



On a product of universal hyperalgebras

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Abstract

We introduce and study a new operation of product of universal hyperalgebras which lies, with respect to set inclusion, between the cartesian product of the hyperalgebras and the cartesian product of their idempotent hulls. We give sufficient conditions for the validity of the first exponential law and a weak form of the second exponential law for the direct power of universal hyperalgebras with respect to the product introduced.

1 Introduction

In his pioneering paper [1], G. Birkhoff introduced the cardinal (i.e., direct) arithmetic of partially ordered sets and showed that it behaves analogously to the arithmetic of natural numbers. Several authors then extended the cardinal arithmetic to relational systems - see e.g. [6-7] and [9-10]. Conversely, the cardinal arithmetic has been restricted from relational systems to universal algebras in [11], to partial algebras in [14] and to hyperalgebras in [13]. In the present paper, we continue the study of direct arithmetic of hyperalgebras from [13].

In [7], M. Novotný and J. Šlapal introduced and studied a new operation of product of relational systems which is a combination of the direct sum and the direct product of the systems. In the present note, we restrict the operation of product discussed in [7] from relational systems to hyperalgebras. We will

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study the behavior of the product restricted and will find conditions under which the direct power of hyperalgebras satisfies the first exponential law and a weak form of the second exponential law with respect to the product.

Hyperalgebras proved to be useful for many application in various branches of mathematics and computer science (automata theory). This lead to a rapid development of the theory of hyperalgebras since the beginning of 90's of the last century - see e.g. [2]. The aim of this paper is to contribute to the development by introducing and studying a new operation of product of hyperalgebras.

2 Combined product of hyperalgebras

Given sets G and H , we denote by G^H the set of all mappings of H into G . The bijection $\varphi : (G^H)^K \rightarrow G^{H \times K}$ (where \times denotes the cartesian product) given by $\varphi(h)(x, y) = h(y)(x)$ whenever $h \in (G^H)^K, x \in H$ and $y \in K$ will be called *canonical*.

Throughout the paper, maps $f : G \rightarrow H$ (G, H sets) will often be denoted as indexed sets $(f_i; i \in G)$ where f_i means $f(i)$ for every $i \in G$.

Let Ω be a nonempty set. A family $\tau = (K_\lambda; \lambda \in \Omega)$ of sets will be called a *type*. By a *universal hyperalgebra* (briefly, a *hyperalgebra*) of type τ we understand a pair $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ where G is a nonempty set, the so-called *carrier* of \mathbf{G} , and $p_\lambda : G^{K_\lambda} \rightarrow \exp G \setminus \{\emptyset\}$ is a map, the so called K_λ -ary *hyperoperation* on G , for every $\lambda \in \Omega$. Of course, if $K_\lambda = \emptyset$, then p_λ is nothing but a nonempty subset of G . To avoid some nonwanted singularities, all hyperalgebras $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ are supposed to have a type $(K_\lambda; \lambda \in \Omega)$ with $K_\lambda \neq \emptyset$ for every $\lambda \in \Omega$. A hyperalgebra $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ with the property $\text{card } p_\lambda(x_i; i \in K_\lambda) = 1$ for every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}$ is called a (*universal*) *algebra*. If $\tau = (K)$ where K is a finite set with $\text{card } K = n$, then hyperalgebras of type τ are called *n-ary hyperalgebras*. In the case $\text{card } K = 2$, hyperalgebras of type τ are called *hypergroupoids*.

Given a hyperalgebra $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ of type $\tau = (K_\lambda; \lambda \in \Omega)$, $\lambda \in \Omega$ and a family $X_i, i \in K_\lambda$, of subsets of G , we put $p_\lambda(X_i; i \in K_\lambda) = \bigcup \{p_\lambda(x_i; i \in K_\lambda); x_i \in X_i \text{ for every } i \in K_\lambda\}$.

Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ be a pair of hyperalgebras of type τ . Then \mathbf{G} is called a *subhyperalgebra* of \mathbf{H} provided that $G \subseteq H$ and $p_\lambda(x_i; i \in K_\lambda) = q_\lambda(x_i; i \in K_\lambda)$ whenever $\lambda \in \Omega$ and $x_i \in H$ for each $i \in K_\lambda$. We will write $\mathbf{G} \leq \mathbf{H}$ if $G = H$ and $p_\lambda(x_i; i \in K_\lambda) \subseteq q_\lambda(x_i; i \in K_\lambda)$ whenever $\lambda \in \Omega$ and $(x_i; i \in K_\lambda) \in H^{K_\lambda}$. A map $f : H \rightarrow G$ is called a *homomorphism* of \mathbf{H} into \mathbf{G} if, for each $\lambda \in \Omega$, $f(q_\lambda(x_i; i \in K_\lambda)) \subseteq p_\lambda(f(x_i); i \in K_\lambda)$. The set of all homomorphisms of \mathbf{H} into \mathbf{G} will be denoted by $\text{Hom}(\mathbf{H}, \mathbf{G})$. If f is a bijection of H onto G and $f(q_\lambda(x_i; i \in K_\lambda)) = p_\lambda(f(x_i); i \in K_\lambda)$

whenever $x_i \in H$ for each $i \in K_\lambda$, then f is called an *isomorphism* of \mathbf{H} onto \mathbf{G} . In other words, an isomorphism of \mathbf{H} onto \mathbf{G} is a bijection $f : H \rightarrow G$ such that f is a homomorphism of \mathbf{H} into \mathbf{G} and f^{-1} is a homomorphism of \mathbf{G} into \mathbf{H} . If there is an isomorphism of \mathbf{H} onto \mathbf{G} , then we write $\mathbf{H} \cong \mathbf{G}$ and say that \mathbf{H} and \mathbf{G} are *isomorphic*. We say that \mathbf{H} may be *embedded* into \mathbf{G} and write $\mathbf{H} \preceq \mathbf{G}$ if there exists a subhyperalgebra \mathbf{G}' of \mathbf{G} such that $\mathbf{H} \cong \mathbf{G}'$.

The *direct product* of a nonempty family $\mathbf{G}_i = \langle G_i, (p_\lambda^i; \lambda \in \Omega) \rangle, i \in I$, of hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ is the hyperalgebra $\prod_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (q_\lambda; \lambda \in \Omega) \rangle$ of type τ where $\prod_{i \in I} G_i$ denotes the cartesian product and, for any $\lambda \in \Omega$ and any $(f_k; k \in K_\lambda) \in (\prod_{i \in I} G_i)^{K_\lambda}$, $q_\lambda(f_k; k \in K_\lambda) = \prod_{i \in I} p_\lambda^i(f_k(i); k \in K_\lambda)$. If the set I is finite, say $I = \{1, \dots, m\}$, then we write $\mathbf{G}_1 \times \dots \times \mathbf{G}_m$ instead of $\prod_{i \in I} \mathbf{G}_i$. If $\mathbf{G}_i = \mathbf{G}$ for every $i \in I$, then we write \mathbf{G}^I instead of $\prod_{i \in I} \mathbf{G}_i$.

The *direct sum* of a nonempty family $\mathbf{G}_i = \langle G, (p_\lambda^i; \lambda \in \Omega) \rangle, i \in I$, of hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ is the hyperalgebra $\sum_{i \in I} \mathbf{G}_i = \langle G, (q_\lambda; \lambda \in \Omega) \rangle$ of type τ where, for every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}$, $q_\lambda(x_i; i \in K_\lambda) = \bigcup_{i \in I} p_\lambda^i(x_i; i \in K_\lambda)$. If the set I is finite, say $I = \{1, \dots, m\}$, then we write $\mathbf{G}_1 \uplus \dots \uplus \mathbf{G}_m$ instead of $\sum_{i \in I} \mathbf{G}_i$.

Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be a hyperalgebra of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let $\lambda_0 \in \Omega$. An element $x \in G$ is called an *idempotent* with respect to the hyperoperation p_{λ_0} if $x \in p_{\lambda_0}(x_i; i \in K_{\lambda_0})$ whenever $x_i = x$ for every $i \in K_{\lambda_0}$. If every element of G is an idempotent with respect to each hyperoperation $p_\lambda, \lambda \in \Omega$, then the hyperalgebra \mathbf{G} is said to be *idempotent*. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be a hyperalgebra of type $\tau = (K_\lambda; \lambda \in \Omega)$. For every $\lambda \in \Omega$ and every $(x_i; i \in K_\lambda) \in G^{K_\lambda}$ we put

$$\bar{p}_\lambda(x_i; i \in K_\lambda) = \begin{cases} p_\lambda(x_i; i \in K_\lambda) \cup \{x\} & \text{if there is } x \in G \text{ with } x_i = x \text{ for all} \\ & i \in K_\lambda, \\ p_\lambda(x_i; i \in K_\lambda) & \text{if there are } i, j \in K_\lambda \text{ with } x_i \neq x_j. \end{cases}$$

The hyperalgebra $\langle G, (\bar{p}_\lambda; \lambda \in \Omega) \rangle$ is called the *idempotent hull* of \mathbf{G} and is denoted by $\bar{\mathbf{G}}$.

Let $\mathbf{G}_i = \langle G_i, (p_{i\lambda}; \lambda \in \Omega) \rangle, i \in I$, be a nonempty family of hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$. The *combined product* of the family $\mathbf{G}_i, i \in I$, is the hyperalgebra $\bigotimes_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (r_\lambda; \lambda \in \Omega) \rangle$ of type τ given by $\bigotimes_{i \in I} \mathbf{G}_i = \sum_{i \in I} \prod_{j \in I} \mathbf{G}_{ij}$ where

$$\mathbf{G}_{ij} = \begin{cases} \bar{\mathbf{G}}_j & \text{if } i = j, \\ \mathbf{G}_j & \text{if } i \neq j. \end{cases}$$

Thus, for any $\lambda \in \Omega$ and any $((x_k^i; i \in I); k \in K_\lambda) \in (\prod_{i \in I} \mathbf{G}_i)^{K_\lambda}$, we have $(x^i; i \in I) \in r_\lambda((x_k^i; i \in I); k \in K_\lambda)$ if and only if there exists a subset $J \subseteq I$,

card $J \leq 1$, such that $x^i \in p_\lambda^i(x_k^i; k \in K_\lambda)$ for every $i \in I \setminus J$ and $x_k^i = x^i$ for every $k \in K_\lambda$ and every $i \in J$.

If the set I is finite, say $I = \{1, \dots, m\}$, we write $\mathbf{G}_1 \otimes \dots \otimes \mathbf{G}_m$ instead of $\bigotimes_{i \in I} \mathbf{G}_i$. We then clearly have $\mathbf{G}_1 \otimes \dots \otimes \mathbf{G}_m = (\bar{\mathbf{G}}_1 \times \mathbf{G}_2 \times \dots \times \mathbf{G}_m) \uplus (\mathbf{G}_1 \times \bar{\mathbf{G}}_2 \times \dots \times \mathbf{G}_m) \uplus \dots \uplus (\mathbf{G}_1 \times \mathbf{G}_2 \times \dots \times \bar{\mathbf{G}}_m)$.

In particular, if $I = \{1, 2\}$, then, for every $\lambda \in \Omega$ and every $((x_k, y_k); k \in K_\lambda) \in (\mathbf{G}_1 \times \mathbf{G}_2)^{K_\lambda}$, $(x, y) \in r_\lambda((x_k, y_k); k \in K_\lambda)$ if and only if one of the following three conditions is satisfied:

- (i) $x \in p_\lambda^1(x_k; k \in K_\lambda)$ and $y \in p_\lambda^2(y_k; k \in K_\lambda)$,
- (ii) $x = x_k$ for every $k \in K_\lambda$ and $y \in p_\lambda^2(y_k; k \in K_\lambda)$,
- (iii) $x \in p_\lambda^1(x_k; k \in K_\lambda)$ and $y = y_k$ for every $k \in K_\lambda$.

If I is a singleton, say $I = \{j\}$, then $\bigotimes_{i \in I} \mathbf{G}_i = \bar{\mathbf{G}}_j$.

Remark 2.1. Let $\mathbf{G}_i, i \in I$, be a family of hyperalgebras of the same type. We clearly have $\prod_{i \in I} \mathbf{G}_i \leq \bigotimes_{i \in I} \mathbf{G}_i \leq \prod_{i \in I} \bar{\mathbf{G}}_i = \bigotimes_{i \in I} \bar{\mathbf{G}}_i$ and $\bigotimes_{i \in I} \mathbf{G}_i \leq \bigotimes_{i \in I} \bar{\mathbf{G}}_i$. If \mathbf{G}_i is idempotent for every $i \in I$, then all the previous inequalities become equalities. If the hyperalgebras $\mathbf{G}_i, i \in I$, are idempotent with the exception of at most one of them, then $\bigotimes_{i \in I} \mathbf{G}_i$ is idempotent.

Theorem 2.2. Let $\mathbf{G}_i, i \in I$, be a nonempty family of hyperalgebras of the same type τ and let \mathbf{H} be a hyperalgebra of type τ . Then $\sum_{i \in I} \mathbf{H} \otimes \mathbf{G}_i = \mathbf{H} \otimes \sum_{i \in I} \mathbf{G}_i$.

Proof. Let $\mathbf{G}_i = \langle G_i, (p_\lambda^i; \lambda \in \Omega) \rangle$ for all $i \in I$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ and let $\tau = (K_\lambda; \lambda \in \Omega)$. Let $\sum_{i \in I} \mathbf{G}_i = \langle G, (s_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} \otimes \mathbf{G}_i = \langle H \times G, (r_\lambda^i; \lambda \in \Omega) \rangle$ for each $i \in I$, $\mathbf{H} \otimes \sum_{i \in I} \mathbf{G}_i = \langle H \times G, (u_\lambda; \lambda \in \Omega) \rangle$ and $\sum_{i \in I} \mathbf{H} \otimes \mathbf{G}_i = \langle H \times G, (v_\lambda; \lambda \in \Omega) \rangle$. We will show that $u_\lambda((x_k, y_k); k \in K_\lambda) = v_\lambda((x_k, y_k); k \in K_\lambda)$ for every $((x_k, y_k); k \in K_\lambda) \in (H \times G)^{K_\lambda}$. It is easy to see that the following five conditions are equivalent:

- (a) $(x, y) \in u_\lambda((x_k, y_k); k \in K_\lambda)$;
- (b) one of the following three cases occurs:
 - (i) $x \in q_\lambda(x_k; k \in K_\lambda)$ and $y \in s_\lambda(y_k; k \in K_\lambda)$;
 - (ii) $x = x_k$ for every $k \in K_\lambda$ and $y \in s_\lambda(y_k; k \in K_\lambda)$;
 - (iii) $x \in q_\lambda(x_k; k \in K_\lambda)$ and $y = y_k$ for every $k \in K_\lambda$;
- (c) one of the following three cases occurs:
 - (i) $x \in q_\lambda(x_k; k \in K_\lambda)$ and $y \in p_\lambda^i(y_k; k \in K_\lambda)$ for some $i \in I$;

- (ii) $x = x_k$ for every $k \in K_\lambda$ and $y \in p_\lambda^i(y_k; k \in K_\lambda)$ for some $i \in I$;
- (iii) $x \in q_\lambda(x_k; k \in K_\lambda)$ and $y = y_k$ for every $k \in K_\lambda$;
- (d) $(x, y) \in r_\lambda^i((x_k, y_k); k \in K_\lambda)$ for some $i \in I$;
- (e) $(x, y) \in v_\lambda((x_k, y_k); k \in K_\lambda)$.

This proves the statement. \square

3 Direct power of hyperalgebras

In the sequel, we will work with $K \times L$ -matrices over G where K, L, G are nonempty sets. These matrices are nothing but the maps $x : K \times L \rightarrow G$, i.e., the indexed sets $(x_{ij}; i \in K, j \in L)$ denoted briefly by (x_{ij}) .

Definition 3.1. A hyperalgebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ of type $\tau = (K_\lambda; \lambda \in \Omega)$ is called *medial* if, for every $\mu, \nu \in \Omega$ and every $K_\mu \times K_\nu$ -matrix (a_{ij}) over G , from $x_i \in p_\nu(a_{ij}; j \in K_\nu)$ for each $i \in K_\mu$ and $y_j \in p_\mu(a_{ij}; i \in K_\mu)$ for each $j \in K_\nu$ it follows that $p_\nu(x_i; i \in K_\mu) = p_\mu(y_j; j \in K_\nu)$.

The medial universal algebras are often called commutative or the algebras satisfying the interchange law and they were dealt with e.g. in [4], [5] and [11]. The medial n -ary algebras are studied in [12] and the medial groupoids in [3].

Example 3.2. (1) Let (X, \leq) be a partially ordered set with a least element 0 and let A be the set of all atoms of (X, \leq) . Put $0' = \{0\}$ and, for any $x \in X$ with $x \neq 0$, put $x' = \{y \in X; y < x \text{ and } y \in A \cup \{0\}\}$. Further, for any pair $x, y \in X$ put $x * y = x' \cap y'$. Then $(X, ', *)$ is a medial hyperalgebra of type (1, 2). Indeed, it can easily be seen that, for any $x \in X$, we have $a' = \{0\} = b'$ whenever $a \in x'$ and $b \in x'$. Further, for any $a, b, c, d \in X$, we have $x * y = f * g$ whenever $x \in a * b$, $y \in c * d$, $f \in a * c$ and $g \in b * d$. Finally, for any $a, b \in X$, we have $x' = y * z$ whenever $x \in a * b$, $y \in a'$ and $z \in b'$.

(2) Every unary algebra \mathbf{G} consisting of two-element cycles only is medial and, moreover, the unary hyperalgebra $\bar{\mathbf{G}}$ is medial, too.

Lemma 3.3. Let \mathbf{H} and $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ be hyperalgebras of the same type $\tau = (K_\lambda; \lambda \in \Omega)$, let $\lambda \in \Omega$ and let $f_i \in \text{Hom}(\mathbf{H}, \mathbf{G})$ for each $i \in K_\lambda$. Let $f : H \rightarrow G$ be a map such that $f(x) \in p_\lambda(f_i(x); i \in K_\lambda)$ for every $x \in H$. If \mathbf{G} is medial, then f is a homomorphism from \mathbf{H} into \mathbf{G} .

Proof. Let $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\mu \in \Omega$ and $x_i \in H$ for each $i \in K_\mu$. We will show that $f(q_\mu(x_i; i \in K_\mu)) \subseteq p_\mu(f(x_i); i \in K_\mu)$. Let $y \in f(q_\mu(x_i; i \in K_\mu))$. Then there exists $x \in q_\mu(x_i; i \in K_\mu)$ such that $y = f(x)$, so $f(x) \in f(q_\mu(x_i; i \in K_\mu))$. Since $f_j \in \text{Hom}(\mathbf{H}, \mathbf{G})$ for each $j \in K_\lambda$ and $x \in q_\mu(x_i; i \in K_\mu)$, we

have $f_j(x) \in f_j(q_\mu(x_i; i \in K_\mu)) \subseteq p_\mu(f_j(x_i); i \in K_\mu)$ for each $j \in K_\lambda$. By the assumptions of the statement, $f(x_i) \in p_\lambda(f_j(x_i); j \in K_\lambda)$ for each $i \in K_\mu$ and $f(x) \in p_\lambda(f_j(x); j \in K_\lambda)$. Since \mathbf{G} is medial, $p_\lambda(f_j(x); j \in K_\lambda) = p_\mu(f(x_i); i \in K_\mu)$. Thus, $y = f(x) \in p_\mu(f(x_i); i \in K_\mu)$. \square

Definition 3.4. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ and \mathbf{H} be hyperalgebras of the same type $\tau = (K_\lambda; \lambda \in \Omega)$ and let \mathbf{G} be medial. The *direct power* of \mathbf{G} and \mathbf{H} is the hyperalgebra $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$ of type τ where, for any $\lambda \in \Omega$ and any $(f_i; i \in K_\lambda) \in (\text{Hom}(\mathbf{H}, \mathbf{G}))^{K_\lambda}$, $r_\lambda(f_i; i \in K_\lambda) = \{f \in G^{\mathbf{H}}; f(x) \in p_\lambda(f_i(x); i \in K_\lambda) \text{ for each } x \in H\}$.

It is easy to see that the power $\mathbf{G}^{\mathbf{H}}$ of hyperalgebras of the same type is idempotent whenever \mathbf{G} is idempotent (and medial).

Theorem 3.5. *Let \mathbf{G}, \mathbf{H} be hyperalgebras of the same type. If \mathbf{G} is medial, then so is $\mathbf{G}^{\mathbf{H}}$.*

Proof. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$. Let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of \mathbf{G} and \mathbf{H} and let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$. Let $\mu, \nu \in \Omega$ and let (f_{ij}) be a $K_\mu \times K_\nu$ -matrix over $\text{Hom}(\mathbf{H}, \mathbf{G})$. Suppose that $h_i \in r_\nu(f_{ij}; j \in K_\nu)$ for all $i \in K_\mu$ and $g_j \in r_\mu(f_{ij}; i \in K_\mu)$ for all $j \in K_\nu$. For every $x \in H$, there holds $h_i(x) \in p_\nu(f_{ij}(x); j \in K_\nu)$ for all $i \in K_\mu$ and $g_j(x) \in p_\mu(f_{ij}(x); i \in K_\mu)$ for all $j \in K_\nu$. Since \mathbf{G} is medial, we have $p_\mu(h_i(x); i \in K_\mu) = p_\nu(g_j(x); j \in K_\nu)$ for every $x \in H$. Therefore, $r_\mu(h_i; i \in K_\mu) = r_\nu(g_j; j \in K_\nu)$, so that $\mathbf{G}^{\mathbf{H}}$ is medial. \square

Definition 3.6. A hyperalgebra $\langle G, (p_\lambda; \lambda \in \Omega) \rangle$ of type τ is called *diagonal* if, for every $\lambda \in \Omega$ and every $K_\lambda \times K_\lambda$ -matrix (a_{ij}) over G , we have

$$p_\lambda(p_\lambda(a_{ij}; j \in K_\lambda); i \in K_\lambda) \cap p_\lambda(p_\lambda(a_{ij}; i \in K_\lambda); j \in K_\lambda) \subseteq p_\lambda(a_{ii}; i \in K_\lambda).$$

For idempotent n -ary algebras, the diagonality introduced coincides with the diagonality studied in [8].

Example 3.7. It may easily be seen that the hyperalgebra from Example 3.2(1) is diagonal.

Theorem 3.8. *Let \mathbf{G}, \mathbf{H} be hyperalgebras of the same type. If \mathbf{G} is medial and diagonal, then $\mathbf{G}^{\mathbf{H}}$ is diagonal.*

Proof. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$. Let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of \mathbf{G} and \mathbf{H} and let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$. Let $\lambda \in \Omega$ and let (f_{ij}) be a $K_\lambda \times K_\lambda$ -matrix over $\text{Hom}(\mathbf{H}, \mathbf{G})$. Suppose that $f \in r_\lambda(r_\lambda(f_{ij}; j \in K_\lambda); i \in K_\lambda) \cap r_\lambda(r_\lambda(f_{ij}; i \in K_\lambda); j \in K_\lambda)$. Then $f(x) \in p_\lambda(p_\lambda(f_{ij}(x); j \in K_\lambda); i \in K_\lambda) \cap p_\lambda(p_\lambda(f_{ij}(x); i \in K_\lambda); j \in K_\lambda)$ for every $x \in H$. Since \mathbf{G} is diagonal, we have $f(x) \in p_\lambda(f_{ii}(x); i \in K_\lambda)$ for every $x \in H$. Thus, $f \in r_\lambda(f_{ii}; i \in K_\lambda)$. Hence, $\mathbf{G}^{\mathbf{H}}$ is diagonal. \square

We will find sufficient conditions for the validity of the first exponential law and a weak form of the second exponential law for the direct power of hyperalgebras with respect to the combined product, namely, the laws

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \otimes \mathbf{K}} \text{ and}$$

$$\bigotimes_{i \in I} \mathbf{G}_i^{\mathbf{H}} \simeq \left(\bigotimes_{i \in I} \mathbf{G}_i \right)^{\mathbf{H}}.$$

Lemma 3.9. *Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be partial algebras of the same type and let \mathbf{G} be medial and diagonal. Then the canonical bijection $\varphi : (G^H)^K \rightarrow G^{H \times K}$ restricted to $\text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ is a bijection of $\text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ onto $\text{Hom}(\mathbf{H} \otimes \mathbf{K}, \mathbf{G})$.*

Proof. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{K} = \langle K, (s_\lambda; \lambda \in \Omega) \rangle$. Let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of \mathbf{G}, \mathbf{H} and \mathbf{K} . Let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} \otimes \mathbf{K} = \langle H \times K, (v_\lambda; \lambda \in \Omega) \rangle$ and let \mathbf{G} be diagonal and medial. Let $\lambda \in \Omega$, $h \in \text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ and let $(y_i, z_i) \in H \times K$ for every $i \in K_\lambda$. To show that $\varphi(h)(v_\lambda((y_i, z_i); i \in K_\lambda)) \subseteq p_\lambda(\varphi(h)(y_i, z_i); i \in K_\lambda)$, let $x \in \varphi(h)(v_\lambda((y_i, z_i); i \in K_\lambda))$. Then there exists $(y, z) \in v_\lambda((y_i, z_i); i \in K_\lambda)$ such that $x = \varphi(h)(y, z)$. Thus, one of the following three conditions is satisfied:

- (i) $y \in q_\lambda(y_i; i \in K_\lambda)$ and $z \in s_\lambda(z_i; i \in K_\lambda)$;
- (ii) $y = y_i$ for every $i \in K_\lambda$ and $z \in s_\lambda(z_i; i \in K_\lambda)$;
- (iii) $y \in q_\lambda(x_i; i \in K_\lambda)$ and $z = z_i$ for every $i \in K_\lambda$.

If $z \in s_\lambda(z_i; i \in K_\lambda)$, then $h(z) \in h(s_\lambda(z_i; i \in K_\lambda)) \subseteq r_\lambda(h(z_i); i \in K_\lambda)$, hence $\varphi(h)(y, z) = h(z)(y) \in p_\lambda(h(z_i)(y); i \in K_\lambda) = p_\lambda(\varphi(h)(y, z_i); i \in K_\lambda)$ for every $y \in H$. Thus, $\varphi(h)(y, z) \in p_\lambda(\varphi(h)(y_i, z_i); i \in K_\lambda)$ provided that (ii) is satisfied.

If $y \in q_\lambda(y_i; i \in K_\lambda)$, then $\varphi(h)(y, z) = h(z)(y) \in h(z)(q_\lambda(y_i; i \in K_\lambda)) \subseteq p_\lambda(h(z)(y_i); i \in K_\lambda) = p_\lambda(\varphi(h)(y_i, z); i \in K_\lambda)$ for every $z \in K$. Thus, $\varphi(h)(y, z) \in p_\lambda(\varphi(h)(y_i, z_i); i \in K_\lambda)$ provided that (iii) is satisfied.

Suppose that (i) is satisfied. Since $z \in s_\lambda(z_i; i \in K_\lambda)$, by the above considerations we have $\varphi(h)(y_i, z) \in p_\lambda(\varphi(h)(y_i, z_j); j \in K_\lambda)$ for every $i \in K_\lambda$ and $\varphi(h)(y, z) \in p_\lambda(\varphi(h)(y, z_j); j \in K_\lambda)$. Since $y \in q_\lambda(y_i; i \in K_\lambda)$, by the above considerations we have $\varphi(h)(y, z_j) \in p_\lambda(\varphi(h)(y_i, z_j); i \in K_\lambda)$ for every $j \in K_\lambda$ and $\varphi(h)(y, z) \in p_\lambda(\varphi(h)(y_i, z); i \in K_\lambda)$. The diagonality of \mathbf{G} implies $\varphi(h)(y, z) \in p_\lambda(\varphi(h)(y_i, z_i); i \in K_\lambda)$. We have shown that $\varphi(h) \in \text{Hom}(\mathbf{H} \otimes \mathbf{K}, \mathbf{G})$.

Let $g \in \text{Hom}(\mathbf{H} \otimes \mathbf{K}, \mathbf{G})$ and $z_i \in K$ for every $i \in K_\lambda$. To show that $\varphi^{-1}(g)(s_\lambda(z_i; i \in K_\lambda)) \subseteq r_\lambda(\varphi^{-1}(g)(z_i); i \in K_\lambda)$, let $x \in \varphi^{-1}(g)(s_\lambda(z_i; i \in$

K_λ). Then there exists $z \in s_\lambda(z_i; i \in K_\lambda)$ such that $x \in \varphi^{-1}(g)(z)$. Let $y \in H$. Then $(y, z) \in v_\lambda((y, z_i); i \in K_\lambda)$. Since $g \in \text{Hom}(\mathbf{H} \otimes \mathbf{K}, \mathbf{G})$, we have $g(y, z) \in g(v_\lambda((y, z_i); i \in K_\lambda)) \subseteq p_\lambda(g(y, z_i); i \in K_\lambda)$. It follows that $\varphi^{-1}(g)(z)(y) \in p_\lambda(\varphi^{-1}(g)(z_i)(y); i \in K_\lambda)$, hence $\varphi^{-1}(g)(z) \in p_\lambda(\varphi^{-1}(g)(z_i); i \in K_\lambda)$. Therefore, $\varphi^{-1}(s_\lambda(z_i); i \in K_\lambda) \subseteq r_\lambda(\varphi^{-1}(g)(z_i); i \in K_\lambda)$. Consequently, $\varphi^{-1}(g) \in \text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$. \square

Theorem 3.10. *Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ be hyperalgebras of the same type. If \mathbf{G} is medial and diagonal, then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \otimes \mathbf{K}}.$$

Proof. Let $\mathbf{G} = \langle G, (p_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$ and $\mathbf{K} = \langle K, (s_\lambda; \lambda \in \Omega) \rangle$. Let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of \mathbf{G}, \mathbf{H} and \mathbf{K} . Let $\mathbf{G}^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}), (r_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{H} \otimes \mathbf{K} = \langle H \times K, (v_\lambda; \lambda \in \Omega) \rangle$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} = \langle \text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}}), (t_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}^{\mathbf{H} \otimes \mathbf{K}} = \langle \text{Hom}(\mathbf{H} \otimes \mathbf{K}, \mathbf{G}), (u_\lambda; \lambda \in \Omega) \rangle$ and let \mathbf{G} be diagonal and medial. By Lemma 3.9, we are to show that $\varphi(t_\lambda(h_i; i \in K_\lambda)) = u_\lambda(\varphi(h_i); i \in K_\lambda)$ whenever $h_i \in \text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ for every $i \in K_\lambda$.

Let $x \in \varphi(t_\lambda(h_i; i \in K_\lambda))$. Then there exists $h \in t_\lambda(h_i; i \in K_\lambda)$ such that $x = \varphi(h)$. Since $h \in t_\lambda(h_i; i \in K_\lambda)$, we have $h(z) \in r_\lambda(h_i(z); i \in K_\lambda)$ for every $z \in K$. Therefore, $h(z)(y) \in p_\lambda(h_i(z)(y); i \in K_\lambda)$ for every $y \in H$. It follows that $\varphi(h)(y, z) \in p_\lambda(\varphi(h_i)(y, z); i \in K_\lambda)$ for every $(y, z) \in H \times K$. Hence, $\varphi(h) \in u_\lambda(\varphi(h_i); i \in K_\lambda)$. Thus, $\varphi(t_\lambda(h_i; i \in K_\lambda)) \subseteq u_\lambda(\varphi(h_i); i \in K_\lambda)$.

Let $x \in u_\lambda(\varphi(h_i); i \in K_\lambda)$. By Lemma 3.9, there exists $h \in \text{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ with $x = \varphi(h)$, so that $\varphi(h) \in u_\lambda(\varphi(h_i); i \in K_\lambda)$. Thus, $\varphi(h)(y, z) \in p_\lambda(\varphi(h_i)(y, z); i \in K_\lambda)$ for every $(y, z) \in H \times K$, which implies $h(z)(y) \in p_\lambda(h_i(z)(y); i \in K_\lambda)$ for every $y \in H$ and every $z \in K$. It follows that $h(z) \in r_\lambda(h_i(z); i \in K_\lambda)$ for every $z \in K$. Therefore, $h \in t_\lambda((h_i); i \in K_\lambda)$, hence $\varphi(h) \in \varphi(t_\lambda((h_i); i \in K_\lambda))$. Consequently, $u_\lambda(\varphi(h_i); i \in K_\lambda) \subseteq \varphi(t_\lambda((h_i); i \in K_\lambda))$.

We have shown that $\varphi(t_\lambda(h_i; i \in K_\lambda)) = u_\lambda(\varphi(h_i); i \in K_\lambda)$. Thus, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ is isomorphic to $\mathbf{G}^{\mathbf{H} \otimes \mathbf{K}}$. \square

It may easily be seen that the direct product of a family of medial hyperalgebras of the same type is a medial hyperalgebra. But this is not generally true for the combined product of hyperalgebras. With respect to the combined product, we have the following weak form of the second exponential law:

Theorem 3.11. *Let $\mathbf{G}_i, i \in I$, be a nonempty family of medial hyperalgebras of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let \mathbf{H} be a hyperalgebra of type τ . If $\bigotimes_{i \in I} \mathbf{G}_i$ is medial, then*

$$\bigotimes_{i \in I} \mathbf{G}_i^{\mathbf{H}} \preceq (\bigotimes_{i \in I} \mathbf{G}_i)^{\mathbf{H}}.$$

Proof. Let $\tau = (K_\lambda; \lambda \in \Omega)$. Let $\mathbf{G}_i = \langle G_i, (p_\lambda^i; \lambda \in \Omega) \rangle$ for every $i \in I$, $\mathbf{H} = \langle H, (q_\lambda; \lambda \in \Omega) \rangle$, $\bigotimes_{i \in I} \mathbf{G}_i = \langle \prod_{i \in I} G_i, (r_\lambda; \lambda \in \Omega) \rangle$, $\mathbf{G}_i^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \mathbf{G}_i), (u_\lambda^i; \lambda \in \Omega) \rangle$ for every $i \in I$, $\bigotimes_{i \in I} \mathbf{G}_i^{\mathbf{H}} = \langle \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i), (s_\lambda; \lambda \in \Omega) \rangle$ and $(\bigotimes_{i \in I} \mathbf{G}_i)^{\mathbf{H}} = \langle \text{Hom}(\mathbf{H}, \bigotimes_{i \in I} \mathbf{G}_i), (t_\lambda; \lambda \in \Omega) \rangle$. We define a map $\alpha : \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i) \rightarrow (\prod_{i \in I} G_i)^{\mathbf{H}}$ by $\alpha(f^i; i \in I)(z) = (f^i(z); i \in I)$ for each $z \in H$.

Let $(f^i; i \in I) \in \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i)$ and $h \in \alpha(f^i; i \in I)(q_\lambda(x_j; j \in K_\lambda))$. Then $h = \alpha(f^i; i \in I)(x)$ where $x \in q_\lambda(x_j; j \in K_\lambda)$. Thus, $h = \alpha(f^i; i \in I)(x) = (f^i(x); i \in I)$. Since $x \in q_\lambda(x_j; j \in K_\lambda)$ and $f^i \in \text{Hom}(\mathbf{H}, \mathbf{G}_i)$ for every $i \in I$, we have $f^i(x) \in p_\lambda^i(f^i(x_j); j \in K_\lambda)$ for every $i \in I$. Thus, $h = (f^i(x); i \in I) \in r_\lambda((f^i(x_j); i \in I); j \in K_\lambda) = r_\lambda(\alpha(f^i; i \in I)(x_j); j \in K_\lambda)$. Consequently, $\alpha(f^i; i \in I)(q_\lambda(x_j; j \in K_\lambda)) \subseteq r_\lambda(\alpha(f^i; i \in I)(x_j); j \in K_\lambda)$. Therefore, $\alpha(f^i; i \in I) \in \text{Hom}(\mathbf{H}, \bigotimes_{i \in I} \mathbf{G}_i)$. We have shown that α maps $\prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i)$ into $\text{Hom}(\mathbf{H}, \bigotimes_{i \in I} \mathbf{G}_i)$.

Suppose that $\alpha(f^i; i \in I) = \alpha(g^i; i \in I)$ where $(f^i; i \in I), (g^i; i \in I) \in \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i)$. Then $(f^i(x); i \in I) = \alpha(f^i; i \in I)(x) = \alpha(g^i; i \in I)(x) = (g^i(x); i \in I)$ for every $x \in H$. Therefore, $f^i(x) = g^i(x)$ for every $i \in I$ and every $x \in H$. Hence, $f^i = g^i$ for every $i \in I$. Thus, $\alpha : \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i) \rightarrow \text{Hom}(\mathbf{H}, \bigotimes_{i \in I} \mathbf{G}_i)$ is an injection.

Finally, let $(f_j^i; i \in I) \in \prod_{i \in I} \text{Hom}(\mathbf{H}, \mathbf{G}_i)$. We will show that $\alpha(s_\lambda((f_j^i; i \in I); j \in K_\lambda)) = t_\lambda(\alpha(f_j^i; i \in I); j \in K_\lambda)$. It is easy to see that the following seven conditions are equivalent:

- (a) $f \in \alpha(s_\lambda((f_j^i; i \in I); j \in K_\lambda))$;
- (b) $f = \alpha(f^i; i \in I)$ where $(f^i; i \in I) \in s_\lambda((f_j^i; i \in I); j \in K_\lambda)$;
- (c) $f = \alpha(f^i; i \in I)$ and there exists $J \subseteq I$, $\text{card } J \leq 1$, such that $f^i \in u_\lambda^i(f_j^i; j \in K_\lambda)$ for every $i \in I \setminus J$ and every $f^i = f_j^i$ for every $j \in K_\lambda$ and $i \in J$;
- (d) $f(x) = (f^i(x); i \in I)$ for every $x \in H$ and there exists $J \subseteq I$, $\text{card } J \leq 1$, such that $f^i(x) \in p_\lambda^i(f_j^i(x); j \in K_\lambda)$ for every $i \in I \setminus J$ and every $x \in H$ and $f^i(x) = f_j^i(x)$ for every $j \in K_\lambda$, every $i \in J$ and every $x \in H$;
- (e) $f(x) = (f^i(x); i \in I) \in r_\lambda((f_j^i(x); i \in I); j \in K_\lambda)$ for every $x \in H$;
- (f) $f(x) \in r_\lambda(\alpha(f_j^i; i \in I)(x); j \in K_\lambda)$ for every $x \in H$;
- (g) $f \in t_\lambda(\alpha(f_j^i; i \in I); j \in K_\lambda)$.

Consequently, $\alpha(s_\lambda((f_j^i; i \in I); j \in K_\lambda)) = t_\lambda(\alpha(f_j^i; i \in I); j \in K_\lambda)$, which yields $\bigotimes_{i \in I} \mathbf{G}_i^{\mathbf{H}} \preceq (\bigotimes_{i \in I} \mathbf{G}_i)^{\mathbf{H}}$. \square

It may easily be shown that, in Theorem 3.12, we may write \cong instead of \preceq provided that \mathbf{G}_i is idempotent for every $i \in I$. We then obtain the second exponential law for the direct product $\prod_{i \in I} \mathbf{G}_i^{\mathbf{H}} \cong (\prod_{i \in I} \mathbf{G}_i)^{\mathbf{H}}$.

4 Conclusion

The paper contributes to the development of the arithmetic of universal hyperalgebras. We proved that a naturally defined direct sum of universal hyperalgebras of the same type is compatible with their direct product in the sense that the direct product distributes over it (Theorem 2.2). The main results concern the behavior of the operation of combined product introduced for universal algebras (of the same type). In particular, for the direct power of universal hyperalgebras, we studied the validity of the first and second exponential laws with respect to the combined product. Theorem 3.10 states that, with respect to the combined product, the direct power of the hyperalgebras satisfies the first exponential law $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \otimes \mathbf{K}}$ whenever \mathbf{G} is medial and diagonal. This is not true when replacing the combined product with the direct product. It follows from Theorem 3.10 that the direct power of hyperalgebras satisfies the first exponential law with respect to the direct product if \mathbf{G} is medial and diagonal and, moreover, \mathbf{H} and \mathbf{K} are idempotent (this result was proved in [13]). Theorem 3.11 says that the direct power of universal hyperalgebras satisfies the weak form of the second exponential law $\bigotimes_{i \in I} \mathbf{G}_i^{\mathbf{H}} \preceq (\bigotimes_{i \in I} \mathbf{G}_i)^{\mathbf{H}}$. It may easily be shown that, in Theorem 3.11, we may write \cong instead of \preceq (thus obtaining the second exponential law with respect to the combined product) provided that \mathbf{G}_i is idempotent for every $i \in I$ (in which case the combined product coincides with the direct one). It is an open problem to find a necessary and sufficient condition for the validity of the second exponential law for the direct power of hyperalgebras with respect to the combined product.

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