Hop Domination in Graphs-II

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subset V(G)$ is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set of G is called a hop domination number of G and is denoted by $\gamma_h(G)$. In this paper we characterize the family of trees and unicyclic graphs for which $\gamma_h(G) = \gamma_t(G)$ and $\gamma_h(G) = \gamma_c(G)$ where $\gamma_t(G)$ and $\gamma_c(G)$ are the total domination and connected domination numbers of G respectively. We then present the strong equality of hop domination and hop independent domination numbers for trees. Hop domination numbers of shadow graph and mycielskian graph of graph are also discussed.

1 Introduction

Domination in graphs is one of the fastest growing areas in Graph theory. Many authors contribute several interesting domination parameters to nurture the growth of this research area. An excellent treatment of several topics in domination can be found in two books [4, 5] written by Haynes et al. The following are some basic definitions and results to discuss further.

By an *ntc graph* G we mean a non trivial connected graph $G = (V, E)$ where V is the set of vertices and E is the set of edges of G. The *distance* between two vertices u and v of a graph G is the length of the shortest path joining u and v in G and is denoted by $d(u, v)$. A graph G is said to be *unicyclic* if it has exactly one cycle. A *double star* is a tree obtained by joining the centers of two stars $K_{1,r}$ and $K_{1,s}$. We denote a double star of order $r + s + 2$ by $D(r, s)$. A set $D \subset V$ is a *dominating set* of G if every vertex $v \in V - D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The upper domination number $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G. A set $D \subset V$ is a *total dominating set* of G if every vertex $v \in V$ is dominated by at least one vertex in D. The minimum cardinality of such a set is called the *total domination number* of G and is denoted by $\gamma_t(G)$. A dominating set D is a *connected dominating set* of G if the subgraph $< D >$ induced by D, is connected. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G which we denote by $\gamma_c(G)$. S.K. Ayyaswamy et al. [1] have recently defined a new domination parameter called *hop domination number* of a graph. The definition is as follows: A set $S \subset V$ of a graph G is a *hop dominating set*(hd-set, in short) of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hd-set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. We present the hop domination number of a few well known graphs in the following proposition.

Proposition 1. *([1])*

(i) For a complete graph K_n, $\gamma_h(K_n) = n$.

(ii)For a complete bipartite graph $K_{m,n}$, $\gamma_h(K_{m,n}) = 2$.

(iii) For a path P_n on n vertices $\gamma_h(P_n) = \begin{cases} 2r, & \text{if } n=6r; \\ 2r + 1, & \text{if } n=6r+1; \\ 2r + 2, & \text{if } n=6r+s \; ; 2 \leq s \leq 5. \end{cases}$

(iv) For a cycle C_n of length n, $\gamma_h(C_n) = \begin{cases} 2r, & \text{if } n=6r; \\ 2r + 1, & \text{if } n=6r+1; \\ 2r + 2, & \text{if } n=6r+s; 2 \leq s \leq 5. \end{cases}$

(v) $\gamma_h(W_n)=3$ where W_n is a wheel with $n - 1$ spokes.

(vi) $\gamma_h(P) = 2$ *where P denotes the Petersen graph.*

Proposition 2. *[1] For every traingle-free graph G without isolated vertices, $\gamma_h(G) \leq \gamma_t(G)$.*

2   Main Results

It is in the usual practice to study the relation between the new domination parameter and the existing one. Likewise we study the equality of hop domination number and other domination parameters such as connected domination number and total domination number.

2.1   Characterization of graphs with $\gamma_h = \gamma_t$

We now construct a family of $\mathcal{T}$ trees having $\gamma_h = \gamma_t$ in the following.
We define a family $\mathcal{T}$ of trees T that can be obtained form the disjoint union of $k \geq 1$ double stars. Let C be the set of central vertices of the double stars. Add $k - 1$ edges between the vertices of $V - C$ so that the resulting graph is a tree.
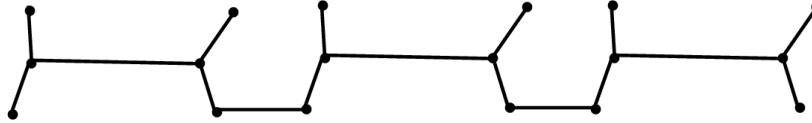

Figure 1: Tree T in $\mathcal{T}$

Theorem 3. *For any tree T, $\gamma_h(T) = \gamma_t(T)$ if and only if $T \in \mathcal{T}$.*

Proof. Assume that $\gamma_h(T) = \gamma_t(T)$.
Let D be a total dominating set. By Proposition 2 D is a hop dominating set as well. Choose a vertex a_1 which is support vertex of leaves only. This is possible since T is a tree. If $a_1 \in P_l \subset D$ where $P_l = < a_1, b_1, ..., x_l >$ is a path, then $l = 2$. Suppose $l \geq 3$. Then c_1 hop dominates a_1 and all those vertices hop dominated by a_1. Therefore, $D - \{a_1\}$ is a hop dominating set

and so $\gamma_h(T) \leq |D| - 1 < \gamma_t(T)$, a contradiction. Thus $P_l = < a_1, b_1 >$. Let $a_2 \in D$ be such that $d(a_2, b_1)$ is minimum. As D is a hop dominating set also, we have $d(a_2, b_1) = 2$ or 3. We claim that $d(a_2, b_1) = 3$. Suppose $b_1 v a_2$ is a path in T. Then there exists $b_2 \in N(a_2) \cap D$, as D is a total dominating set.
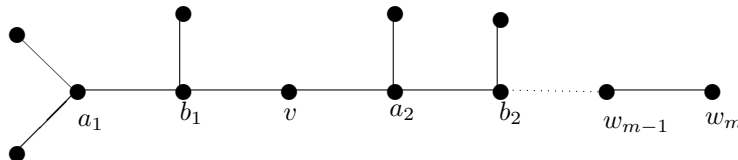

Figure 2:

If b_2 is a support vertex of leaves only, then $D - \{a_1, b_2\} \cup \{v\}$ is a hop dominating set since v hop dominates b_2 and all those vertices hop dominated by a_1 and b_2. This implies that $\gamma_h(T) < \gamma_t(T)$, a contradiction. If b_2 is not a support of leaves only, let $P_m = < b_2 = w_1, w_2, ..., w_m > \subset D$ be a path, then w_{m-1} is in D whereas w_{m-1} is not needed to hop dominate any vertex of T and hence $D - \{a_1, w_{m-1}\} \cup \{v\}$ is a hop dominating set and consequently $\gamma_h(T) < \gamma_t(T)$, a contradiction. Thus $d(b_1, a_2) = 3$. Now, let $a_2 \in P_s \subset D$ where $P_s = < a_2 = u_1, u_2, ..., u_s >$, $s \geq 3$, then as proved earlier we can show that $D - \{a_2\}$ is a hop dominating set which leads to a contradiction to our assumption that $\gamma_h(T) = \gamma_t(T)$. Proceeding like this, we get $D = \cup\{a_i, b_i\}$ and $d(b_i, a_{i+1}) = 3$. This shows that $T \in \mathcal{T}$.

$\square$

We define a family of unicyclic graphs $\mathcal{T}^*$ as follows:

$\mathcal{T}^* = \{T \cup e : T \in \mathcal{T}, e$ is an edge joining any two leaves or a leaf and an internal vertex which is not a support vertex in $T\}$.

Theorem 4. *Let G be a unicyclic graph with the unique cycle C. Then $\gamma_h(G) = \gamma_t(G)$ if and only if G is C_4 or $G \in \mathcal{T}^*$.*

Proof. Let $\gamma_h(G) = \gamma_t(G)$.
If G is just a cycle C, then $\gamma_h(C) = \gamma_t(C)$ if and only if $G = C_4$. So we assume

that G is not a mere cycle. Then G has at least one pendant vertex. Let D be a total dominating set of G and let $P = < a_0, a_1, ..., a_m >$ be a longest path in G. If $a_1 \in P_l \subset D$, then as discussed in Theorem 3, $l = 2$. That is, $P_l = < a_1, b_1 > \subset D$ and $(N(b_1) - \{a_1\}) \cap D = \emptyset$ and $(N(a_1) - \{b_1\}) \cap D = \emptyset$. Let $a_2, b_2 \in D$ be such that a_2 is nearest to b_1. Then $d(b_1, a_2) \leq 3$. We claim that $d(b_1, a_2) = 3$. Suppose $d(b_1, a_2) = 2$ and consider the path $b_1 v a_2$ in D.

Case 1: There is a path $P = \{b_2 = w_1, w_2, ..., w_m\}$ from b_2 such that $b_2 \in C$, $a_2 \notin P$ and $P \cap C = \{b_2\}$ only where C is the unique cycle in G. Then as discussed in Theorem 3, $w_{m-1} \in D$ whereas w_{m-1} is not in any hd-set so that $D - \{a_1, w_{m-1}\} \cup \{v\}$ is hd-set which contradicts our assumption that $\gamma_h(G) = \gamma_t(G)$.

Case 2: Let $b_2 \in C$ and there is no path from b_2. If the cycle C does not contain a_2, then clearly $D - \{a_1, a_2\} \cup \{v\}$ is a hd-set. So, assume that C contains a_2 and b_2. Let C be $< b_2, a_2, w_1, w_2, ..., w_{m-1}, w_m = b_2 >$. We shall assume that $deg(w_i) = 2$ for all $i = 1, 2, ..., m$; otherwise as discussed in Case 1 we can arrive at a contradiction. If $w_{m-2} \in D$, then w_{m-2} hop dominates b_2 and a_2 hop dominates the neighbours of b_2. So $D - \{a_1, b_2\} \cup \{v\}$ is a hd-set of G. If $w_{m-1} \in D$, then w_{m-1} hop dominates a_2 and v hop dominates all neighbours of a_2. So $D - \{a_1, a_2\} \cup \{v\}$ is a hd-set. If $w_{m-3} \in D$, then w_{m-3} hop dominates w_{m-1} and v hop dominates b_2. So $D - \{a_1, b_2\} \cup \{v\}$ is a hd-set. All these cases imply $\gamma_h(G) < \gamma_t(G)$. Thus $d(b_1, a_2) = 3$. If $a_2 b_2 c_2$ is a path in D, then $D - \{a_2\}$ is a hd-set since a_2 is hop dominated by c_2 and all neighbours of a_2 are hop dominated by b_2. Thus $a_2 \in P_l \subset D$ implies $l = 2$, that is, neither the neighbours of a_2 nor the neighbours of b_2 other than a_2 and b_2 are in D. Proceeding like this we get $D = \cup \{a_i, b_i\}$ such that a_i, b_i forms K_2 and $d(b_{i+1}, a_i) = 3$. Again, as $D = \cup \{a_i, b_i\}$, the vertices a_is and b_is are the only supports of leaves in G. This together with the fact that $d(b_i, a_{i+1}) = 3$ implies $G \in \mathcal{T}^*$. $\square$

2.2 Characterization of a family of graphs for which $\gamma_h(G) = \gamma_c(G)$

Theorem 5. *Let T be a tree of order $n > 3$. Then $\gamma_h(T) = \gamma_c(T)$ if and only if T is a double star.*

Proof. Assume that $\gamma_h(T) = \gamma_c(T)$.
Let l denote the number of leaves in T. We first show that every internal

vertex of T is a support vertex.

Suppose T has internal vertices which are support vertices. Choose one such vertex, say v such that $N(v)$ contains a support vertex u. Since T has no cycle, there exists $w \in N(v)$ such that $d(u,w) = 2$. Then v hop dominates all the leaves at u and w hop dominates u. Therefore we can choose a hop dominating set of T not containing u and its leaves so that $\gamma_h(G) \leq n-l-1 < \gamma_c(G)$, a contradiction. Hence every internal vertex in T is a support vertex.

We next prove that T has exactly two support vertices. Suppose T has more than two support vertices. Let $P : v_1, v_2, \cdots v_n$. Then v_1 and v_i are leaves and all other v_is are support vertices in T.

Clearly, v_2 is hop dominated by v_4 and the leaves of v_2 including v_1 are hop dominated by v_3. Hence we can find a hop dominating set S with $S \leq n-l-1 < n-l = \gamma_c$ $\square$

We construct a family $\mathcal{G}$ of unicyclic graphs as follows:

Operation U_1: Join any two non-adjacent vertices of a double star $D(r,s)$ where $r,s \geq 2$.

Operation U_2: If $r = 1$, then we have a double star $(K_{1,1}, K_{1,s})$. In this case we join any two non-adjacent vertices except v and u_1.
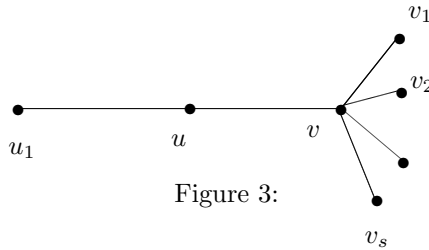

Figure 3:

Theorem 6. *Let G be a unicyclic graph. Then $\gamma_h(G) = \gamma_c(G)$ if and only if $G \in \mathcal{G}$.*

Proof. Assume that $\gamma_h(G) = \gamma_c(G)$. Let C be the cycle in G of length m and let X denote the set of all vertices of degree 2 in C. Let t denote the number of trees in G of diameter ≥ 2 from a vertex in C and l be the number of leaves in G.

We first show that $m \leq 4$.

Case 1: $m = 6r_1 + s_1$; $r_1 > 0$ and $0 \leq s_1 \leq 5$.

   Case 1.1: Let $m = 6r_1$.

      If $u = u_1 u_2 ... u_x$ is a longest path of length ≥ 2 in $G - C$ from $u \in C$, then we can find a hop dominating set S not containing u_{x-1} and u_x. As $m \geq 4$, this S can be chosen such that it does not contain any leaves of G. Also we know that $\gamma_h(C) = 2r_1$. Therefore, we have $\gamma_h(G) \leq 2r_1 + n - m - t - l$.

      i.e., $\leq 2r_1 + n - 6r_1 - t - l < n - l - 2 \leq \gamma_c(G)$.

   Case 1.2: Let $m = 6r_1 + 1$.

      In this case we have $\gamma_h(G) \leq 2r_1 + 1 + n - m - t - l$.

      i.e., $\leq n - l - (4r_1 + 2) < n - l - 2 \leq \gamma_c(G)$.

   Case 1.3: Let $m = 6r_1 + a$; $2 \leq a \leq 5$.

      Then $\gamma_h(G) \leq 2r_1 + 2 + n - m - l - t$.

      i.e., $\leq n - l - (4r_1 + t + a - 2) < n - l - 2 \leq \gamma_c(G)$.

Case 2: Let $m = 5$.

Then $\gamma_h(G) \leq 2 + \gamma_h(G - C) < 3 + \gamma_c(G - C) = \gamma_c(G)$, a contradiction.

   Therefore, if $\gamma_h(G) = \gamma_c(G)$, then $m \leq 4$.

Case 3: Let $m = 3$.

   Case 3.1: Let $m = 3$ and $|X| = 1$.

      Let $u, v \in C - X$ and $w \in X$.

      We claim that u and v can be supports of leaves only.

      If not, let there be a longest path $u = u_1 u_2 ... u_y$; $y \geq 3$. Then we can find a hop dominating set not containing u_{y-1} and u_y. But the unique γ_c-set of G contains all u_is and v and so $\gamma_c(G) > \gamma_h(G)$, a contradiction. Therefore, the γ_h-set of G will consist of u and one of the leaves of u or

v and one of the leaves of v. This implies $\gamma_h(G) = 2$. As $\{u, v\}$ is the γ_c-set of G, we have $\gamma_h(G) = \gamma_c(G)$.

Case 3.2: Let $m = 3$ and $|X| = 2$.

Let $u \in V(C) - X$ and $v, w \in X$.

Claim: u is a support of a tree in $G - C$ with diameter 2.

Suppose u supports a tree in $G - C$ with diameter ¿ 2. Let $u = u_1 u_2 ... u_t$ be the longest path in G. Then we can form a hop dominating set not containing u_{t-1} and u_t; but the unique γ_c-set of G contains $u = u_1 u_2 ... u_{t-1}$. Thus $\gamma_h(G) < \gamma_c(G)$, a contradiction.

We next observe that u cannot be a support of leaves alone, since, in that case $\gamma_c(G) = 1$ as $\{u\}$ is the γ_c-set of G whereas a γ_h-set of G contains two vertices, namely u and one of its leaves.

Case 3.3: Let $m = 3$ and $|X| = 0$.

This case is not possible, since $\gamma_h(G) < 3 + r$ whereas $\gamma_c(G) = 3 + r$, where r is the number of internal vertices in $G - C$.

Case 4: Let $m = 4$.

We claim that $2 \leq |X| \leq 4$.

Suppose $|X| = 0$.

If G has r number of internal vertices other than $V(C)$, then $\gamma_h(G) \leq 2 + r$ whereas $\gamma_c(G) = 4 + r$, a contradiction.

Similarly, we can show that $|X| \neq 1$.
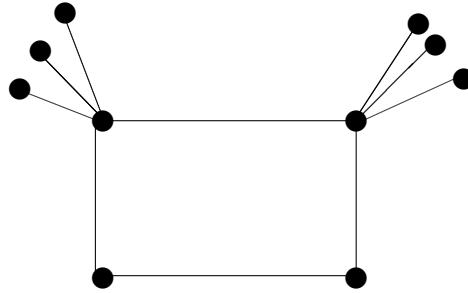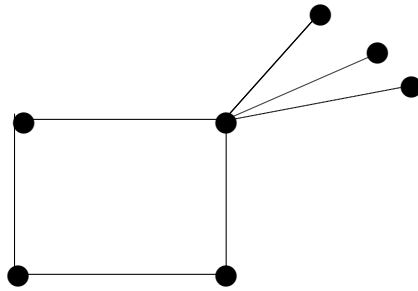
The graph G for $|X| = 2$ is G_1.

The graph G for $|X| = 3$ is G_2.

The graph G for $|X| = 4$ is C_4.

The converse is trivial.   □

Figure 4: G_1

Figure 5: G_2

2.3 Strong equality of hop domination and hop independent domination numbers

In [6], Haynes et al. obtained strong equality of domination number and other domination parameters for trees. Similarly we have the following strong equality of hop domination and hop independent domination numbers for trees.

Definition 7. *For a graph G, we say that $\gamma_h(G)$ strongly equals $\gamma_{hi}(G)$, if $\gamma_h(G) = \gamma_{hi}(G)$ and every γ_h-set of G is a γ_{hi}-set of G. The strong equality of these two sets is denoted by $\gamma_h(G) \equiv \gamma_{hi}(G)$.*

Proposition 8. *Let T be a tree of diameter≤ 5. Then $\gamma_h(T) \equiv \gamma_{hi}(T)$.*

Proof. Let P be a longest path in T of length d. Then $d \leq 5$. **Case** 1 Let

$d = 5$. Then P has the vertices v_1, v_2, v_3, v_4, v_5 and v_6. Clearly $\{v_3, v_4\}$ is the only γ_h-set of P which is also a γ_{hi}-set of P.

Case 2 Let $d = 4$. Then $P = <v_1, v_2, v_3, v_4, v_5>$. Clearly $\{v_3, v_4\}$ is the only γ_h-set of T which is also an h-independent set of T.   □

Remark: The converse of the above proposition is not true. For example, consider the following tree T.
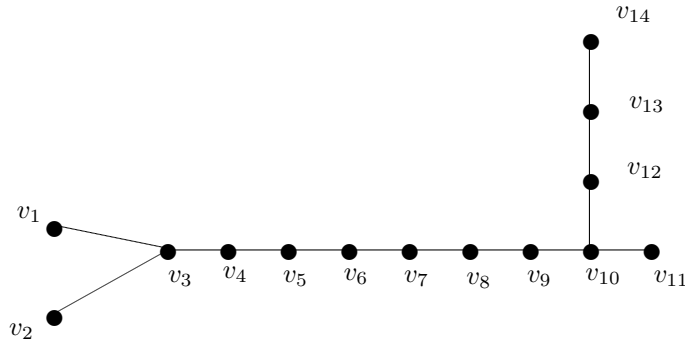

Figure 6: T

For this tree T $\gamma_h(T) \equiv \gamma_{hi}(T)$ since $\{v_4, v_5, v_{10}, v_{12}\}$ is the only set which is hop dominating as well as hop indepedent dominating set. But $diam(T) > 5$.

Proposition 9. *Let P be a path with $n \geq 4$ vertices. Then $\gamma_h(P) \equiv \gamma_{hi}(P)$ if and only if $n \equiv 0, 4, 5 (mod\ 6)$.*

Proof. Let $V(P) = \{v_1, v_2, \cdots, v_n\}$. Then every γ_h-set S of T contains the pairs of vertices v_3, v_4; v_9, v_{10}; $\cdots$, in general v_{6t-3}, v_{6t-2}. Suppose $n = 6t + 3$. Then v_{6t+2} can be hop dominated by v_{6t}. But v_{6t-2} is already in the set S which is of distance 2 from v_{6t}. Therefore the γ_h-set $S = \{v_3, v_4, v_9, v_{10}, \cdots, v_{6t-3}, v_{6t-2}, v_{6t}\}$ will not be h-independent. This is the case for $n = 6t + 1$ and $n = 6t + 2$. Therefore, $n = 6t + 4$ and $n = 6t + 5$ is same as writing $n \equiv 0, 4, 5\ (mod\ 6)$.   □

Definition 10. *A caterpillar T is full footed (FFC) if every internal vertex is a support vertex of at least one leaf. We say that a central path in a FFC is of order n if FFC has n number of internal vertices.*

Theorem 11. *An FFC T of order n has the property $\gamma_h(T) \equiv \gamma_{hi}(T)$ if and only if $n = 4t$.*

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be the set of internal vertices of T. If $n = 4t$, then the only γ_h-set is $\{v_2, v_3, v_6, v_{10}, \cdots, v_{4t-2}, v_{4t-1}\}$ where v_{4t-1} is needed to hop dominate the leaves of v_{4t}. Therefore $\gamma_h(T) \equiv \gamma_{hi}(T)$. If $n \neq 4t$, for instance $n = 4t - 1$, then the set $S = \{v_2, v_3, v_6, v_7, \cdots, v_{4t-6}, v_{4t-5}, v_{4t-3}, v_{4t-2}\}$ is a γ_h-set of T with $d_G(v_{4t-5}, v_{4t-3}) = 2$ and so S is not a γ_{hi}-set of T implying $\gamma_h(T)$ is not strongly equal to $\gamma_{hi}(T)$. Similarly, if $n = 4t + 1$, then we can find a γ_h-set containing the leaves v_{4t-1} and v_{4t-3} which are of distance 2. Other cases $n = 4t + 2$ and $n = 4t + 3$ can be argued similarly. $\qquad \square$

2.4 Hop domination number of shadow graph and mycielskian graph of a graph

Theorem 12. *For any graph G, $\gamma_h(Sh(G)) = \gamma_h(G)$.*

Proof. Let D be a hop dominating set of G. Let v' be the twin of a vertex $v \in G$. If $v \in D$, then $d(v, v') = 2$ and hence v hop dominates v'. Otherwise, let $u \in D$ hop dominates v. Then there exists $w \in G$ such that $w \in N(u) \cap N(v)$. Now, $w \in N(v)$ implies w and v' are adjacent. This together with $d(u, w) = 1$ implies v' is hop dominated by u. This proves that $\gamma_h(Sh(G)) = \gamma_h(G)$. $\quad \square$

Corollary 13. *For any graph G, $\gamma_h(\mu(G)) = \gamma_h(G)$.*

Proof. Let D be a hop dominating set of G. As $\mu(G) = Sh(G) \cup \{c\}$ where c is a vertex adjacent to all the twins of $v \in V(G)$, any vertex in D hop dominates c. Thus $\gamma_h(\mu(G)) = \gamma_h(G)$. $\qquad \square$

3 Open Problems

We conclude this study on hop domination with the following open problems which lead to further research in this topic.

1. Characterize the family of graphs for which

    (i) $\gamma_h(G) = \gamma_{hi}(G)$

    (ii) $\Gamma_h(G) = \beta_h(G)$

    (iii) $\gamma_h(G) = \gamma_c(G)$ (other than trees and unicyclic graphs)

    (iv) $\gamma_h(G) = \gamma_t(G)$ (other than trees and unicyclic graphs)

2. Determine complexity of these problems mentioned in 1.

3. Characterize connected graphs G with $\gamma_h(G) = n - 1$.

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