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Construction of composition (m, n, k)-hyperrings

B. Davvaz, N. Rakhsh-Khorshid and K. P. Shum

Abstract

In this paper, our aim is to introduce the notion of a composition (m, n, k)-hyperring and to analyze its properties. We also consider the algebraic structure of (m, n, k) hyperrings which is a generalization of composition rings and composition hyperrings. Also, the isomorphism theorems of ring theory are derived in the context of composition (m, n, k)-hyperrings.

1 Introduction

We first consider several definitions for a hyperring by replacing at least one of the two operations by hyperoperations. A well known type of a hyperring which is called the Krasner hyperring [8] is obtained by considering the addition as a hyperoperation such that the structure $(R, +)$ is a canonical hypergroup. A comprehensive review of the theory of hyperrings appears in [4]. Based on the notion of a composition ring introduced by Adler [1] in [2]. Crista and Jančić-Rašović defined the concept of a composition hyperring as a quadruple $(R, +, \cdot, \circ)$ such that $(R, +, \cdot)$ is a commutative hyperring in the general sense and the composition hyperoperation $\circ$ is an associative hyperoperation which is distributive to the right side with respect to the addition and multiplication.

A suitable generalization of a hypergroup which is called an n-hypergroup was introduced and studied by Davvaz and Vougiouklis [6]. In [5], Davvaz et al. further considered a class of algebraic hypersystems which represent

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a generalization of semigroups, hypersemiroups and n-semigroups. Then, Leoreanu-Fotea in [9] continued to study the canonical n-hypergroups. Recently, the Krasner (m, n)-hyperrings are introduced and analyzed by Mirvakili and Davvaz [11]. Iin fact, the Krasner (m, n)-hyperrings are suitable generalizations of the Krasner hyperrings. The notion of (m, n)-ary hyperring in the general form was introduced in [3, 10], as the strong distributive structure. Then, in [7], Jančić-Rašović and Dašić further generalized such structure by introducing the notion of (m, n)-hyperring with the inclusive distributivity.

In this paper, we introduce the notion of composition (m, n, k)-hyperring as a generalization of the composition rings and composition hyperrings.

2 n-hypergroups and (m, n)-hyperrings

Let H be a non-empty set and f be a mapping from H^n to $\mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. Then, f is called an *n-hyperoperation*. If f is an n-hyperoperation defined on H, then (H, f) is called an *n-hypergroupoid*. If for all $x_1, \ldots, x_n \in H$ the set $f(x_1, \ldots, x_n)$ is a singleton, then f is called an *n-operation* We now calll (H, f) is called an *n-groupoid*. The sequence $x_i, \ldots, x_j$ will be denoted by x_i^j. For non-empty subsets $A_1, \ldots, A_n$ of H. Now, we define $f(A_1, \ldots, A_n) = f(A_1^n) = \cup\{f(x_1^n) \mid x_i \in A_i, i = 1, \ldots, n\}$. The n-hyperoperation f is said to be *associative* if $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$ holds for every $i, j \in \{1, \ldots, n\}$ and all $x_1^{2n-1} \in H$. An n-hypergroupoid with the associative hyperoperation is called an *n-semihypergroup*. An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$ is called an *n-quasihypergroup*. An n-semihypergroup which is an n-quasihypergroup, is called an *n-hypergroup*. An n-hypergroupoid (H, f) is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$ we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. An element e of H is called an *identity element* if $x \in f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$ for all $x \in H$ and all $1 \leq i \leq n$. An element 0 of an n-semihypergroup (H, f) is called a *zero element* if for every $x_2^n \in H$ we have $f(0, x_2^n) = f(x_2, 0, x_3^n) = \cdots = f(x_2^n, 0) = 0$. A commutative n-hypergroup (H, f) is called an *n-canonical hypergroup* if the following three conditions are satisfied:

(1) there exists a unique $e \in H$ such that for each $x \in H$, $f(x, e^{(n-1)}) = x$,

(2) for all $x \in H$ there exists a unique $x^{-1} \in H$ such that $e \in f(x, x^{-1}, e^{(n-2)})$,

(3) if $x \in f(x_1^n)$, then for all $1 \leq i \leq n$, we have

$$x_i \in f(x, x_1^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1}).$$

Now, we recall the definition of (m, n)-hyperring.

Definition 2.1. [10] An (m, n)-*hyperring* is a hyperstructure (R, f, g), which satisfies the following axioms: (1) (R, f) is an m-hypergroup, (2) (R, g) is an n-semihypergroup, (3) the n-hyperoperation g is distributive with respect to the m-hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$ and $1 \leq i \leq n$, $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n))$. If the (m, n)-hyperring R is commutative with respect to both m-hyperoperation f and n-hyperoperation g, then it is called a *commutative* (m, n)-hyperring. A non-empty subset $S \subseteq R$ is called an (m, n)-*subhyperring* of R if (S, f, g) is an (m, n)-hyperring. An element 0 is called a *zero element* of (R, f, g) if it is an identity of (R, f) and for every $x_2^m \in R$, we have $f(0, x_2^m) = f(x_2, 0, x_3^m) = \cdots = f(x_2^m, 0) = 0$.

Definition 2.2. [11] A *Krasner* (m, n)-*hyperring* is a hyperstructure (R, f, g) which satisfies the following axioms: (1) (R, f) is a canonical m-hypergroup, (2)(R, g) is an n-semigroup,(3) the n-operation g is distributive with respect to the m-hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$ and $1 \leq i \leq n$, we have

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)),$$

(4) 0 is a zero element of the n-operation g, i.e., for every $x_2^n \in R$ we have $g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0) = 0$.

A non-empty subset I of a Krasner (m, n)-hyperring R is called an (m, n)-*hyperideal* if (1) $e \in I$, (2) for every $x \in I$, $-x \in I$, (3) for every $a_1^m \in I$, $f(a_1^m) \subseteq I$, (4) for every $x_1^n \in R$ and $1 \leq i \leq n$, $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$.

Lemma 2.3. *[11] Let (R, f, g) be a Krasner (m, n)-hyperring. Then, the following statements hold.*

(1) For every $x \in R$, we have $-(-x) = x$ and $-0 = 0$.

(2) For every $x \in R$, $0 \in f(x, -x, \overset{(m-2)}{0})$.

(3) For every $x_1^m \in R$, $-f(x_1, \ldots, x_m) = f(-x_1, \ldots, -x_m)$, where $-A = \{-a \mid a \in A\}$.

Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two (m, n)-hyperrings. Then, we define a *homomorphism* from R_1 to R_2 be a mapping $\phi : R_1 \to R_2$ such that $\phi(f_1(a_1^m)) = f_2(\phi(a_1), \ldots, \phi(a_m))$ and $\phi(g_1(b_1^n)) = g_2(\phi(b_1), \ldots, \phi(b_n))$ hold, for all $a_1^m, b_1^n \in R_1$. The map ϕ is an *isomorphism* if it is one to one and onto too. In this case, we say R_1 is isomorphic to R_2 and we denote $R_1 \cong R_2$. The *kernel* of ϕ is defined by $ker(\phi) = \{(a, b) \in R_1 \times R_1 \mid \phi(a) = \phi(b)\}$. If ϕ is a homomorphism of Krasner (m, n)-hyperrings, then the kernel of ϕ is as following $ker(\phi) = \{x \in R_1 \mid \phi(x) = 0_{R_2}\}$.

3 Composition (m, n, k)-hyperrings

In this section, we present the notion of a composition (m, n, k)-hyperring which is a generalization of the composition hyperring introduced by Crista and Jančić-Rašović[2]. Some examples of this new hyperstructure are found and will be expressed.

Definition 3.1. A *composition* (m, n, k)*-hyperring* is an algebraic composition hyperstructure (R, f, g, h), where (R, f, g) is a commutative (m, n)-hyperring and k-hyperoperation h (called *composition*) satisfies the following properties:

(1) h is the right distributive with respect to f;

(2) h is the right distributive with respect to g,

(3) h is associative.

An element 0 of composition (m, n, k)-hyperring R is called a *zero element* if it is a zero element of (m, n)-hyperring R and $g(a_1^{i-1}, 0, a_{i+1}^n) = 0$, for every $a_1^{i-1}, a_{i+1}^n \in R$ and $1 \leq i \leq n$. An element $c \in R$ is called a *constant* if $h(a_1^{i-1}, c, a_{i+1}^k) = c$, holds for all $a_1^{i-1}, a_{i+1}^k \in R$ and $1 \leq i \leq k$. If A is an arbitrary subset of R, then, the set of all constants in A is called a *foundation* of A, denoted by $Found(A)$.

EXAMPLE 1. Every composition hyperring is an composition $(2, 2, 2)$-hyperring.

One can see several examples of composition hyperrings in [2].

EXAMPLE 2. Let $(R, +, \cdot, \circ)$ be a composition hyperring. If we define $f(x_1^m) = x_1 + \ldots + x_m$, $g(x_1^n) = x_1 \cdot \ldots \cdot x_n$ and $h(x_1^k) = x_1 \circ \ldots \circ x_k$. Then (R, f, g, h) is a composition (m, n, k)-hyperring.

EXAMPLE 3. Let (R, f, g) be a commutative (m, n)-hyperring. If we define the k-hyperoperation h by $h(x_1^k) = 0$, for all $x_1^k \in R$, then (R, f, g, h) is a composition (m, n, k)-hyperring. In this case, we shall call R a *null composition* (m, n, k)*-hyperring*.

Throughout the rest of the paper, (R, f, g, h) is always a composition (m, n, k)-hyperring such that (R, f, g) be a Krasner (m, n)-hyperring.

Definition 3.2. Let R be a composition (m, n, k)-hyperring and N be a non-empty subset of R. Then, we call N a *composition* (m, n, k)*-hyperideal* of R if the following conditions are satisfied:

(1) N is an (m, n)-hyperideal of Krasner (m, n)-hyperring R,

(2) $h(r_1^{i-1}, n, r_{i+1}^k) \subseteq N$, for all $n \in N$, $r_1^{i-1}, r_{i+1}^k \in R$ and $1 \leq i \leq k$,

(3) if $f(r_1^{i-1}, -r_i^m) \cap N \neq \emptyset$, then

$$f(h(t_1^{j-1}, r_1, t_{j+1}^k), \ldots, h(t_1^{j-1}, r_{i-1}, t_{j+1}^k), h(t_1^{j-1}, -r_i, t_{j+1}^k), \\ \ldots, h(t_1^{j-1}, -r_m, t_{j+1}^k))$$

is a subset of N.

Let N be a composition (m,n,k)-hyperideal of R. Define on R the following relation:

$$x\ N^*\ y \Leftrightarrow f(x, N, \overset{(m-2)}{0}) = f(y, N, \overset{(m-2)}{0}),$$

for all $x, y \in R$. Clearly, N^* is an equivalence relation on R. Consider $x \in R$. The equivalence class of x is defined by $N^*[x] = \{y \in R \mid y\ N^*\ x\}$. Then, we have $N^*[x] = f(x, N, \overset{(m-2)}{0})$. The set of all equivalence classes of the elements of R with respect to the equivalence relation N^* is denoted by $[R : N]$ and it defined as follows: $[R : N] = \{N^*[x] \mid x \in R\}$.

Proposition 3.3. *Let R be a composition (m,n,k)-hyperring and N be a composition (m,n,k)-hyperideal of R. Then, we consider F, G and H as it follows:*

$$F(f(x_1, N, \overset{(m-2)}{0}), \ldots, f(x_m, N, \overset{(m-2)}{0})) = \{f(z, N, \overset{(m-2)}{0}) \mid z \in f(x_1, \ldots, x_m)\},$$
$$G(f(x_1, N, \overset{(m-2)}{0}), \ldots, f(x_n, N, \overset{(m-2)}{0})) = \{f(z, N, \overset{(m-2)}{0}) \mid z \in g(x_1, \ldots, x_n)\},$$
$$H(f(x_1, N, \overset{(m-2)}{0}), \ldots, f(x_k, N, \overset{(m-2)}{0})) = \{f(z, N, \overset{(m-2)}{0}) \mid z \in h(x_1, \ldots, x_k)\}.$$

Then, the hyperoperations F, G and H are well defined.

Proof. The proof is is straightforward and is hence omited. □

Theorem 3.4. *$([R : N], F, G, H)$ is a composition (m,n,k)-hyperring.*

Proof. The proof is straightforward. □

The above hyperstructure is called the *quotient composition (m,n,k)-hyperring* related to the equivalence relation N^*.

Definition 3.5. Let (R_1, f_1, g_1, h_1) and (R_2, f_2, g_2, h_2) be two composition (m,n,k)-hyperrings. A mapping $\phi : R_1 \to R_2$ is called a *strong homomorphism* if the following conditions are satisfied, for all $x_1^m, y_1^n, z_1^k \in R_1$:

(1) $\phi(f_1(x_1, \ldots, x_m)) = f_2(\phi(x_1), \ldots, \phi(x_m))$,

(2) $\phi(g_1(y_1, \dots, y_n)) = g_2(\phi(y_1), \dots, \phi(y_n))$,

(3) $\phi(h_1(z_1, \dots, z_k)) = h_2(\phi(z_1), \dots, \phi(z_k))$,

(4) $\phi(0_{R_1}) = 0_{R_2}$.

A strong homomorphism ϕ is called an *isomorphism* if ϕ is one to one and onto. We write $R_1 \cong R_2$ if R_1 is isomorphic with R_2.

Proposition 3.6. *If $\phi : R_1 \to R_2$ is a strong homomorphism, then for all $x \in R_1$, it holds $\phi(-x) = -\phi(x)$.*

Proof. Since $0 \in f(x, -x, \overset{(m-2)}{0})$, it follows that $\phi(0) \in \phi(f(x, -x, \overset{(m-2)}{0}))$. We conclude that

$$\begin{aligned} 0 &= \phi(0) \in \phi(f(x, -x, \overset{(m-2)}{0})) = f(\phi(x), \phi(-x), \overset{(m-2)}{\phi}(0)) \\ &= f(\phi(x), \phi(-x), \overset{(m-2)}{0}) \end{aligned}$$

Hence, $0 \in f(\phi(x), \phi(-x), \overset{(m-2)}{0})$ and so $\phi(-x) = -\phi(x)$. □

The kernel of ϕ is defined by $ker(\phi) = \{x \in R_1 \mid \phi(x) = 0_{R_2}\}$.

Proposition 3.7. *Let $\phi : R_1 \to R_2$ be a strong homomorphism of(m, n)-hyperrings. Then, $ker(\phi)$ is a hyperideal of R_1.*

Proof. Set $K := ker(\phi)$. (1) $\phi(0) = 0$. Thus, $0 \in K$. (2) Let $x \in K$ be an arbitrary element. Then, $\phi(x) = 0$ and by Proposition 3.6, we have $\phi(-x) = -\phi(x)$. It follows that $\phi(-x) = -0 = 0$. So, $-x \in K$. (3) Suppose that $a_1^m \in K$. We have $\phi(a_1) = \phi(a_2) = \cdots = \phi(a_m) = 0$. Consider $x \in f(a_1^m)$. Then,

$$\phi(x) \in \phi(f(a_1^m)) = f(\phi(a_1), \dots, \phi(a_m)) = f(\overset{(m)}{0}) = 0 \Rightarrow x \in K \Rightarrow f(a_1^m) \subseteq K.$$

(4) Let $x_1^{i-1}, x_{i+1}^n \in R_1$ and $1 \leq i \leq n$. Consider $y \in g(x_1^{i-1}, K, x_{i+1}^n)$. Then, there exists $k \in K$ such that $y \in g(x_1^{i-1}, k, x_{i+1}^n)$. Thus,

$$\begin{aligned} \phi(y) \in \phi(g(x_1^{i-1}, k, x_{i+1}^n)) &= g(\phi(x_1), \dots, \phi(x_{i-1}), \phi(k), \phi(x_{i+1}), \dots, \phi(x_n)) \\ &= g(\phi(x_1), \dots, \phi(x_{i-1}), 0, \phi(x_{i+1}), \dots, \phi(x_n)) = 0. \end{aligned}$$

It follows that $y \in K$ and so $g(x_1^{i-1}, k, x_{i+1}^n) \subseteq K$. □

Notice that, in generally, $ker(\phi)$ is not a composition (m, n, k)-hyperideal.

In the following, we will state and prove the isomorphism theorems for composition (m, n, k)-hyperrings.

Theorem 3.8. *Let (R_1, f, g, h) and (R_2, f, g, h) be two composition (m, n, k)-hyperrings. If $\phi : R_1 \to R_2$ is a strong homomorphism with the kernel K such that K is composition(m, n, k)-hyperideal of R_1, then $[R_1 : K] \cong Im(\phi)$.*

Proof. We define $\Psi : [R_1 : K] \to Im(\phi)$ by $\Psi(f(x, K, \overset{(m-2)}{0})) = \phi(x)$, for all $x \in R_1$. First, we prove that Ψ is well defined. Suppose that $f(x, K, \overset{(m-2)}{0}) = f(y, K, \overset{(m-2)}{0})$. It is obvious that $x \in f(x, K, \overset{(m-2)}{0})$. Thus, $x \in f(y, K, \overset{(m-2)}{0})$. Hence, there exists $k^{'} \in K$ such that $x \in f(y, k^{'}, \overset{(m-2)}{0})$. It follows that

$$\begin{aligned} k^{'} \in f(x, -y, \overset{(m-2)}{0}) \quad &\Rightarrow \phi(k^{'}) \in \phi(f(x, -y, \overset{(m-2)}{0})) \\ &\Rightarrow \phi(k^{'}) \in f(\phi(x), \phi(-y), \overset{(m-2)}{0}) \\ &\Rightarrow 0 \in f(\phi(x), -\phi(y), \overset{(m-2)}{0}) \\ &\Rightarrow \phi(x) = \phi(y). \end{aligned}$$

Obviously, Ψ is onto. Now, we show that Ψ is one to one. Suppose that $\phi(x) = \phi(y)$. Then, we have

$$0 \in f(\phi(x), -\phi(y), \overset{(m-2)}{0}) = \phi(f(x, -y, \overset{(m-2)}{0})).$$

Thus, there exists $z \in f(x, -y, \overset{(m-2)}{0})$ such that $\phi(z) = 0$. So, $z \in K$. Therefore,

$$\begin{aligned} f(x, K, \overset{(m-2)}{0}) &\subseteq f(f(z, y, \overset{(m-2)}{0}), K, \overset{(m-2)}{0}) = f(y, K, \overset{(m-2)}{0}), \\ f(y, K, \overset{(m-2)}{0}) &\subseteq f(f(x, z, \overset{(m-2)}{0}), K, \overset{(m-2)}{0}) = f(x, K, \overset{(m-2)}{0}). \end{aligned}$$

It follows that $f(x, K, \overset{(m-2)}{0}) = f(y, K, \overset{(m-2)}{0})$. Moreover, Ψ is a strong homomorphism, because

$$\begin{aligned} &\Psi(F(f(x_1, K, \overset{(m-2)}{0}), \ldots, f(x_m, K, \overset{(m-2)}{0}))) \\ &= \Psi(\{f(z, K, \overset{(m-2)}{0}) \mid z \in f(x_1, \ldots, x_m)\}) \\ &= \{\Psi(f(z, K, \overset{(m-2)}{0})) \mid z \in f(x_1, \ldots, x_m)\} \\ &= \{\phi(z) \mid z \in f(x_1, \ldots, x_m)\} \\ &= \phi(f(x_1, \ldots, x_m)) = f(\phi(x_1), \ldots, \phi(x_m)) \\ &= f(\Psi(f(x_1, K, \overset{(m-2)}{0})), \ldots, \Psi(f(x_m, K, \overset{(m-2)}{0}))), \end{aligned}$$

$$\begin{aligned}
&\Psi(G(f(x_1, K, \overset{(m-2)}{0}), \ldots, f(x_n, K, \overset{(m-2)}{0})))\\
&= \Psi(\{f(z, K, \overset{(m-2)}{0}) \mid z \in g(x_1, \ldots, x_n)\})\\
&= \{\Psi(f(z, K, \overset{(m-2)}{0})) \mid z \in g(x_1, \ldots, x_n)\}\\
&= \{\phi(z) \mid z \in g(x_1, \ldots, x_n)\}\\
&= \phi(g(x_1, \ldots, x_n)) = g(\phi(x_1), \ldots, \phi(x_n))\\
&= g(\Psi(f(x_1, K, \overset{(m-2)}{0})), \ldots, \Psi(f(x_n, K, \overset{(m-2)}{0}))),
\end{aligned}$$

$$\begin{aligned}
&\Psi(H(f(x_1, K, \overset{(m-2)}{0}), \ldots, f(x_k, K, \overset{(m-2)}{0})))\\
&= \Psi(\{f(z, K, \overset{(m-2)}{0}) \mid z \in h(x_1, \ldots, x_k)\})\\
&= \{\Psi(f(z, K, \overset{(m-2)}{0})) \mid z \in h(x_1, \ldots, x_k)\}\\
&= \{\phi(z) \mid z \in h(x_1, \ldots, x_k)\}\\
&= \phi(h(x_1, \ldots, x_k)) = h(\phi(x_1), \ldots, \phi(x_k))\\
&= h(\Psi(f(x_1, K, \overset{(m-2)}{0})), \ldots, \Psi(f(x_k, K, \overset{(m-2)}{0}))),
\end{aligned}$$

and $\Psi(0_{[R_1:K]}) = \Psi(f(0, K, \overset{(m-2)}{0})) = \phi(0_{R_1}) = 0_{R_2}$. Hence, it is clear that Ψ is isomorphism, i.e., $[R_1 : K] \cong Im(\phi)$. □

Theorem 3.9. *If $I_1, \ldots, I_m$ are composition (m, n, k)-hyperideals of a composition (m, n, k)-hyperring R and $1 \leq j \leq m$, then*

$$[f(I_1^{J-1}, 0, I_{j+1}^m) : f(I_1^{J-1}, 0, I_{j+1}^m) \cap I_j] \cong [f(I_1^m) : I_j].$$

Proof. For all $1 \leq j \leq m$, I_j is a composition (m, n, k)-hyperideal of $f(I_1^m)$ and so $[f(I_1^m) : I_j]$ is defined. Let us take $\Psi : f(I_1^{j-1}, 0, I_{j+1}^m) \to [f(I_1^m) : I_j]$ by $\Psi(a) = f(a, I_j, \overset{(m-2)}{0})$. It is easy to verify that Ψ is well-defined. We prove that Ψ is a strong homomorphism.

$$\begin{aligned}
\Psi(f(a_1, \ldots, a_m)) &= \bigcup_{v \in f(a_1^m)} \Psi(v)\\
&= \bigcup_{v \in f(a_1^m)} f(v, I_j, \overset{(m-2)}{0})\\
&= f(f(a_1, \ldots, a_m), I_j, \overset{(m-2)}{0})\\
&= f(f(a_1, I_j, \overset{(m-2)}{0}), \ldots, f(a_m, I_j, \overset{(m-2)}{0}))\\
&= f(\Psi(a_1), \ldots, \Psi(a_m)),
\end{aligned}$$

$$\begin{aligned}
\Psi(g(a_1,\dots,a_n)) &= \bigcup_{v\in g(a_1^n)} \Psi(v)\\
&= \bigcup_{v\in g(a_1^n)} f(v,I_j,\overset{(m-2)}{0}) = f(g(a_1,\dots,a_n),I_j,\overset{(m-2)}{0})\\
&= g(f(a_1,I_j,\overset{(m-2)}{0}),\dots,f(a_n,I_j,\overset{(m-2)}{0}))\\
&= g(\Psi(a_1),\dots,\Psi(a_n)),
\end{aligned}$$

$$\begin{aligned}
\Psi(h(a_1,\dots,a_k)) &= \bigcup_{v\in h(a_1^k)} \Psi(v)\\
&= \bigcup_{v\in h(a_1^k)} f(v,I_j,\overset{(m-2)}{0})\\
&= f(h(a_1,\dots,a_k),I_j,\overset{(m-2)}{0})\\
&= h(f(a_1,I_j,\overset{(m-2)}{0}),\dots,f(a_k,I_j,\overset{(m-2)}{0}))\\
&= h(\Psi(a_1),\dots,\Psi(a_k)),
\end{aligned}$$

$$\begin{aligned}
\Psi(0_{f(I_1^{j-1},0,I_{j+1}^m)}) &= f(0_{f(I_1^{j-1},0,I_{j+1}^m)},I_j,\overset{(m-2)}{0})\\
&= f(f(\overset{m}{0}),I_j,\overset{(m-2)}{0}) = I_j = 0_{[f(I_1^m):I_j]}.
\end{aligned}$$

Obviously, Ψ is onto. Suppose that $a\in ker(\Psi)$. Hence, we have

$$a\in Ker(\Psi)\Leftrightarrow \Psi(a)=I_j \Leftrightarrow f(a,I_j,\overset{(m-2)}{0})=I_j \Leftrightarrow a\in I_j\cap f(I_1^{j-1},0,I_{j+1}^m).$$

by Theorem 3.8, we get the isomorphism

$$[f(I_1^{J-1},0,I_{j+1}^m):f(I_1^{J-1},0,I_{j+1}^m)\cap I_j]\cong[f(I_1^m):I_j] \qquad \square$$

Theorem 3.10. *If A and B are composition (m,n,k)-hyperideals of R such that $A\subseteq B$, then $[B:A]$ is a composition (m,n,k)-hyperideal of $[R:A]$ and $[[R:A]:[B:A]]\cong[R:B]$.*

Proof. First, we prove that $[B:A]$ is a composition (m,n,k)-hyperideal of $[R:A]$. We define $\phi:[R:A]\to[R:B]$ by $\phi(f(r,A,\overset{(m-2)}{0}))=f(r,B,\overset{(m-2)}{0})$.

Obviously, ϕ is well-defined. Moreover, ϕ is a strong homomorphism, because

$$\begin{aligned}
&\phi(F(f(r_1, A, \overset{(m-2)}{0}), \ldots, f(r_m, A, \overset{(m-2)}{0})))\\
&= \phi(\{f(z, A, \overset{(m-2)}{0}) \mid z \in f(r_1, \ldots, r_m)\})\\
&= \{\phi(f(z, A, \overset{(m-2)}{0})) \mid z \in f(r_1, \ldots, r_m)\}\\
&= \{f(z, B, \overset{(m-2)}{0}) \mid z \in f(r_1, \ldots, r_m)\}\\
&= F(f(r_1, B, \overset{(m-2)}{0}), \ldots, f(r_m, B, \overset{(m-2)}{0}))\\
&= F(\phi(f(r_1, A, \overset{(m-2)}{0})), \ldots, \phi(f(r_m, A, \overset{(m-2)}{0}))),
\end{aligned}$$

$$\begin{aligned}
&\phi(G(f(r_1, A, \overset{(m-2)}{0}), \ldots, f(r_n, A, \overset{(m-2)}{0})))\\
&= \phi(\{f(z, A, \overset{(m-2)}{0}) \mid z \in g(r_1, \ldots, r_n)\})\\
&= \{\phi(f(z, A, \overset{(m-2)}{0})) \mid z \in g(r_1, \ldots, r_n)\}\\
&= \{f(z, B, \overset{(m-2)}{0}) \mid z \in g(r_1, \ldots, r_n)\}\\
&= G(f(r_1, B, \overset{(m-2)}{0}), \ldots, f(r_n, B, \overset{(m-2)}{0}))\\
&= G(\phi(f(r_1, A, \overset{(m-2)}{0})), \ldots, \phi(f(r_n, A, \overset{(m-2)}{0}))),
\end{aligned}$$

$$\begin{aligned}
&\phi(H(f(r_1, A, \overset{(m-2)}{0}), \ldots, f(r_k, A, \overset{(m-2)}{0})))\\
&= \phi(\{f(z, A, \overset{(m-2)}{0}) \mid z \in h(r_1, \ldots, r_k)\})\\
&= \{\phi(f(z, A, \overset{(m-2)}{0})) \mid z \in h(r_1, \ldots, r_k)\}\\
&= \{f(z, B, \overset{(m-2)}{0}) \mid z \in h(r_1, \ldots, r_k)\}\\
&= H(f(r_1, B, \overset{(m-2)}{0}), \ldots, f(r_n, B, \overset{(m-2)}{0}))\\
&= H(\phi(f(r_1, A, \overset{(m-2)}{0})), \ldots, \phi(f(r_k, A, \overset{(m-2)}{0}))).
\end{aligned}$$

$$\begin{aligned} ker(\phi) &= \{f(r, A, \overset{(m-2)}{0}) \in [R : A] \mid \phi(f(r, A, \overset{(m-2)}{0})) = 0_{[R:B]}\} \\ &= \{f(r, A, \overset{(m-2)}{0}) \in [R : A] \mid f(r, B, \overset{(m-2)}{0}) = B\} \\ &= \{f(r, A, \overset{(m-2)}{0}) \in [R : A] \mid r \in B\} = [B : A]. \end{aligned}$$

by Theorem 3.8, we conclude that $[[R : A] : [B : A]] \cong [R : B]$. □

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B. DAVVAZ,
Department of Mathematics,
Yazd University,
Yazd, Iran.
Email: davvaz@yazd.ac.ir, bdavvaz@yahoo.com

N. RAKHSH-KHORSHID,
Department of Mathematics,
Yazd University,
Yazd, Iran.
E-mail: rakhshkhorshid@stu.yazd.ac.ir

K. P. SHUM,
Institute of Mathematics,
Yunnan University
Kunming, 650091, PR. China
E-mail: kpshum@ynu.edu.cn