On Modules over Local Rings

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Abstract

This paper is dealed with a special local ring **A** and modules over **A**. Some properties of modules, that are constructed over the real plural algebra, are investigated. Moreover a module is constructed over the linear algebra of matrix $M_{mm}(\mathbb{R})$ and one of its basis is found.

1 Introduction.

The structure of a field which has many simplicities in its operations, in fact is a generalization of the system of real numbers $\mathbb{R}$. The structure of a ring which does not have some properties that a field has, is also a generalization of the system of integers $\mathbb{Z}$. For example, while all linear equations have solutions according to both addition and multiplication in $\mathbb{R}$ or in a field, every linear equation according to multiplication in $\mathbb{Z}$ or in a ring does not necessarily have any solutions because of non-existence of inverse element. In the study of vector spaces constructed over fields, certainly there are many simplicities compared to the study of modules constructed over rings. Modules are more general structures than vector spaces. For more detailed study on modules, we refer to [1]. For the Algebra and Linear Algebra that will be used throughout this paper, we refer to [4] and [5].

In this paper, we investigate some properties of modules constructed over the real plural algebra $\boldsymbol{A}$. We construct a module over the linear algebra of matrix $M_{mm}(\mathbb{R})$ and we find one of its basis. Furthermore, we give a theorem that describes the linear independent vectors in a module constructed over a local ring.

2   Preliminaries.

In this section, we will recall some basic definitons and propositions from [2] and [3].

Definition 1 ([2, Def. 1.1]). *A real plural algebra of order m is a linear algebra $\boldsymbol{A}$ on $\mathbb{R}$ having, as a vector space over $\mathbb{R}$, a basis $\left\{1, \eta, \eta^2, \eta^3, \cdots, \eta^{m-1}\right\}$, where $\eta^m = 0$.*

A ring with identity element is called *local* if the set of its non-units form an ideal. A module that is constructed over a local ring $\boldsymbol{A}$ is called an $\boldsymbol{A} - module.$

Definition 2 ([3, Def. 1]). *Let $\boldsymbol{A}$ be a local ring. Let M be a finitely generated $\boldsymbol{A} - module$. Then M is an $\boldsymbol{A} - space$ of finite dimension if there exists $E_1, E_2, \cdots, E_n$ in M with*
   i) $M = \boldsymbol{A}E_1 \oplus \boldsymbol{A}E_2 \oplus \cdots \oplus \boldsymbol{A}E_n$
   ii) the map $\boldsymbol{A} \to \boldsymbol{A}E_i$ defined by $x \to xE_i$ is an isomorphism for $1 \leq i \leq n$.

Definition 3 ([2, Def. 1.2]). *By a system of projections $\boldsymbol{A} \to \mathbb{R}$, it is meant a system of mappings $p_k : \boldsymbol{A}$ onto $\mathbb{R}$, defined for $k = 0, 1, \cdots, m-1$, as follows:*

$$\forall \beta \in \boldsymbol{A}, \ \beta = \sum_{i=0}^{m-1} b_i \eta^i, \ p_k(\beta) := b_k.$$

Now we can introduce the following propositions without proof, from [2].

Proposition 4 ([2, Prop. 1.3]). *An element $\varepsilon \in \boldsymbol{A}$ is a unit if and only if $p_0(\varepsilon) \neq 0$.*

Proposition 5 ([2, Prop. 1.5]). *$\boldsymbol{A}$ is a local ring with the maximal ideal $\eta\boldsymbol{A}$. The ideals $\eta^j \boldsymbol{A}$, $1 \leq j \leq m$, are all ideals in $\boldsymbol{A}$.*

3   Modules Constructed Over The Real Plural Algebra.

In this section, we examine some properties of modules constructed over the real plural algebra $\boldsymbol{A}$. First we give a theorem that tells us the relation between units and zero divisors in $\boldsymbol{A}$. Then we give a short proof of the isomorphism between $\boldsymbol{A}$ and linear algebra of matrix $\boldsymbol{K} = M_{mm}(\mathbb{R})$. After that we investigate a basis for $\boldsymbol{K}$ and we construct a module over $\boldsymbol{K}$ and find one of its basis. We give a theorem that describes the linear independent vectors in a module constructed over a local ring. Finally we give a detailed proof of Remark 2 given in [3].

Theorem 6. *None of the units of* **A** *are zero divisors, namely for every* $\alpha, \beta \in \mathbf{A}$;

$$\alpha = \sum_{i=0}^{m-1} a_i \eta^i, a_0 \neq 0 \quad and \quad \beta = \sum_{i=0}^{m-1} b_i \eta^i \ if \ \alpha \cdot \beta = 0 \quad or \quad \beta \cdot \alpha = 0,$$

then $\beta = 0$. *Also for* $1 \leq k \leq m-1$ *and* $\alpha = a_k \eta^k + a_{k+1} \eta^{k+1} + \cdots + a_{m-1} \eta^{m-1}$ *if* $\alpha \cdot \beta = 0$ *or* $\beta \cdot \alpha = 0$, *then* $\beta = b_{m-k} \eta^{m-k} + b_{m-k+1} \eta^{m-k+1} + \cdots + b_{m-1} \eta^{m-1}$.

Proof. If α is a unit, then there is an inverse element α^{-1} and since there is an associative property in the real algebra;

$$\alpha \cdot \beta = 0 \Rightarrow \alpha^{-1}(\alpha \cdot \beta) = \alpha^{-1} \cdot 0 \Rightarrow \beta = 0.$$

For $\beta \cdot \alpha = 0$, it is easily seen that $\beta = 0$ by similar calculations.

Now we show that for $1 \leq k \leq m-1$, $\alpha \cdot \beta = (a_k \eta^k + a_{k+1} \eta^{k+1} + \cdots + a_{m-1} \eta^{m-1})(b_0 + b_1 \eta + b_2 \eta^2 + \cdots + b_{m-1} \eta^{m-1}) = 0 \Rightarrow \beta = b_{m-k} \eta^{m-k} + b_{m-k+1} \eta^{m-k+1} + \cdots + b_{m-1} \eta^{m-1}$.

First let $k = 1$, so we deal with $\alpha = a_1 \eta + a_2 \eta^2 + \cdots + a_{m-1} \eta^{m-1}$. We have $\alpha \cdot \beta = (a_1 \eta + a_2 \eta^2 + \cdots + a_{m-1} \eta^{m-1}) \cdot (b_0 + b_1 \eta + b_2 \eta^2 + \cdots + b_{m-1} \eta^{m-1})$ and

$$\begin{aligned}
\alpha \cdot \beta &= 0 \\
&\Rightarrow (a_1 b_0)\eta + (a_1 b_1 + a_2 b_0)\eta^2 + (a_1 b_2 + a_2 b_1 + a_3 b_0)\eta^3 \\
&\quad + \cdots + (a_1 b_{m-2} + a_2 b_{m-3} + \cdots + a_{m-2} b_1 + a_{m-1} b_0)\eta^{m-1} \\
&\quad + (a_1 b_{m-1} + a_2 b_{m-2} + \cdots + a_{m-2} b_2 + a_{m-1} b_1)\eta^m \\
&\quad + (a_2 b_{m-1} + a_3 b_{m-2} + \cdots + a_{m-2} b_3 + a_{m-1} b_2)\eta^{m+1} \\
&\quad + \cdots + (a_{m-1} b_{m-1})\eta^{2m-2} \\
&= 0\eta + 0\eta^2 + \cdots + 0\eta^{m-1} + c_m \eta^m + c_{m+1}\eta^{m+1} + \cdots + c_{2m-2}\eta^{2m-2}.
\end{aligned}$$

Here since $\eta^m = \eta^{m+1} = \cdots = \eta^{2m-2} = 0$, the coefficients $c_m, c_{m+1}, \cdots, c_{2m-2}$ need not to be zero. Forcing the coefficient of η to be zero, we obtain $a_1 b_0 = 0$, and since $a_1 \neq 0$, we find $b_0 = 0$. Forcing the coefficient of η^2 to be zero, we obtain $a_1 b_1 + a_2 b_0 = 0$. If we put $b_0 = 0$ in this equation and use the fact that $a_1 \neq 0$, we obtain $b_1 = 0$. Forcing the coefficient of η^3 to be zero, we get $a_1 b_2 + a_2 b_1 + a_3 b_0 = 0$. Putting $b_0 = b_1 = 0$ in this equation and using $a_1 \neq 0$, we find $b_2 = 0$. Carrying on this process, we get $b_k = 0$ for $0 \leq k \leq m-3$. Finally forcing the coefficient of η^{m-1} to be zero, we get $a_1 b_{m-2} + a_2 b_{m-3} + \cdots + a_{m-2} b_1 + a_{m-1} b_0 = 0$. Putting $b_0 = b_1 = \cdots = b_{m-3} = 0$ in this equation and using $a_1 \neq 0$, we find $b_{m-2} = 0$. If we put $b_0 = b_1 = \cdots = b_{m-3} = b_{m-2} = 0$ in the following equation, we have $(a_1 b_{m-1})\eta^m = 0$. Since $\eta^m = 0$, then $b_{m-1} \in \mathbb{R}$. Thus for $k = 1$, we obtain

$\alpha \cdot \beta = (a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1}) \cdot (b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1}) = 0 \Rightarrow \beta = b_{m-1}\eta^{m-1}$.

Now let $k = 2$ so we deal with $\alpha = a_2\eta^2 + a_3\eta^3 + \cdots + a_{m-1}\eta^{m-1}$. We have $\alpha \cdot \beta = (a_2\eta^2 + a_3\eta^3 + \cdots + a_{m-1}\eta^{m-1}) \cdot (b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1})$ and

$$\alpha \cdot \beta = 0$$
$$\Rightarrow (a_2b_0)\eta^2 + (a_2b_1 + a_3b_0)\eta^3 + (a_2b_2 + a_3b_1 + a_4b_0)\eta^4$$
$$+ \cdots + (a_2b_{m-4} + a_3b_{m-5} + \cdots + a_{m-3}b_1 + a_{m-2}b_0)\eta^{m-2}$$
$$+ (a_2b_{m-3} + a_3b_{m-4} + \cdots + a_{m-2}b_1 + a_{m-1}b_0)\eta^{m-1}$$
$$+ (a_2b_{m-2} + a_3b_{m-3} + \cdots + a_{m-2}b_2 + a_{m-1}b_1)\eta^{m}$$
$$+ (a_2b_{m-1} + a_3b_{m-2} + \cdots + a_{m-2}b_3 + a_{m-1}b_2)\eta^{m+1}$$
$$+ \cdots + (a_{m-1}b_{m-1})\eta^{2m-2}$$
$$= 0\eta + 0\eta^2 + \cdots + 0\eta^{m-1} + c_m\eta^m + c_{m+1}\eta^{m+1} + \cdots + c_{2m-2}\eta^{2m-2}.$$

Here since $\eta^m = \eta^{m+1} = \cdots = \eta^{2m-2} = 0$, the coefficients $c_m, c_{m+1}, \cdots, c_{2m-2}$ need not be zero. Forcing the coefficient of η^2 to be zero, we obtain $a_2b_0 = 0$ and since $a_2 \neq 0$, we find $b_0 = 0$. Forcing the coefficient of η^3 to be zero, we obtain $a_2b_1 + a_3b_0 = 0$. If we use $b_0 = 0$ in this equation and use the fact that $a_2 \neq 0$, we obtain $b_1 = 0$. Forcing the coefficient of η^4 to be zero, we get $a_2b_2 + a_3b_1 + a_4b_0 = 0$. If we put $b_0 = b_1 = 0$ in this equation and use $a_2 \neq 0$, we obtain $b_2 = 0$. Continuing on this process, we get $b_k = 0$ for $0 \leq k \leq m - 3$. If we put $b_0 = b_1 = \cdots = b_{m-3} = 0$ in the coefficient of η^m, we have $(a_2b_{m-2})\eta^m = 0$. Since $\eta^m = 0$, we find $b_{m-2} \in \mathbb{R}$. Similarly if we put $b_0 = b_1 = \cdots = b_{m-3} = 0$ and $b_{m-2} \in \mathbb{R}$, in the coefficient of η^{m+1}, we get $(a_2b_{m-1} + a_3b_{m-2})\eta^{m+1} = 0$. Since $\eta^{m+1} = 0$, we have $b_{m-1} \in \mathbb{R}$. Thus for $k = 2$, we find $\alpha \cdot \beta = (a_2\eta^2 + a_3\eta^3 + \cdots + a_{m-1}\eta^{m-1}) \cdot (b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1}) = 0 \Rightarrow \beta = b_{m-2}\eta^{m-2} + b_{m-1}\eta^{m-1}$.

Finally we generalize this process for k. Let $\alpha = a_k\eta^k + a_{k+1}\eta^{k+1} + \cdots + a_{m-1}\eta^{m-1}$. Then we have
$\alpha \cdot \beta = (a_k\eta^k + a_{k+1}\eta^{k+1} + \cdots + a_{m-1}\eta^{m-1}) \cdot (b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1})$
and

$$\alpha \cdot \beta = 0$$
$$\Rightarrow (a_kb_0)\eta^k + (a_kb_1 + a_{k+1}b_0)\eta^{k+1} + (a_kb_2 + a_{k+1}b_1 + a_{k+2}b_0)\eta^{k+2}$$

$$+ \cdots + (a_k b_{m-(k+1)} + a_{k+1} b_{m-(k+2)} + \cdots + a_{m-1} b_0) \eta^{m-1}$$
$$+ (a_k b_{m-k} + a_{k+1} b_{m-(k+1)} + \cdots + a_{m-2} b_2 + a_{m-1} b_1) \eta^m$$
$$+ (a_k b_{m-(k-1)} + a_{k+1} b_{m-k} + \cdots + a_{m-2} b_3 + a_{m-1} b_2) \eta^{m+1}$$
$$+ \cdots + (a_{m-1} b_{m-1}) \eta^{2m-2},$$
$$= 0\eta + 0\eta^2 + \cdots + 0\eta^{m-1} + c_m \eta^m + c_{m+1} \eta^{m+1} + \cdots + c_{2m-2} \eta^{2m-2}.$$

Here since $\eta^m = \eta^{m+1} = \cdots = \eta^{2m-2} = 0$, the coefficients $c_m, c_{m+1}, \cdots, c_{2m-2}$ need not be zero. Forcing the coefficient of η^k to be zero, we obtain $a_k b_0 = 0$ and since $a_k \neq 0$, we find $b_0 = 0$. Forcing the coefficient of η^{k+1} to be zero, we obtain $a_k b_1 + a_{k+1} b_0 = 0$. If we put $b_0 = 0$ in this equation and use $a_k \neq 0$, we obtain $b_1 = 0$. Forcing the coefficient of η^{k+2} to be zero, we get $a_k b_2 + a_{k+1} b_1 + a_{k+2} b_0 = 0$. If we use $b_0 = b_1 = 0$ in this last equation and use $a_k \neq 0$, we obtain $b_2 = 0$. Carrying on this process, we can easily find $b_0 = b_1 = \cdots = b_{m-(k+1)} = 0$ by forcing the coefficients of $\eta^k, \eta^{k+1}, \eta^{k+2}, \cdots, \eta^{m-1}$ to be zero. Now if we put $b_0 = b_1 = \cdots = b_{m-(k+1)} = 0$ in the coefficient of η^m, we obtain $(a_k b_{m-k}) \eta^m = 0$. Since $\eta^m = 0$, we have $b_{m-k} \in \mathbb{R}$. By similar calculations, if we use $b_0 = b_1 = \cdots = b_{m-(k+1)} = 0$ and $b_{m-k} \in \mathbb{R}$ in the coefficient of η^{m+1}, we obtain $(a_k b_{m-(k-1)} + a_{k+1} b_{m-k}) \eta^{m+1} = 0$. Since $\eta^{m+1} = 0$, we get $b_{m-(k-1)} \in \mathbb{R}$. Carrying on this process by using $\eta^m = \eta^{m+1} = \cdots = \eta^{2m-2} = 0$ we can easily find $b_{m-k}, b_{m-(k-1)}, b_{m-(k-2)}, \cdots, b_{m-1} \in \mathbb{R}$. Thus for $\alpha = a_k \eta^k + a_{k+1} \eta^{k+1} + \cdots + a_{m-1} \eta^{m-1}$ and $\beta = b_0 + b_1 \eta + b_2 \eta^2 + \cdots + b_{m-1} \eta^{m-1}$, $\alpha \cdot \beta = 0$ implies $\beta = b_{m-k} \eta^{m-k} + b_{m-k+1} \eta^{m-k+1} + \cdots + b_{m-1} \eta^{m-1}$. $\square$

Let $\boldsymbol{A}$ be a real plural algebra having a basis $\left\{1, \eta, \eta^2, \eta^3, \cdots, \eta^{m-1}\right\}$ with $\eta^m = 0$. Let $\boldsymbol{K} = M_{mm}(\mathbb{R})$ be the linear algebra of matrix of the form

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_0 \end{pmatrix},$$

where $b_i \in \mathbb{R}$ for $0 \leq i \leq m-1$. In [2] it is stated that without proof the map $f : \boldsymbol{A} \to \boldsymbol{K}$ which is given as

$$f(\alpha) = (a_{ij}) = \begin{cases} a_{ij} = 0 & , \quad j < i \\ a_{ij} = a_{j-i} , & j \geq i. \end{cases}$$

for every $\alpha = \sum_{i=0}^{m-1} a_i \eta^i \in \boldsymbol{A}$ is isomorphism. Here we give the short proof of this proposition.

Proposition 7. *The ring $\boldsymbol{A}$ is isomorphic to the linear algebra of matrix $\boldsymbol{K}$ of the form:*

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ 0 & b_0 & \cdots & b_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_0 \end{pmatrix}.$$

Proof. The map $f : \boldsymbol{A} \to \boldsymbol{K}$ is defined in the following way: For every $\alpha = \sum\limits_{i=0}^{m-1} a_i \eta^i \in \boldsymbol{A}$,

$$f(\alpha) = (a_{ij}) = \left\{ \begin{array}{ll} a_{ij} = 0 & , \quad j < i \\ a_{ij} = a_{j-i} , & j \geq i. \end{array} \right.$$

Then $f(\alpha) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ 0 & a_0 & a_1 & \cdots & a_{m-2} \\ 0 & 0 & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}.$

Now for the considered mapping f, it is easily seen that f is one to one, onto and for every $\alpha = a_0 + a_1\eta + a_2\eta^2 + \cdots + a_{m-1}\eta^{m-1}$ and $\beta = b_0 + b_1\eta + b_2\eta^2 + \cdots + b_{m-1}\eta^{m-1} \in \boldsymbol{A}$ and for every $c \in \mathbb{R}$

$$f(\alpha + \beta) = f(\alpha) + f(\beta) \text{ and } f(c\alpha) = cf(\alpha).$$

Now we show the multiplication are preserved under this map. For every α, $\beta \in \boldsymbol{A}$, $\alpha \cdot \beta = (a_0 b_0) + (a_0 b_1 + a_1 b_0)\eta + (a_0 b_2 + a_1 b_1 + a_2 b_0)\eta^2 + \cdots + (a_0 b_{m-1} + a_1 b_{m-2} + \cdots + a_{m-1} b_0)\eta^{m-1}$. Then we obtain

$$f(\alpha\beta) = \begin{pmatrix} a_0 b_0 & a_0 b_1 + a_1 b_0 & \cdots & a_0 b_{m-1} + a_1 b_{m-2} + \cdots + a_{m-1} b_0 \\ 0 & a_0 b_0 & \cdots & a_0 b_{m-2} + a_1 b_{m-3} + \cdots + a_{m-2} b_0 \\ 0 & 0 & \ddots & a_0 b_{m-3} + a_1 b_{m-2} + \cdots + a_{m-3} b_0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 b_0 \end{pmatrix}$$

and by using multiplication of matrices, we get

$$\begin{aligned} f(\alpha \cdot \beta) &= \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ 0 & a_0 & a_1 & \cdots & a_{m-2} \\ 0 & 0 & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \cdot \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-2} \\ 0 & 0 & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} \\ &= f(\alpha) \cdot f(\beta). \end{aligned}$$

$\square$

Now we investigate one of the basis of $\boldsymbol{K}$. If we take any element of $\boldsymbol{K}$ such that

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-2} \\ 0 & 0 & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} \in \boldsymbol{K},$$

we can write it as follows:

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} \\ 0 & b_0 & b_1 & \cdots & b_{m-2} \\ 0 & 0 & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} = b_0 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$+ b_1 \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + b_{m-1} \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Here if we put

$$\eta_0 = (a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m} \quad \text{where } a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\eta_1 = (a_{ij}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m} \quad \text{where } a_{ij} = \begin{cases} 1, & j = i+1 \\ 0, & j \neq i+1 \end{cases}$$

$$\eta_2 = (a_{ij}) = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m} \quad \text{where } a_{ij} = \begin{cases} 1, & j = i+2 \\ 0, & j \neq i+2 \end{cases}$$

$$\vdots$$

$$\eta_{m-1} = (a_{ij}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ where } a_{ij} = \begin{cases} 1 , & j = i + (m-1) \\ 0 , & j \neq i + (m-1) \end{cases}$$

then it is seen that $B = b_0\eta_0 + b_1\eta_1 + b_2\eta_2 + \cdots + b_{m-1}\eta_{m-1}$ and also linear independence of the set $\{\eta_0, \eta_1, \eta_2, \cdots, \eta_{m-1}\}$ is obvious. Thus the set $\{\eta_0, \eta_1, \eta_2, \cdots, \eta_{m-1}\}$ is a basis of $\mathbf{K}$. We can express any element of this set in general as follows:

For $0 \leq i, j, k \leq m-1$, $\eta_k = (a_{ij})_{m \times m}$ where $a_{ij} = \begin{cases} 1 , & j = i + k \\ 0 , & j \neq i + k \end{cases}$.

Also if we take

$$\eta = \eta_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

we see that $\eta_k = \eta^k$ for every η_k where $1 \leq k \leq m-1$. For example

$$\eta^2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \eta_2 .$$

Now we construct a module M over the linear algebra of matrix $\mathbf{K}$, then we will find one of the basis of it.

Proposition 8. $M = \mathbb{R}_n^m$ *is a module over the linear algebra of matrix* $\mathbf{K}$. *Then the following set is a basis of* $\mathbf{K}-$*module* M.

$$\left\{ E_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \right.$$

$$\left. E_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0\cdots & 0 \end{pmatrix}, \cdots, E_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right\}.$$

Proof. Linear independence of this set is obvious. Moreover for every $X \in M$, X can be written as follows:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} x_{m1} & x_{(m-1)1} & \cdots & x_{11} \\ 0 & x_{m1} & \cdots & x_{21} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} x_{m2} & x_{(m-1)2} & \cdots & x_{12} \\ 0 & x_{m2} & \cdots & x_{22} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} x_{mn} & x_{(m-1)n} & \cdots & x_{1n} \\ 0 & x_{mn} & \cdots & x_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{mn} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Thus $[E_1, E_2, \cdots, E_n] = M$. Consequently, the set $\{E_1, E_2, \cdots, E_n\}$ is a basis of $\boldsymbol{K}-$module M. $\square$

For some $a_1, a_2, \cdots, a_n \in \mathbb{R}$, we write

$$a_1 E_1 + a_2 E_2 + \cdots + a_n E_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}.$$

So if M is thought as a vector space over $\mathbb{R}$, it is seen that

$$[E_1, E_2, \cdots, E_n] = P_0$$

$$= \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix} \right| a_i \in \mathbb{R}, 1 \le i \le n \right\}.$$

Thus $\{E_1, E_2, \cdots, E_n\}$ can not be a basis for module M over $\mathbb{R}$. However if we take the system of generators $B = \{IE_1, IE_2, \cdots, IE_n, \eta E_1, \eta E_2, \cdots, \eta E_n, \eta^2 E_1, \eta^2 E_2, \cdots, \eta^2 E_n, \cdots, \eta^{m-1} E_1, \eta^{m-1} E_2, \cdots, \eta^{m-1} E_n\}$, B is obviously a basis of the module M over $\mathbb{R}$. Consequently M is an mn-dimensional vector space over $\mathbb{R}$.

From Proposition 8 we have seen that for every $X = (x_{ij}) \in \mathbb{R}_n^m$, X can be written as follows:

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \cdots & x_{mn} \end{pmatrix} = \begin{pmatrix} x_{m1} & x_{(m-1)1} & \cdots & x_{11} \\ 0 & x_{m1} & \cdots & x_{21} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m1} \end{pmatrix} E_1$$

$$+ \begin{pmatrix} x_{m2} & x_{(m-1)2} & \cdots & x_{12} \\ 0 & x_{m2} & \cdots & x_{22} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m2} \end{pmatrix} E_2 + \cdots$$

$$+ \begin{pmatrix} x_{mn} & x_{(m-1)n} & \cdots & x_{1n} \\ 0 & x_{mn} & \cdots & x_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{mn} \end{pmatrix} E_n.$$

Here by using

$$\begin{pmatrix} x_{mk} & x_{(m-1)k} & \cdots & x_{1k} \\ 0 & x_{mk} & \cdots & x_{2k} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{mk} \end{pmatrix} = x_{mk}\eta_0 + x_{(m-1)k}\eta_1 + \cdots + x_{1k}\eta_{m-1},$$

X also can be expressed as follows:

$$X = (x_{m1}\eta_0 + x_{(m-1)1}\eta_1 + \cdots + x_{11}\eta_{m-1})E_1 + (x_{m2}\eta_0 + x_{(m-1)2}\eta_1 + \cdots + x_{12}\eta_{m-1})E_2 + \cdots + (x_{mn}\eta_0 + x_{(m-1)n}\eta_1 + \cdots + x_{1n}\eta_{m-1})E_n.$$

The set $M = \mathbb{R}_n^m$ is a vector space over the field $\mathbb{R}$. If $m \neq n$ for the matrices $A_{m \times n}$ and $B_{m \times n}$ multiplication is not defined but if $m = n$ then M is an associative algebra with identity. Furthermore if we use a commutative ring with identity then $M = \mathbb{R}_n^m$ is a module over the ring R. Multiplication is defined when $m = n$, so the operation is associative and it has unity.

Theorem 9. *Let $\mathbf{A}$ be a local ring with a maximal ideal J and $M = \mathbf{A}^n$. Then for $u_1, u_2, \cdots, u_k \in \mathbf{A} \setminus J$ and $x_{ij} \in J$, there are linearly independent vectors such that $\alpha_1 = (u_1, x_{21}, x_{31}, \cdots, x_{n1})$, $\alpha_2 = (x_{12}, u_2, x_{32}, \cdots, x_{n2})$, $\alpha_3 = (x_{13}, x_{23}, u_3, \cdots, x_{n3})$, $\cdots$, $\alpha_k = (x_{1k}, x_{2k}, x_{3k}, \cdots, u_k, x_{(k+1)k}, \cdots, x_{nk})$, and for $k = n$, the set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a basis for M.*

Proof. First let $k = 1$ so we deal with only $\alpha_1 = (u_1, x_{21}, x_{31}, \cdots, x_{n1}) \in M$.

For $a_1 \in \mathbf{A}$, let $a_1 \alpha_1 = 0 \Rightarrow a_1 \alpha_1 = (a_1 u_1, a_1 x_{21}, a_1 x_{31}, \cdots, a_1 x_{n1}) = (0, 0, 0, \cdots, 0)$, then from the equality of first components, we get $a_1 u_1 = 0$. Since u_1 is a unit, we obtain $a_1 = 0$. Thus the vector α_1 is linearly independent.

Let $k = 2$ so we deal with

$$\alpha_1 = (u_1, x_{21}, x_{31}, \cdots, x_{n1}) \text{ and } \alpha_2 = (x_{12}, u_2, x_{32}, \cdots, x_{n2}).$$

Then for $a_1, a_2 \in \mathbf{A}$, let

$$\begin{aligned} a_1 \alpha_1 + a_2 \alpha_2 &= 0 \Rightarrow a_1(u_1, x_{21}, x_{31}, \cdots, x_{n1}) + a_2(x_{12}, u_2, x_{32}, \cdots, x_{n2}) \\ &= (0, 0, 0, \cdots, 0). \end{aligned}$$

From the equality of first two components, we get $a_1 u_1 + a_2 x_{12} = 0$, $a_1 x_{21} + a_2 u_2 = 0$. If we compose these equations, we obtain

$$a_1 = -a_2 x_{12} u_1^{-1} \tag{1}$$

and

$$a_2(u_2 - x_{12} u_1^{-1} x_{21}) = 0 \tag{2}$$

Since $u_2 \in \mathbf{A} \setminus J$ and $x_{12} u_1^{-1} x_{21} \in J$, we have $u_2 - x_{12} u_1^{-1} x_{21} \in \mathbf{A} \setminus J$, so we find $a_2 = 0$ from the equation (2). If we put $a_2 = 0$ in equation (1), we get $a_1 = 0$. Thus the vectors α_1 and α_2 are linearly independent.

Finally to understand the process, let $k = 3$ so we have deal with $a_1 = (u_1, x_{21}, x_{31}, \cdots, x_{n1})$, $\alpha_2 = (x_{12}, u_2, x_{32}, \cdots, x_{n2})$, $\alpha_3 = (x_{13}, x_{23}, u_3, \cdots, x_{n3}) \in M$. Let $a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 0$. From the equality of first three components, we obtain the following equations:

$$\begin{aligned} a_1 u_1 + a_2 x_{12} + a_3 x_{13} &= 0 & (3) \\ a_1 x_{21} + a_2 u_2 + a_3 x_{23} &= 0 & (4) \\ a_1 x_{31} + a_2 x_{32} + a_3 u_3 &= 0. & (5) \end{aligned}$$

We get

$$a_1 = -(a_2 x_{12} + a_3 x_{13})u_1^{-1} \tag{6}$$

from equation (3). Then if we put a_1 in equation (4), we obtain $a_2(-x_{12}u_1^{-1}x_{21}+ u_2) + a_3(-x_{13}u_1^{-1}x_{21} + x_{23}) = 0$.

Since $x_{12}u_1^{-1}x_{21} \in J$ and $u_2 \in \boldsymbol{A} \setminus J$, we get $x_{12}u_1^{-1}x_{21} + u_2 \in \boldsymbol{A} \setminus J$ from which we obtain

$$a_2 = a_3(x_{13}u_1^{-1}x_{21} - x_{23})(-x_{12}u_1^{-1}x_{21} + u_2)^{-1}. \tag{7}$$

Now if we put a_1 and a_2 in equation (5) and compose the equations we get

$$a_3 \left[\begin{array}{c} (x_{23} - x_{13}u_1^{-1}x_{21})(u_2 - x_{12}u_1^{-1}x_{21})x_{12}u_1^{-1}x_{31} - x_{13}u_1^{-1}x_{31} \\ + (x_{13}u_1^{-1}x_{21} - x_{23})(u_2 - x_{12}u_1^{-1}x_{21})^{-1}x_{32} + u_3 \end{array} \right] = 0.$$

Since $(x_{23} - x_{13}u_1^{-1}x_{21})(u_2 - x_{12}u_1^{-1}x_{21})x_{12}u_1^{-1}x_{31} - x_{13}u_1^{-1}x_{31} + (x_{13}u_1^{-1}x_{21} - x_{23})(u_2 - x_{12}u_1^{-1}x_{21})^{-1}x_{32} + u_3 \in \boldsymbol{A} \setminus J$, we obtain $a_3 = 0$. By using $a_3 = 0$ in equations (6) and (7), we get $a_2 = 0$ and $a_1 = 0$. Thus the vectors α_1, α_2 and α_3 are linearly independent.

For a positive integer k, we have

$$\alpha_1 = (u_1, x_{21}, x_{31}, \cdots, x_{n1}), \alpha_2 = (x_{12}, u_2, x_{32}, \cdots, x_{n2}),$$
$$\alpha_3 = (x_{13}, x_{23}, u_3, \cdots, x_{n3}), \cdots, \alpha_k = (x_{1k}, x_{2k}, \cdots, u_k, \cdots, x_{nk}).$$

Let $\sum_{i=1}^{k} a_i \alpha_i = 0$. Then from the equality of first k components to zero, we get exactly k equations. By similar calculations, we obtain all a_i's to be zero. Thus the set $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ is linearly independent. $\square$

Finally we express Remark 2 from [3] as a proposition and we give detailed proof of this proposition.

Proposition 10. *An $\boldsymbol{A}-$module M over a local ring $\boldsymbol{A}$ is an $\boldsymbol{A}-$space if and only if it is a free finitely dimensional module.*

Proof. Let M be a free module over $\boldsymbol{A}$ with a basis $\{E_1, E_2, \cdots, E_n\}$. Now we will show that the $\boldsymbol{A}-$module M is an $\boldsymbol{A}-$space. For every $\beta \in M$, β can be written as

$$\beta = b_1 E_1 + b_2 E_2 + \cdots + b_n E_n \in \boldsymbol{A}E_1 + \boldsymbol{A}E_2 + \cdots + \boldsymbol{A}E_n.$$

Let $\beta \in \sum_{i \neq j} \boldsymbol{A}E_i \cap \boldsymbol{A}E_j$. Then we can write

$$\beta = a_1 E_1 + a_2 E_2 + \cdots + a_{j-1}E_{j-1} + a_{j+1}E_{j+1} + \cdots + a_n E_n = a_j E_j.$$

So we get $a_1E_1 + a_2E_2 + \cdots + a_{j-1}E_{j-1} - a_jE_j + a_{j+1}E_{j+1} + \cdots + a_nE_n = 0$. Here for $1 \leq i \leq n$, all a_i's must be zero because of the linearly independence of the vectors $E_1, E_2, \cdots, E_n$. Thus β must be zero. Then since the following conditions are satisfied:

$$
\begin{aligned}
&\text{i)} \quad M = \boldsymbol{A}E_1 + \boldsymbol{A}E_2 + \cdots + \boldsymbol{A}E_n \\
&\text{ii)} \quad \sum_{i \neq j} \boldsymbol{A}E_i \cap \boldsymbol{A}E_j = \{0\}.
\end{aligned}
$$

We can say $M = \boldsymbol{A}E_1 \oplus \boldsymbol{A}E_2 \oplus \cdots \oplus \boldsymbol{A}E_n$. Now we show that for $1 \leq i \leq n$, all functions $f_i : \boldsymbol{A} \to \boldsymbol{A}E_i$ are isomorphisms. For every $X, Y \in \boldsymbol{A}$,

$$f_i(X + Y) = (X + Y)E_i = XE_i + YE_i = f_i(X) + f_i(Y)$$

and also for every $X \in \boldsymbol{A}$ and for every $\lambda \in \mathbb{R}$,

$$f_i(\lambda X) = (\lambda X)E_i = \lambda(XE_i) = \lambda f_i(X).$$

So each f_i preserves the operations. Let $f_i(X) = f_i(Y)$ for some $X, Y \in \boldsymbol{A}$. So we get $XE_i = YE_i$ and hence $(X-Y)E_i = 0$. By using the linear independence of the vectors E_i for $1 \leq i \leq n$,

$$X - Y = 0 \quad \text{and hence} \quad X = Y$$

is obtained. Thus for $1 \leq i \leq n$, every f_i is one-to-one. For every $XE_i \in \boldsymbol{A}E_i$, there is obviously an $X \in \boldsymbol{A}$ such that $f(X) = XE_i$. Thus for $1 \leq i \leq n$, every f_i is onto.

Conversely let M be a finite dimensional $\boldsymbol{A}$−space over the local ring $\boldsymbol{A}$. Now we show that M is a finite dimensional free module. Since M is an $\boldsymbol{A}$−space, there are the vectors $E_1, E_2, \cdots, E_n$, and they form a finite basis of M. So M is a finite dimensional free module. $\square$

References

[1]   B. R. McDonald, *Geometric algebra over local rings*, Marcel Dekker, New York, 1976.

[2]   M. Jukl, *Linear forms on free modules over certain local ring*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 32 (1993), 49-62.

[3] M. Jukl, *Grassmann formula for certain type of modules*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 34 (1995), 69-74.

[4] T. W. Hungerford, *Algebra*, Holt, Rinehart and Winston, New York, 1974.

[5] K. Nomizu, *Fundamentals of Linear Algebra*, McGraw-Hill, New York, 1966.

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