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On some Properties of Tribonacci Quaternions

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Abstract

In this paper, we give some properties of the Tribonacci and Tribonacci-Lucas quaternions and obtain some identities for them.

1 Introduction

Quaternions are fundamental objects of various parts of mathematics. They have applications in both theoretical and applied mathematics such as group theory, computer science and even also physics, see [4, 14, 16]. Let $\mathcal{H}$ be the real division quaternion algebra. A natural basis of this algebra is formed by the elements $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. So all quaternions are of the form

$$q = a_0 + \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3,$$

where the coefficients a_n, $0 \leq n \leq 3$ are all real. The multiplication table for the basis of $\mathcal{H}$ is

$\cdot$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

Every $q \in \mathcal{H}$ can be simply written as $q = \mathrm{Re}(q)+\mathrm{Im}(q)$, where $\mathrm{Re}(q) = a_0$ and $\mathrm{Im}(q) = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ are called the real and imaginary parts, respectively. The conjugate of the real quaternion q is the quaternion denoted by q^*, and

$$q^* = \mathrm{Re}(q) - \mathrm{Im}(q).$$

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This operation satisfies

$$(q^*)^* = q,\ (q_1 + q_2)^* = q_1^* + q_2^*,\ (q_1 q_2)^* = q_2^* q_1^*$$

for all q_1, $q_2 \in \mathcal{H}$. Also, any quaternion $q \in \mathcal{H}$ can be written as

$$q = q_1 + \mathbf{j} q_2,$$

where q_1, $q_2 \in \mathbb{C}$. The addition and multiplication of any two quaternions,

$$q = q_1 + \mathbf{j} q_2, q' = q_1' + \mathbf{j} q_2',$$

are defined by

$$q + q' = (q_1 + q_1') + \mathbf{j}(q_2 + q_2')$$

and

$$qq' = [q_1 q_1' - (q_2')^* q_2] + \mathbf{j}[(q_2')^* q_1^* + q_2^* q_1'].$$

The norm of the quaternion q is defined by

$$N(q) = q\overline{q}.$$

Thus the inverse of a nonzero quaternion q is given by

$$q^{-1} = \frac{\overline{q}}{N(q)}.$$

For all $p, q \in \mathcal{H}$, we have

$$\begin{aligned} N(pq) &= N(p)N(q), \\ (pq)^{-1} &= q^{-1}p^{-1}. \end{aligned}$$

There are various types of quaternion sequences which are determined by their components taken from different types of sequences and they have been studied by many researchers. One of the well-known sequence is given by see [6]. In [6], Horadam defined the n^{th} Fibonacci and Lucas quaternions as the quaternions whose components are Fibonacci and Lucas numbers respectively. After that several authors were interested in these structures and obtained some results, see [5, 7, 9, 10, 11, 13, 15]. Recently Cerda-Morales considered the generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$ defined by

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},\ n \geq 3$$

where r, s, t are real numbers and $V_0 = a$, $V_1 = b$, $V_2 = c$ are arbitrary integers, see [1]. For $r = s = t = 1$ and $V_0 = 0$, $V_1 = 1$, $V_2 = 1$, the sequence $\{V_n\}_{n\geq 0}$

is the well-known Tribonacci sequence denoted by $\{T_n\}_n$, see [2, 3, 12]. For $r = s = t = 1$ and $V_0 = 3$, $V_1 = 1$, $V_2 = 3$, we obtain the Tribonacci-Lucas sequence $\{K_n\}_n$, see [17]. The first few Tribonacci numbers and Tribonacci Lucas numbers are given in the following table.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757

The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called the generating function for the sequence $\{a_0, a_1, a_2, \ldots\}$. The generating functions of the Tribonacci sequence $\{T_n\}_n$ and the Tribonacci-Lucas sequence $\{K_n\}_n$ are

$$\begin{aligned} f(x) &= \frac{x}{1-x-x^2-x^3}, \\ h(x) &= \frac{3-2x-x^2}{1-x-x^2-x^3}, \end{aligned}$$

respectively. The Binet formulas of T_n and K_n are given as

$$\begin{aligned} T_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \qquad (1.1) \\ K_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively, where

$$\begin{aligned} \alpha &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3} \\ \beta &= \frac{1+w\sqrt[3]{19+3\sqrt{33}}+w^2\sqrt[3]{19-3\sqrt{33}}}{3} \\ \gamma &= \frac{1+w^2\sqrt[3]{19+3\sqrt{33}}+w\sqrt[3]{19-3\sqrt{33}}}{3}, (w = \frac{-1+i\sqrt{3}}{2}), \end{aligned}$$

see [17].

In [1], we see a new type of quaternion whose coefficients are generalized Tribonacci numbers as follows,

$$Q_{v,n} = V_n + V_{n+1}\mathbf{i} + V_{n+2}\mathbf{j} + V_{n+3}\mathbf{k},\ n \geq 0.$$

In this paper, we are interested in the quaternions with Tribonacci number and Tribonacci-Lucas number components denoted by Q_n and $\tilde{Q}_n$, respectively. We give some properties of these quaternions and obtain some identities for them.

2 Quaternions with Tribonacci Number Components

For $n \geq 0$, the n^{th} Tribonacci quaternion Q_n and n^{th} Tribonacci-Lucas quaternion $\tilde{Q}_n$ are defined by

$$Q_n = T_n + \mathbf{i}T_{n+1} + \mathbf{j}T_{n+2} + \mathbf{k}T_{n+3}$$

and

$$\tilde{Q}_n = K_n + \mathbf{i}K_{n+1} + \mathbf{j}K_{n+2} + \mathbf{k}K_{n+3}$$

where T_n and K_n are the n^{th} Tribonacci and Tribonacci-Lucas numbers, respectively.

Note that for $n \geq 0$,

$$Q_{n+3} = Q_{n+2} + Q_{n+1} + Q_n,$$

and

$$\tilde{Q}_{n+3} = \tilde{Q}_{n+2} + \tilde{Q}_{n+1} + \tilde{Q}_n.$$

The conjugate of the Tribonacci quaternion Q_n is denoted by Q_n^* and

$$Q_n^* = T_n - \mathbf{i}T_{n+1} - \mathbf{j}T_{n+2} - \mathbf{k}T_{n+3},$$

and the conjugate of the Tribonacci-Lucas quaternion $\tilde{Q}_n$ is denoted by $\tilde{Q}_n^*$ and

$$\tilde{Q}_n^* = K_n - \mathbf{i}K_{n+1} - \mathbf{j}K_{n+2} - \mathbf{k}K_{n+3}.$$

Let $f(x)$ be a series in powers of x. Then by the symbol $[x^n]f(x)$ we will mean the coefficient of x^n in the series $f(x)$. Hence the norm of the quaternion Q_n is

$$Q_n Q_n^* = \sum_{i=0}^{3} T_{n+i}^2 = [x^n]\frac{2(3 + 5x + 4x^2 - 2x^3 - x^4 - x^5)}{(1 - 3x - x^2 - x^3)(1 + x + x^2 - x^3)}.$$

For $n \geq 2$, let $A_n = T_{-n}$ and $B_n = K_{-n}$. Then Tribonacci and Tribonacci-Lucas sequences with negative indices are defined by the following equations (see [17]):

$$\begin{aligned} A_n &= -A_{n-1} - A_{n-2} + A_{n-3}\ ; \quad A_{-1} = 1,\ A_0 = A_1 = 0, \\ B_n &= -B_{n-1} - B_{n-2} + B_{n-3}\ ; \quad B_{-1} = 1,\ B_0 = 3,\ B_1 = -1. \end{aligned}$$

Hence we can give the following definition.

Definition 1. *The Tribonacci and Tribonacci-Lucas quaternions with negative subscripts are defined by*

$$\begin{aligned} Q_{-n} &= A_n + \mathbf{i}A_{n-1} + \mathbf{j}A_{n-2} + \mathbf{k}A_{n-3}, \\ \tilde{Q}_{-n} &= B_n + \mathbf{i}B_{n-1} + \mathbf{j}B_{n-2} + \mathbf{k}B_{n-3}. \end{aligned}$$

The generating function and Binet formula for generalized Tribonacci quaternions are given in [1]. For the completeness of the paper, we give the generating function and Binet formula for the Tribonacci quaternions.

Theorem 1. *The generating function for the Tribonacci quaternion Q_n is*

$$G(x)=\frac{x+\mathbf{i}+\mathbf{j}(1+x+x^2)+\mathbf{k}(2+2x+x^2)}{1-x-x^2-x^3}.$$

Proof. Let

$$G(x)=Q_0+Q_1x+Q_2x^2+\cdots+Q_nx^n+\cdots$$

be the generating function of the Tribonacci quaternion Q_n. Since the orders of Q_{n-1}, Q_{n-2} and Q_{n-3} are 1, 2 and 3 less than the order of Q_n, respectively, find $xG(x)$, $x^2G(x)$ and $x^3G(x)$:

$$\begin{aligned}
xG(x) &= Q_0x+Q_1x^2+Q_2x^3+\cdots+Q_{n-1}x^n+\cdots,\\
x^2G(x) &= Q_0x^2+Q_1x^3+Q_2x^4+\cdots+Q_{n-2}x^n+\cdots,\\
x^3G(x) &= Q_0x^3+Q_1x^4+Q_2x^5+\cdots+Q_{n-3}x^n+\cdots.
\end{aligned}$$

Thus

$$G(x)=\frac{Q_0+x(Q_1-Q_0)+x^2(Q_2-Q_1-Q_0)}{1-x-x^2-x^3},$$

and so

$$G(x)=\frac{x+\mathbf{i}+\mathbf{j}(1+x+x^2)+\mathbf{k}(2+2x+x^2)}{1-x-x^2-x^3}.$$

□

Theorem 2. *The Binet formulas for the Tribonacci and Tribonacci-Lucas quaternions are given by*

$$\begin{aligned}
Q_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma},\\
\tilde{Q}_n &= \alpha^n\underline{\alpha}+\beta^n\underline{\beta}+\gamma^n\underline{\gamma}
\end{aligned}$$

where $\underline{\alpha}=1+\mathbf{i}\alpha+\mathbf{j}\alpha^2+\mathbf{k}\alpha^3$, $\underline{\beta}=1+\mathbf{i}\beta+\mathbf{j}\beta^2+\mathbf{k}\beta^3$ *and* $\underline{\gamma}=1+\mathbf{i}\gamma+\mathbf{j}\gamma^2+\mathbf{k}\gamma^3$.

Proof. Using the Binet formulæ for T_n and K_n given in (1.1) and the definition of Q_n and $\tilde{Q}_n$, we obtain the Binet formulæ for Q_n and $\tilde{Q}_n$ as follows,

$$\begin{aligned}
Q_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma},\\
\tilde{Q}_n &= \alpha^n\underline{\alpha}+\beta^n\underline{\beta}+\gamma^n\underline{\gamma}.
\end{aligned}$$

□

3 Some Identities on Tribonacci Quaternions

3.1 Identities 1.

$$Q_n^2 = 2T_nQ_n - Q_nQ_n^*,$$
$$Q_n + Q_n^* = 2T_n,$$
$$\tilde{Q}_n = Q_n + 2Q_{n-1} + 3Q_{n-2}.$$

3.2 Identities 2.

$$\begin{aligned} Q_{m+n} &= Q_mK_n - Q_{m-n}C_n + Q_{m-2n}, \\ \tilde{Q}_{m+n} &= \tilde{Q}_mK_n - \tilde{Q}_{m-n}C_n + \underline{C}_{2n-m}, \end{aligned}$$

where

$$C_n = \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n$$

and

$$\underline{C}_{2n-m} = C_{2n-m} + \mathbf{i}C_{2n-m-1} + \mathbf{j}C_{2n-m-2} + \mathbf{k}C_{2n-m-3}.$$

Another identity can be given as

$$Q_{n+2m} = K_mQ_{n+m} - K_{-m}Q_n + Q_{n-2m}.$$

3.3 Identity 3.

For $n \geq 0$, $m \geq 3$ we have

$$Q_{n+m} = T_{m-2}Q_n + (T_{m-3} + T_{m-2})Q_{n+1} + T_{m-1}Q_{n+2}.$$

3.4 Identities 4.

Let $\widehat{S}_n = \sum\limits_{k=0}^{n} Q_k$. Then we have

$$Q_n = \frac{1}{2}\left[\widehat{S}_n - \widehat{S}_{n-4}\right]$$

and for $n \geq 0$, $m \geq 5$ we have

$$\widehat{S}_{n+m} = -S_{m-3}\widehat{S}_n - S_{m-4}\widehat{S}_{n+1} - S_{m-5}\widehat{S}_{n+2} + S_{m-2}\widehat{S}_{n+3},$$

where $S_m = \sum\limits_{k=0}^{m} T_k$.

3.5 Identity 5.

$$(Q_nQ_{n+4})^2 + (2(Q_{n+1} + Q_{n+2})Q_{n+3})^2 = (Q_n^2 + 2(Q_{n+1} + Q_{n+2})Q_{n+3})^2.$$

3.6 Identities 6.

Let

$$\begin{aligned} R_n &= 3T_{n+1} - T_n \text{ for } n \geq 0, \\ \tilde{R}_n &= R_n + \mathbf{i}R_{n+1} + \mathbf{j}R_{n+2} + \mathbf{k}R_{n+3} \end{aligned}$$

and

$$\begin{aligned} U_n &= T_{n-1} + T_{n-2}\ ; \quad U_0 = U_1 = 0 \text{ for } n \geq 2, \\ \tilde{U}_n &= U_n + \mathbf{i}U_{n+1} + \mathbf{j}U_{n+2} + \mathbf{k}U_{n+3}. \end{aligned}$$

Then we have

$$\tilde{R}_{n+3} = \tilde{R}_{n+2} + \tilde{R}_{n+1} + \tilde{R}_n$$

and

$$\tilde{U}_{n+3} = \tilde{U}_{n+2} + \tilde{U}_{n+1} + \tilde{U}_n.$$

We also obtain the following identities:

$$Q_n^2 - Q_{n-1}^2 = \tilde{U}_{n+1}\tilde{U}_{n-1} \text{ for } n \geq 2,$$

$$\tilde{U}_{n+1}^2 + \tilde{U}_{n-1}^2 = 2(Q_{n-1}^2 + Q_n^2) \text{ for } n \geq 2.$$

3.7 Identities 7.

Now we will give some identities about the finite sums of various quaternions that we obtained.

$$\sum_{k=0}^{n} Q_k = \frac{Q_{n+2} + Q_n + Q_0 - Q_2}{2},$$

$$\sum_{k=0}^{n} Q_{2k} = \frac{Q_{2n+1} + Q_{2n} - (1 + \mathbf{j} + 2\mathbf{k})}{2},$$

$$\sum_{k=0}^{n} Q_{2k+1} = \frac{Q_{2n+2} + Q_{2n+1} - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})}{2},$$

$$\begin{aligned}\sum_{k=0}^{n} Q_{3k} &= \sum_{k=0}^{3n-1} Q_k + Q_0 \\ &= \frac{Q_{3n+2} - Q_{3n} - (1 - \mathbf{i} + \mathbf{j} + \mathbf{k})}{2},\end{aligned}$$

$$\sum_{k=0}^{n} Q_{4k} = \frac{Q_{4n+2} + Q_{4n} - (1 - \mathbf{i} + \mathbf{j} + \mathbf{k})}{4}.$$

We also have

$$\sum_{k=0}^{n} \tilde{U}_n = Q_{n+1} - (1 + \mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

$$\sum_{k=1}^{n} \tilde{Q}_n = 2\tilde{U}_{n+2} + \tilde{U}_n - (3 + 4\mathbf{i} + 7\mathbf{j} + 14\mathbf{k}),$$

$$\sum_{k=0}^{n} Q_k = \frac{\tilde{U}_{n+2} + \tilde{U}_{n+1} - (1 + \mathbf{i} + 3\mathbf{j} + 5\mathbf{k})}{2},$$

$$\sum_{k=0}^{n} \tilde{R}_k = \frac{3\tilde{U}_{n+3} + 2\tilde{U}_{n+2} - \tilde{U}_{n+1} - (2 + 8\mathbf{i} + 12\mathbf{j} + 22\mathbf{k})}{2},$$

$$\sum_{k=0}^{n} \tilde{U}_{3k} = Q_{3n} - \mathbf{i},$$

$$\sum_{k=0}^{n} \tilde{U}_{3k+1} = Q_{3n+1} - (1+\mathbf{k}).$$

4 Proofs

In order to keep this paper within reasonable length, we restricted ourselves to a short selection. Thus we prove some identities using the Binet formulæ and mathematical induction. The other identities can be shown similarly.

4.1 Proof of the Identities 1:

We will give the proof of identity

$$Q_n^2 = 2T_nQ_n - Q_nQ_n^*.$$

We have

$$Q_n^2 = T_n^2 - T_{n+1}^2 - T_{n+2}^2 - T_{n+3}^2 + 2(\mathbf{i}T_nT_{n+1} + \mathbf{j}T_nT_{n+2} + \mathbf{k}T_nT_{n+3}).$$

On the other hand since

$$Q_nQ_n^* = T_n^2 + T_{n+1}^2 + T_{n+2}^2 + T_{n+3}^2$$

and

$$2T_nQ_n = 2T_n^2 + 2(\mathbf{i}T_nT_{n+1} + \mathbf{j}T_nT_{n+2} + \mathbf{k}T_nT_{n+3}),$$

we get the result.

Now we will prove the identity

$$\tilde{Q}_n = Q_n + 2Q_{n-1} + 3Q_{n-2}.$$

The Binet formula of the Tribonacci quaternion is given as

$$Q_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}.$$

Then we have

$$\begin{aligned}
Q_n + 2Q_{n-1} + 3Q_{n-2} &= \left[\tfrac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \tfrac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \tfrac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}\right] \\
&\quad +2\left[\tfrac{\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \tfrac{\beta^{n}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \tfrac{\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}\right] \\
&\quad +3\left[\tfrac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \tfrac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \tfrac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}\right] \\
&= \left[\tfrac{\alpha^{n+1}+2\alpha^n+3\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}\right]\underline{\alpha} + \left[\tfrac{\beta^{n+1}+2\beta^n+3\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}\right]\underline{\beta} \\
&\quad + \left[\tfrac{\gamma^{n+1}+2\gamma^n+3\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\right]\underline{\gamma} \\
&= \alpha^n\left[\tfrac{\alpha^2+2\alpha+3}{\alpha(\alpha-\beta)(\alpha-\gamma)}\right]\underline{\alpha} + \beta^n\left[\tfrac{\beta^2+2\beta+3}{\beta(\beta-\alpha)(\beta-\gamma)}\right]\underline{\beta} \\
&\quad +\gamma^n\left[\tfrac{\gamma^2+2\gamma+3}{\gamma(\gamma-\alpha)(\gamma-\beta)}\right]\underline{\gamma} \\
&= \alpha^n\underline{\alpha} + \beta^n\underline{\beta} + \gamma^n\underline{\gamma} \\
&= \tilde{Q}_n.
\end{aligned}$$

4.2 Proof of the Identities 2:

It is known that the Tribonacci numbers and Tribonacci-Lucas numbers satisfy the equalities,

$$\begin{aligned} T_{m+n} &= T_m K_n - T_{m-n} C_n + T_{m-2n}, \\ K_{m+n} &= K_m K_n - K_{m-n} C_n + C_{2n-m}, \end{aligned}$$

see [17]. Then we have

$$\begin{aligned} Q_{m+n} &= T_{m+n} + \mathbf{i} T_{m+n+1} + \mathbf{j} T_{m+n+2} + \mathbf{k} T_{m+n+3} \\ &= (T_m K_n - T_{m-n} C_n + T_{m-2n}) \\ &\quad + \mathbf{i}(T_{m+1} K_n - T_{m+1-n} C_n + T_{m+1-2n}) \\ &\quad + \mathbf{j}(T_{m+2} K_n - T_{m+2-n} C_n + T_{m+2-2n}) \\ &\quad + \mathbf{k}(T_{m+3} K_n - T_{m+3-n} C_n + T_{m+3-2n}) \\ &= (T_m + \mathbf{i} T_{m+1} + \mathbf{j} T_{m+2} + \mathbf{k} T_{m+3}) K_n - \\ &\quad (T_{m-n} + \mathbf{i} T_{m-n+1} + \mathbf{j} T_{m-n+2} + \mathbf{k} T_{m-n+3}) C_n \\ &\quad + (T_{m-2n} + \mathbf{i} T_{m-2n+1} + \mathbf{j} T_{m-2n+2} + \mathbf{k} T_{m-2n+3}) \\ &= Q_m K_n - Q_{m-n} C_n + Q_{m-2n}. \end{aligned}$$

For all n and m, Tribonacci and Tribonacci-Lucas sequences also satisfy the following equality,

$$T_{n+2m} = K_m T_{n+m} - K_{-m} T_n + T_{n-2m},$$

see [8]. Similarly we obtain the identity

$$Q_{n+2m} = K_m Q_{n+m} - K_{-m} Q_n + Q_{n-2m}.$$

4.3 Proof of the Identity 3:

For $m = 3$, we have

$$\begin{aligned} Q_{n+3} &= Q_n + Q_{n+1} + Q_{n+2} \\ &= T_1 Q_n + (T_0 + T_1) Q_{n+1} + T_2 Q_{n+2}. \end{aligned}$$

Suppose that the equality holds for all $m \leq k$. For $m = k+1$, we have

$$\begin{aligned} Q_{n+k+1} &= Q_{n+k} + Q_{n+k-1} + Q_{n+k-2} \\ &= T_{k-2}Q_n + (T_{k-3} + T_{k-2})Q_{n+1} + T_{k-1}Q_{n+2} + T_{k-3}Q_n \\ &\quad +(T_{k-4} + T_{k-3})Q_{n+1} + T_{k-2}Q_{n+2} + T_{k-4}Q_n \\ &\quad +(T_{k-5} + T_{k-4})Q_{n+1} + T_{k-3}Q_{n+2} \\ &= (T_{k-2} + T_{k-3} + T_{k-4})Q_n + \\ &\quad (T_{k-3} + T_{k-2} + T_{k-4} + T_{k-3} + T_{k-5} + T_{k-4})Q_{n+1} \\ &\quad +(T_{k-1} + T_{k-2} + T_{k-3})Q_{n+2} \\ &= T_{k-1}Q_n + (T_{k-2} + T_{k-1})Q_{n+1} + T_k Q_{n+2}. \end{aligned}$$

By induction on m, we get the result.

4.4 Proof of the Identities 4:

Since

$$\begin{aligned} \widehat{S}_n &= Q_0 + Q_1 + \cdots + Q_n \\ &= Q_0 + Q_1 + \cdots + Q_{n-4} + Q_{n-3} + Q_{n-2} + Q_{n-1} + Q_n \\ &= \widehat{S}_{n-4} + Q_n + Q_n \\ &= \widehat{S}_{n-4} + 2Q_n, \end{aligned}$$

we obtain that

$$Q_n = \frac{1}{2}\left[\widehat{S}_n - \widehat{S}_{n-4}\right].$$

For the other identity the proof will be done by induction on n and m. First we will prove the identity

$$\widehat{S}_{n+5} = -2\widehat{S}_n - \widehat{S}_{n+1} + 4\widehat{S}_{n+3}.$$

For $n = 0$, we have

$$\begin{aligned} \widehat{S}_5 &= Q_0 + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \\ &= Q_0 + Q_1 + Q_2 + Q_3 + (Q_1 + Q_2 + Q_3) + (Q_2 + Q_3 + Q_4) \\ &= Q_0 + Q_1 + Q_2 + Q_3 + (Q_1 + Q_2 + Q_3) + (Q_2 + Q_3 + Q_1 + Q_2 + Q_3) \\ &= Q_0 + 3Q_1 + 4Q_2 + 4Q_3 \\ &= -2Q_0 - Q_0 - Q_1 + 4(Q_0 + Q_1 + Q_2 + Q_3) \\ &= -2\widehat{S}_0 - \widehat{S}_1 + 4\widehat{S}_3. \end{aligned}$$

Suppose the equality holds for $n = k$, that is,

$$\widehat{S}_{k+5} = -2\widehat{S}_k - \widehat{S}_{k+1} + 4\widehat{S}_{k+3}.$$

For $n = k + 1$, we have

$$\begin{aligned}
\widehat{S}_{k+6} &= \widehat{S}_{k+5} + Q_{k+6} \\
&= -2\widehat{S}_k - \widehat{S}_{k+1} + 4\widehat{S}_{k+3} + Q_{k+6} \\
&= -2\widehat{S}_k - \widehat{S}_{k+1} + 4\widehat{S}_{k+3} + (Q_{k+3} + Q_{k+4} + Q_{k+5}) \\
&= -2\widehat{S}_k - \widehat{S}_{k+1} + 4\widehat{S}_{k+3} \\
&\quad +(Q_{k+4} - Q_{k+2} - Q_{k+1} + Q_{k+4} + Q_{k+4} - Q_{k+2} - Q_{k+1} + Q_{k+4}) \\
&= -2\widehat{S}_{k+1} - \widehat{S}_{k+2} + 4\widehat{S}_{k+4}.
\end{aligned}$$

So the equality holds for all $n \geq 0$.

For $m = 5$, we have

$$\begin{aligned}
\widehat{S}_{n+5} &= -2\widehat{S}_{k+1} - \widehat{S}_{k+2} + 4\widehat{S}_{k+4}. \\
&= -S_2\widehat{S}_n - S_1\widehat{S}_{n+1} - S_0\widehat{S}_{n+2} + S_3\widehat{S}_{n+3}.
\end{aligned}$$

Suppose the equality holds for $m = r$, that is

$$\widehat{S}_{n+r} = -S_{r-3}\widehat{S}_n - S_{r-4}\widehat{S}_{n+1} - S_{r-5}\widehat{S}_{n+2} + S_{r-2}\widehat{S}_{n+3}.$$

For $m = r + 1$,

$$\begin{aligned}
\widehat{S}_{n+r+1} &= \widehat{S}_{n+r} + Q_{n+r+1} \\
&= -S_{r-3}\widehat{S}_n - S_{r-4}\widehat{S}_{n+1} - S_{r-5}\widehat{S}_{n+2} + S_{r-2}\widehat{S}_{n+3} \\
&\quad +(Q_{n+r-2} + Q_{n+r-1} + Q_{n+r}) \\
&= -S_{r-2}\widehat{S}_n - S_{r-3}\widehat{S}_{n+1} - S_{r-4}\widehat{S}_{n+2} + S_{r-1}\widehat{S}_{n+3}.
\end{aligned}$$

By induction on m, we get the result.

4.5 Proof of the Identity 5:

We have

$$\begin{aligned}
Q_n^2 &= (Q_{n+3} - (Q_{n+1} + Q_{n+2}))^2 \\
&= Q_{n+3}^2 + (Q_{n+1} + Q_{n+2})^2 - 2(Q_{n+1} + Q_{n+2})Q_{n+3}
\end{aligned}$$

and this gives

$$Q_n^2 + 2(Q_{n+1} + Q_{n+2})Q_{n+3} = Q_{n+3}^2 + (Q_{n+1} + Q_{n+2})^2.$$

Thus

$$\begin{aligned}(Q_n^2+2(Q_{n+1}+Q_{n+2})Q_{n+3})^2 &= Q_{n+3}^4+(Q_{n+1}+Q_{n+2})^4\\ &\quad +2((Q_{n+1}+Q_{n+2})Q_{n+3})^2\\ &= (Q_{n+3}^2-(Q_{n+1}+Q_{n+2})^2)^2\\ &\quad +(2(Q_{n+1}+Q_{n+2})Q_{n+3})^2.\end{aligned}$$

Here

$$\begin{aligned}(Q_{n+3}^2-(Q_{n+1}+Q_{n+2})^2)^2 &= (Q_{n+3}-(Q_{n+1}+Q_{n+2}))^2\\ &\quad +(Q_{n+3}+(Q_{n+1}+Q_{n+2}))^2\\ &= Q_n^2Q_{n+4}^2.\end{aligned}$$

Substituting this gives the result.

4.6 Proof of the Identity 6:

For $n \geq 2$, we have

$$\begin{aligned}&Q_n^2-Q_{n-1}^2=(Q_n+Q_{n-1})(Q_n-Q_{n-1})\\ =\ &[(T_n+T_{n-1})+\mathbf{i}(T_{n+1}+T_n)+\mathbf{j}(T_{n+2}+T_{n+1})+\mathbf{k}(T_{n+3}+T_{n+2})]\\ &\times[(T_n-T_{n-1})+\mathbf{i}(T_{n+1}-T_n)+\mathbf{j}(T_{n+2}-T_{n+1})+\mathbf{k}(T_{n+3}-T_{n+2})]\\ =\ &[(T_n+T_{n-1})+\mathbf{i}(T_{n+1}+T_n)+\mathbf{j}(T_{n+2}+T_{n+1})+\mathbf{k}(T_{n+3}+T_{n+2})]\\ &\times[(T_{n-2}+T_{n-3})+\mathbf{i}(T_{n-1}+T_{n-2})+\mathbf{j}(T_n+T_{n-1})+\mathbf{k}(T_{n+1}+T_n)]\\ =\ &\tilde{U}_{n+1}\tilde{U}_{n-1}.\end{aligned}$$

4.7 Proof of the Identities 7:

We will show the identity

$$\sum_{k=0}^{n} Q_k = \frac{Q_{n+2}+Q_n+Q_0-Q_2}{2}.$$

The others can be done similarly. The proof can be done by induction on n. For $n=0$ we have

$$Q_0=\frac{Q_2+Q_0+Q_0-Q_2}{2}.$$

So equality holds for $n=0$. Assume it is true for $n=m$, that is,

$$\sum_{k=0}^{m} Q_k = \frac{Q_{m+2}+Q_m+Q_0-Q_2}{2}.$$

For $n = m + 1$, we have

$$\sum_{k=0}^{m+1} Q_k = \sum_{k=0}^{m} Q_k + Q_{m+1}.$$

By induction hypothesis we can write

$$\begin{aligned}
\sum_{k=0}^{m} Q_k + Q_{m+1} &= \frac{Q_{m+2} + Q_m + Q_0 - Q_2}{2} + Q_{m+1} \\
&= \frac{Q_{m+2} + Q_m + Q_0 - Q_2 + 2Q_{m+1}}{2} \\
&= \frac{Q_{m+2} + Q_{m+1} + Q_m + Q_{m+1} + Q_0 - Q_2}{2} \\
&= \frac{Q_{m+3} + Q_{m+1} + Q_0 - Q_2}{2}.
\end{aligned}$$

Hence we obtain that

$$\sum_{k=0}^{m+1} Q_k = \frac{Q_{m+3} + Q_{m+1} + Q_0 - Q_2}{2}.$$

This shows that equality holds for all $n \geq 0$.

5 An Isomorphism on the Tribonacci Quaternions

We consider the Tribonacci and Tribonacci-Lucas quaternions. These quaternions can be written as

$$\begin{aligned}
Q_n &= T_n + A \\
\tilde{Q}_n &= K_n + B
\end{aligned}$$

where $A = \mathrm{Im}(Q_n)$ and $B = \mathrm{Im}(\tilde{Q}_n)$. Let

$$\mathcal{QM} = \{Q_n : Q_n \text{ is the } n^{th} \text{ Tribonacci quaternion}\}$$

and $\mathcal{M}$ is the set of 2×2 matrices with entries from $\mathbb{C}$ of the form:

$$\mathcal{M} = \left\{ X_n : X_n = \begin{bmatrix} z & -w \\ \overline{w} & \overline{z} \end{bmatrix} \ ; z, w \in \mathbb{C} \right\}.$$

Then each matrix can be decomposed into a vector space representation with four basis elements. Let Φ be the following map:

$$\begin{aligned}
\Phi \quad &: \quad \mathcal{QM} \to \mathcal{M} \\
Q_n \quad &\mapsto \quad X_n = \begin{bmatrix} T_n + \mathbf{i}T_{n+1} & -T_{n+2} - \mathbf{i}T_{n+3} \\ T_{n+2} - \mathbf{i}T_{n+3} & T_n - \mathbf{i}T_{n+1} \end{bmatrix}.
\end{aligned}$$

Then it can be easily show that Φ is an isomorphism. Thus we can write

$$X_n = T_n E + T_{n+1} I + T_{n+2} J + T_{n+3} K$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix},\ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ K = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}.$$

Since $\det(X_n) \neq 0$, X_n is an invertible matrix and its inverse is in $\mathcal{M}$.

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