_i(G)$$

$$+ 2e(G) \sum_{j=1}^{k} a \binom{a-1}{s_1-1} \binom{a}{s_2} \cdots \binom{a}{s_j} jSW_j(H)$$

From Lemma 2.2, we have $d_{G\square H}(S) \leq \min\{d_G(S_G) + d_H(S_H)(k-2), d_H(S_H)+$

$d_G(S_G)(k-2)\} \leq \lceil\frac{k-1}{2}\rceil[d_G(S_G)+d_H(S_H)]$, and hence

$$SDD_k(G\square H)$$

$$\leq \left\lceil\frac{k-1}{2}\right\rceil \sum_{\substack{S\subseteq V(G\square H)\\|S|=k}} \left(\sum_{i=1}^{k} deg_G(u_i) + \sum_{i=1}^{k} deg_H(v_i)\right)[d_G(S_G)+d_H(S_H)]$$

$$= \left\lceil\frac{k-1}{2}\right\rceil \sum_{i=1}^{k} b\binom{b-1}{r_1-1}\binom{b}{r_2}\cdots\binom{b}{r_i} SDD_i(G) + \frac{k}{2}\sum_{j=1}^{k} a\binom{a-1}{s_1-1}\binom{a}{s_2}$$

$$\cdots\binom{a}{s_j} SDD_j(H) + ke(H)\sum_{i=1}^{k} b\binom{b-1}{r_1-1}\binom{b}{r_2}\cdots\binom{b}{r_i} iSW_i(G)$$

$$+ke(G)\sum_{j=1}^{k} a\binom{a-1}{s_1-1}\binom{a}{s_2}\cdots\binom{a}{s_j} jSW_j(H).$$

This completes the proof of the theorem.

Remark 2.1. *For $k=3$, the lower and upper bounds on $SDD_k(G\square H)$ are same (X) in Theorem 2.2.*

3   Conclusion

The Steiner distance in a graph, introduced by Chartrand et al. is a natural generalization of the concept of classical graph distance. The degree distance seems to have been considered first by Dobrynin and Kochetova, and practically at the same time by Gutman, who used a different name for it. The k-center Steiner degree distance introduced very recently. In this paper, we obtain the k-center Steiner degree distance of complete and Cartesian product graphs. Investigating the other graph products (corona product, composition, disjunction etc.) are our future work.

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