DOI: 10.2478/auom-2018-0010
An. Şt. Univ. Ovidius Constanţa Vol. 26(1),2018, 159–168

# Commutator Subgroups of Generalized Hecke and Extended Generalized Hecke Groups

**Şule Kaymak (Sarıca), Bilal Demir, Özden Koruoğlu and Recep Şahin**

**Abstract**

Let $p$ and $q$ be integers such that $2 \leq p \leq q$, $p+q > 4$ and let $H_{p,q}$ be the generalized Hecke group associated to $p$ and $q$. The generalized Hecke group $H_{p,q}$ is generated by $X(z) = -(z-\lambda_p)^{-1}$ and $Y(z) = -(z+\lambda_q)^{-1}$ where $\lambda_p = 2\cos\frac{\pi}{p}$ and $\lambda_q = 2\cos\frac{\pi}{q}$. The extended generalized Hecke group $\overline{H}_{p,q}$ is obtained by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke group $H_{p,q}$. In this paper, we study the commutator subgroups of generalized Hecke groups $H_{p,q}$ and extended generalized Hecke groups $\overline{H}_{p,q}$.

## 1 Introduction

In [5], Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where $\lambda$ is a fixed positive real number. Let $S = TU$, i.e.,

$$S(z) = -\frac{1}{z+\lambda}.$$

Key Words: Generalized Hecke groups, Extended generalized Hecke groups, commutator subgroups.
2010 Mathematics Subject Classification: 20H10, 11F06, 30F35
Received: 06.02.2016
Revised: 15.06.2017
Accepted: 20.06.2017

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case $q \geq 3$ integer and we denote it by $H_q = H(\lambda_q)$. The Hecke group $H_q$ is isomorphic to the free product of two finite cyclic groups of orders 2 and $q$,

$$H_q =< T, S : T^2 = S^q = I >\simeq C_2 * C_q.$$

The first few Hecke groups $H_q$ are $H_3 = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$, and $H_6 = H(\sqrt{3})$. It is clear from the above that $H_q \subset PSL(2,\mathbb{Z}[\lambda_q])$ unlike in the modular group case (the case $q = 3$), the inclusion is strict and the index $|PSL(2,\mathbb{Z}[\lambda_q]) : H_q|$ is infinite as $H_q$ is discrete whereas $PSL(2,\mathbb{Z}[\lambda_q])$ is not for $q \geq 4$.

Lehner studied in [11] a more general class $H_{p,q}$ of Hecke groups $H_q$, by taking

$$X = \frac{-1}{z-\lambda_p} \text{ and } V = z + \lambda_p + \lambda_q,$$

where $2 \leq p \leq q$, $p + q > 4$. Here if we take $Y = XV = -\frac{1}{z+\lambda_q}$, then we have the presentation,

$$H_{p,q} =< X, Y : X^p = Y^q = I >\simeq C_p * C_q.$$

In particular $H_{p,q}$ has the signature $(0; p, q, \infty)$. We call these groups as *generalized Hecke groups* $H_{p,q}$. We know from [11] that $H_{2,q} = H_q$, $|H_q : H_{q,q}| = 2$, and there is no group $H_{2,2}$. Furthermore all Hecke groups $H_q$ are included in generalized Hecke groups $H_{p,q}$. Generalized Hecke groups $H_{p,q}$ have been also studied by Calta and Schmidt in [2] and [3].

Extended generalized Hecke groups $\overline{H}_{p,q}$ can be defined similar to extended Hecke groups $\overline{H}_q$, by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke group $H_{p,q}$. Hence extended generalized Hecke groups $\overline{H}_{p,q}$ have a presentation

$$\overline{H}_{p,q} =< X, Y, R : X^p = Y^q = R^2 = I,\ RX = X^{-1}R, RY = Y^{-1}R >,$$

that is

$$\overline{H}_{p,q} =< X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I >\cong D_p *_{\mathbb{Z}_2} D_q.$$

The groups $H_{p,q}$ is a subgroup of index 2 in $\overline{H}_{p,q}$.

Now we focus on the commutator subgroups of $H_{p,q}$ and $\overline{H}_{p,q}$. The commutator subgroups of $H_q$, $\overline{H}_q$, $H_q^m$ ($m-$th power subgroup of $H_q$) and $H'_{p,q}$ ($p$ and $q$ are relatively prime) have been studied by many authors see [1], [16], [18], [19], [20] and [22].

In this paper, we study the commutator subgroups of $H_{p,q}$ and $\overline{H}_{p,q}$. We give the generators, the group structures and the signatures of the commutator subgroups of $H_{p,q}$ and $\overline{H}_{p,q}$. Here we use the Reidemeister-Schreier method, the permutation method (see, [21]) and the extended Riemann-Hurwitz condition (see, [10]) to get our results.

**Remark 1.** *i) The Hecke groups $H_q$, the extended Hecke groups $\overline{H}_q$ and their normal subgroups have been studied extensively for many aspects in the literature. For examples, see [4], [8], [9], [12] and [14]. Also, there are many relationships between the groups $H_q$ (or $\overline{H}_q$) and the automorphism groups of Riemann (or Klein) surfaces or of regular maps $\{p, q\}$. Naturally, many results related with $H_q$ and $\overline{H}_q$ can be generalized to the groups $H_{p,q}$ and $\overline{H}_{p,q}$.*

*ii) Generalized Hecke groups $H(p_1, p_2, ..., p_n)$ and extended generalized Hecke groups $H^*(p_1, p_2, ..., p_n)$ have been introduced first by Huang in [6]. Our studied groups are the special cases ($n = 2$) of these groups given in [6] and [7].*

## 2 Commutator Subgroups of Generalized Hecke Groups $H_{p,q}$

In this section, we study the commutator subgroup $H'_{p,q}$ of generalized Hecke groups $H_{p,q}$. We use standard notation, in particular, $G^(n)$ denotes the $n^th$ derived group of a group $G$.

**Theorem 1.** *Let $p$ and $q$ be integers such that $2 \leq p \leq q$, $p + q > 4$.*

*i) $\left|H_{p,q} : H'_{p,q}\right| = pq$.*

*ii) The commutator subgroup $H'_{p,q}$ of $H_{p,q}$ is a free group of rank $(p-1)(q-1)$ with basis $[X, Y]$, $[X, Y^2]$, ..., $[X, Y^{q-1}]$, $[X^2, Y]$, $[X^2, Y^2]$, ..., $[X^2, Y^{q-1}]$, ..., $[X^{p-1}, Y]$, $[X^{p-1}, Y^2]$, ..., $[X^{p-1}, Y^{q-1}]$. And the signature of $H'_{p,q}$ is $(\frac{pq-p-q-(p,q)+2}{2}; \infty^{(p,q)})$ where $(p, q)$ is the greatest common divisor of $p$ and $q$.*

*iii) For $n \geq 2$, $\left|H_{p,q} : H^{(n)}_{p,q}\right| = \infty$.*

*Proof.* *i)* Firstly, we set up the quotient group $H_{p,q}/H'_{p,q}$. The quotient group $H_{p,q}/H'_{p,q}$ is the group obtained by adding the relation $XY = YX$ to the relations of $H_{p,q}$. Thus we have

$$H_{p,q}/H'_{p,q} =< X, Y : X^p = Y^q = I,\ XY = YX >\simeq C_p \times C_q.$$

*ii)* Now we can determine the generators of $H'_{p,q}$ by the Reidemeister-Schreier method. To do this, we choose the set $\Sigma = \{I,\ X,\ X^2,\ ...,\ X^{p-1},\ Y,\ Y^2,\ ...,$

$Y^{q-1}$, $XY$, $XY^2$, ..., $XY^{q-1}$, $X^2Y$, $X^2Y^2$, ..., $X^2Y^{q-1}$, ..., $X^{p-1}Y$, $X^{p-1}Y^2$, ..., $X^{p-1}Y^{q-1}\}$ as a Schreier transversal. All possible products are

$$\begin{array}{l} I.X.(X)^{-1} = I, \\ X.X.(X^2)^{-1} = I, \\ \quad\vdots \\ X^{p-1}.X.(I)^{-1} = I, \end{array}$$

$$\begin{array}{l}
Y.X.(XY)^{-1} = YXY^{-1}X^{-1} = [Y,X], \\
Y^2.X.(XY^2)^{-1} = Y^2X(XY^2)^{-1} = [Y^2,X], \\
\quad\vdots \\
Y^{q-1}.X.(XY^{q-1})^{-1} = Y^{q-1}XY^{-(q-1)}X^{-1} = [Y^{q-1},X], \\
XY.X.(X^2Y)^{-1} = XYXY^{-1}X^{-2} = [X,Y].[Y,X^2], \\
XY^2.X.(X^2Y^2)^{-1} = XY^2XY^{-2}X^{-2} = [X,Y^2].[Y^2,X^2], \\
\quad\vdots \\
XY^{q-1}.X.(X^2Y^{q-1})^{-1} = XY^{q-1}XY^{-(q-1)}X^{-2} = [X,Y^{q-1}].[Y^{q-1},X^2], \\
X^2Y.X.(X^3Y)^{-1} = X^2YXY^{-1}X^{-3} = [X^2,Y].[Y,X^3], \\
X^2Y^2.X.(X^3Y^2)^{-1} = X^2Y^2XY^{-2}X^{-3} = [X^2,Y^2].[Y^2,X^3], \\
\quad\vdots \\
X^2Y^{q-1}.X.(X^3Y^{q-1})^{-1} = X^2Y^{q-1}XY^{-(q-1)}X^{-3} = [X^2,Y^{q-1}].[Y^{q-1},X^3], \\
\quad\vdots \\
X^{p-1}Y.X.(Y)^{-1} = X^{p-1}YXY^{-1} = [X^{-1},Y], \\
X^{p-1}Y^2.X.(Y^2)^{-1} = X^{p-1}Y^2XY^{-2} = [X^{-1},Y^2], \\
\quad\vdots \\
X^{p-1}Y^{q-1}.X.(Y^{q-1})^{-1} = X^{p-1}Y^{q-1}XY^{-(q-1)} = [X^{-1},Y^{q-1}].
\end{array}$$

The other products are equal to the identity. Thus, we find the generators of $H'_{p,q}$ as $[X,Y]$, $[X,Y^2]$, ..., $[X,Y^{q-1}]$, $[X^2,Y]$, $[X^2,Y^2]$, ..., $[X^2,Y^{q-1}]$, ..., $[X^{p-1},Y]$, $[X^{p-1},Y^2]$, ..., $[X^{p-1},Y^{q-1}]$. The signature of $H'_{p,q}$ can be obtained by permutation method and Riemann-Hurtwitz formula.

*iii*) If we take relations and abelianizing, we find that the resulting quotient is infinite. Thus, it follows that $H''_{p,q}$ has infinite index in $H'_{p,q}$. Further since this has infinite index it follows that in each group in the derived series from this point on have infinite index. □

**Example 1.** *If* $p = 3$ *and* $q = 4$, *then* $\left|H_{3,4} : H'_{3,4}\right| = 12$. *We choose* $\Sigma = \{I$, $X$, $X^2$, $Y$, $Y^2$, $Y^3$, $XY$, $XY^2$, $XY^3$, $X^2Y$, $X^2Y^2$, $X^2Y^3\}$ *as a Schreier transversal for* $H'_{3,4}$. *Using the Reidemeister-Schreier method, we get the generators of* $H'_{3,4}$ *as* $[X,Y]$, $[X,Y^2]$, $[X,Y^3]$, $[X^2,Y]$, $[X^2,Y^2]$, $[X^2,Y^3]$. *Also the signature of* $H'_{3,4}$ *is* $(3;\infty)$.

**Corollary 1.** *If $p = 2$, then the generators of $H'_{2,q}$ are $[X,Y]$, $[X,Y^2]$, ..., $[X,Y^{q-1}]$. Also, the signatures of $H'_{2,q}$ is either $(\frac{q-1}{2};\infty)$ if $q$ is odd, or $(\frac{q-2}{2};\infty^{(2)})$ if $q$ is even. These results coincide with the ones given in [1] and [20], for Hecke groups $H_q$.*

## 3 Commutator Subgroups of Extended Generalized Hecke Groups $\overline{H}_{p,q}$

In this section, we study the first commutator subgroups $\overline{H}'_{p,q}$ of extended generalized Hecke groups $\overline{H}_{p,q}$.

**Theorem 2.** *Let $p$ and $q$ be integers such that $2 \leq p \leq q$, $p+q > 4$.*

*i) If $p$ and $q$ are both odd numbers, then $\overline{H}'_{p,q} =< X,Y : X^p = Y^q = I >\simeq C_p * C_q$.*

*ii) If $p$ is an odd number and $q$ is an even number, then $\overline{H}'_{p,q} =< X, YXY^{-1}, Y^2 : X^p = (YXY^{-1})^p = (Y^2)^{\frac{q}{2}} = I >\simeq C_p * C_p * C_{q/2}$.*

*iii) If $p$ is an even number and $q$ is an odd number, then $\overline{H}'_{p,q} =< X^2, Y, XYX^{-1} : (X^2)^{\frac{p}{2}} = Y^q = (XYX^{-1})^q = I >\simeq C_{p/2} * C_q * C_q$.*

*iv) If $p$ and $q$ are both even numbers, then $\overline{H}'_{p,q} =< X^2,\ YX^2Y^{-1},\ Y^2,\ XY^2X^{-1}, XYXY^{-1} : (X^2)^{p/2} = (YX^2Y^{-1})^{p/2} = (Y^2)^{q/2} = (XY^2X^{-1})^{q/2} = I >\simeq C_{p/2} * C_{p/2} * C_{q/2} * C_{q/2} * \mathbb{Z}$.*

*Proof.* The quotient group $\overline{H}_{p,q}/\overline{H}'_{p,q}$ is

$$\begin{aligned} \overline{H}_{p,q}/\overline{H}'_{p,q} =< X,Y,R : X^p = Y^q = R^2 = I,\ XR = RX^{-1},\ YR = RY^{-1}, \\ XY = YX,\ XR = RX,\ YR = RY > . \end{aligned} \tag{3.1}$$

$i)$ If $p$ and $q$ are both odd numbers, then from (3.1), we get

$$\overline{H}_{p,q}/\overline{H}'_{p,q} =< R : R^2 = I >\simeq C_2.$$

Now we can determine the Schreier transversal as $\Sigma = \{I,\ R\}$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll} I.X.(I)^{-1} = X, & I.Y.(I)^{-1} = Y, & I.R.(R)^{-1} = I, \\ R.X.(R)^{-1} = RXR, & R.Y.(R)^{-1} = RYR, & R.R.(I)^{-1} = I. \end{array}$$

Since $RXR = X^{-1}$ and $RYR = Y^{-1}$, the generators of $\overline{H}'_{p,q}$ are $X$ and $Y$. Thus we have $\overline{H}'_{p,q} =< X,Y : X^p = Y^q = I >\simeq H_{p,q}$. The signature of $\overline{H}'_{p,q}$ is $(0;p,q,\infty)$.

*ii*) If $p$ is an odd number and $q$ is an even number, then from (3.1), we obtain

$$\overline{H}_{p,q}/\overline{H}'_{p,q} =< Y, R : Y^2 = R^2 = I, YR = RY >\simeq C_2 \times C_2.$$

Now we can determine the Schreier transversal as $\Sigma = \{I, Y, R, YR\}$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll}
I.X.(I)^{-1} = X, & I.Y.(Y)^{-1} = I, & I.R.(R)^{-1} = I, \\
Y.X.(Y)^{-1} = YXY^{-1}, & Y.Y.(I)^{-1} = Y^2, & Y.R.(YR)^{-1} = I, \\
R.X.(R)^{-1} = RXR, & R.Y.(YR)^{-1} = RYRY^{-1}, & R.R.(I)^{-1} = I, \\
YR.X.(YR)^{-1} = YRXRY^{-1}, & YR.Y.(R)^{-1} = YRYR, & YR.R.(Y)^{-1} = I.
\end{array}$$

After required calculations, we get the generators of $\overline{H}'_{p,q}$ as $X, YXY^{-1}$ and $Y^2$. Thus we obtain $\overline{H}'_{p,q} =< X, YXY^{-1}, Y^2 : X^p = (YXY^{-1})^p = (Y^2)^{\frac{q}{2}} = I >\simeq C_p * C_p * C_{q/2}$. The signature of $\overline{H}'_{p,q}$ is $(0; p, p, (q/2), \infty)$.

*iii*) The proof is similar to ii). Also, the signature of $\overline{H}'_{p,q}$ is $(0; (p/2), q, q, \infty)$.

*iv*) If $p$ and $q$ are both even numbers, then from (3.1), we find

$$\overline{H}_{p,q}/\overline{H}'_{p,q} =< X, Y, R : X^2 = Y^2 = R^2 = (XY)^2 = (XR)^2 = (YR)^2 = I > \simeq C_2 \times C_2 \times C_2.$$

Now we can determine the Schreier transversal as $\Sigma = \{I, X, Y, R, XY, XR, YR, XYR\}$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{ll}
I.X.(X)^{-1} = I, & I.Y.(Y)^{-1} = I, \\
X.X.(I)^{-1} = X^2, & X.Y.(XY)^{-1} = I, \\
Y.X.(XY)^{-1} = YXY^{-1}X^{-1}, & Y.Y.(I)^{-1} = Y^2, \\
R.X.(XR)^{-1} = RXRX^{-1}, & R.Y.(YR)^{-1} = RYRY^{-1}, \\
XY.X.(Y)^{-1} = XYXY^{-1}, & XY.Y.(X)^{-1} = XY^2X^{-1}, \\
XR.X.(R)^{-1} = XRXR, & XR.Y.(XYR)^{-1} = XRYRY^{-1}X^{-1}, \\
YR.X.(XYR)^{-1} = YRXRY^{-1}X^{-1}, & YR.Y.(R)^{-1} = YRYR, \\
XYR.X.(YR)^{-1} = XYRXRY^{-1}, & XYR.Y.(XR)^{-1} = XYRYRX^{-1},
\end{array}$$

The other products are equal to the identity. Since $RXRX^{-1} = X^{-2}$, $XRXR = I$, $YRXRY^{-1}X^{-1} = YXY^{-1}X^{-1}$, $XYRXRY^{-1} = XYXY^{-1}$, $RYRY^{-1} = Y^{-2}$, $XRYRY^{-1}X^{-1} = I$, $YRYR = I$ and $XYRYRX^{-1} = XY^2X^{-1}$, we have the generators of $\overline{H}'_{p,q}$ as $X^2, YX^2Y^{-1}$, $Y^2$, $XY^2X^{-1}$, $XYXY^{-1}$. Then we get $\overline{H}'_{p,q} =< X^2, YX^2Y^{-1}, Y^2, XY^2X^{-1}, XYXY^{-1} : (X^2)^{p/2} = (YX^2Y^{-1})^{p/2} = (Y^2)^{q/2} = (XY^2X^{-1})^{q/2} = I > \simeq C_{p/2} * C_{p/2} * C_{q/2} * C_{q/2} * \mathbb{Z}$. Also, the signature of $\overline{H}'_{p,q}$ is $(0; (p/2)^{(2)}, (q/2)^{(2)}, \infty^{(2)})$. $\square$

**Corollary 2.** 1) *The first commutator subgroup of* $\overline{H}_{p,q}$ *does not contain any reflection and so* $\overline{H}'_{p,q}$ *is a subgroup of* $H_{p,q}$.

2) *If* $p = 2$, *then the first commutator subgroups* $\overline{H}'_{2,q}$ *coincide with the ones given in [16] for extended Hecke groups* $\overline{H}_q$.

3) $H'_{p,q} \leq \overline{H}'_{p,q} \leq \overline{H}_{p,q}$.

Now we study the second commutator subgroup $\overline{H}''_{p,q}$ of $\overline{H}_{p,q}$, except the case $p$ and $q$ are both even numbers, since in this case $\overline{H}''_{p,q}$ has infinite index in $\overline{H}_{p,q}$.

**Theorem 3.** *Let* $p$ *and* $q$ *be integers such that* $2 \leq p \leq q$, $p + q > 4$.

*i) If* $p$ *and* $q$ *are both odd numbers, then* $\overline{H}''_{p,q} = H'_{p,q}$.

*ii) If* $p$ *is an odd number and* $q$ *is an even number, then* $\overline{H}''_{p,q}$ *is a free group of rank* $(q-1)p^2 - pq + 1$.

*iii) If* $p$ *is an even number and* $q$ *is an odd number, then* $\overline{H}''_{p,q}$ *is a free group of rank* $(p-1)q^2 - pq + 1$.

*Proof.* The case *i*) can be seen from the Theorem 2.1. Since the cases *ii*) and *iii*) are similar, we prove only the case *iii*). If we take $a = X^2$, $b = Y$, $c = XYX^{-1}$, then the quotient group $\overline{H}'_{p,q}/\overline{H}''_{p,q}$ is the group obtained by adding the relations $ab = ba$, $ac = ca$ and $bc = cb$ to the relations of $\overline{H}'_{p,q}$.Then

$$\overline{H}'_{p,q}/\overline{H}''_{p,q} \cong C_{p/2} \times C_q \times C_q.$$

Therefore, we obtain $\left|\overline{H}'_{p,q} : \overline{H}''_{p,q}\right| = pq^2/2$. Now we choose $\Sigma = \{I,\ a,\ a^2,\ \cdots,\ a^{(p/2)-1},\ b,\ b^2,\ \cdots,\ b^{q-1},\ c,\ c^2,\ \cdots,\ c^{q-1},\ ab,\ ab^2, \cdots,\ ab^{q-1},\ a^2b,\ a^2b^2,\ \cdots,\ a^2b^{q-1}, \cdots,\ a^{(p/2)-1}b,\ a^{(p/2)-1}b^2,\ \cdots,\ a^{(p/2)-1}b^{q-1},\ ac,\ ac^2, \cdots,\ ac^{q-1},\ a^2c,\ a^2c^2,\ \cdots,\ a^2c^{q-1}, \cdots,\ a^{(p/2)-1}c,\ a^{(p/2)-1}c^2,\ \cdots,\ a^{(p/2)-1}c^{q-1},\ bc,\ bc^2,\ \cdots,\ bc^{q-1},\ b^2c,\ b^2c^2,\ \cdots,\ b^2c^{q-1}, \cdots,\ b^{q-1}c,\ b^{q-1}c^2,\ \cdots,\ b^{q-1}c^{q-1},\ abc,\ abc^2,\ \cdots,\ abc^{q-1},\ ab^2c,\ ab^2c^2, \cdots, ab^2c^{q-1}, \cdots, ab^{q-1}c, ab^{q-1}c^2, \cdots, ab^{q-1}c^{q-1}, \cdots, a^{(p/2)-1}bc, a^{(p/2)-1}bc^2,\ \cdots,\ a^{(p/2)-1}bc^{q-1}, a^{(p/2)-1}b^2c, a^{(p/2)-1}b^2c^2, \cdots, a^{(p/2)-1}b^2c^{q-1}, \cdots, a^{(p/2)-1}b^{q-1}c,\ a^{(p/2)-1}b^{q-1}c^2,\ \cdots,\ a^{(p/2)-1}b^{q-1}c^{q-1}\ \}$ as a Schreier transversal for $\overline{H}''_{p,q}$.

According to the Reidemeister-Schreier method, we get the generators of $\overline{H}''_{p,q}$ as follows. There are total $(p-1)q^2 - pq + 1$ generators obtained by using the theorem of Nielsen in [13]. These generators are $[a,b]$, $[a,b^2]$, $\cdots$, $[a,b^{q-1}]$, $[a^2,b]$, $[a^2,b^2]$, $\cdots$, $[a^2,b^{q-1}]$, $\cdots$, $[a^{(p/2)-1},b]$, $[a^{(p/2)-1},b^2]$, $\cdots$, $[a^{(p/2)-1},b^{q-1}]$, $[a,c]$, $[a,c^2]$, $\cdots$, $[a,c^{q-1}]$, $[a^2,c]$, $[a^2,c^2]$, $\cdots$, $[a^2,c^{q-1}]$, $\cdots$, $[a^{(p/2)-1},c]$, $[a^{(p/2)-1},c^2]$, $\cdots$, $[a^{(p/2)-1},c^{q-1}]$, $[b,c]$, $[b,c^2]$, $\cdots$, $[b,c^{q-1}]$, $[b^2,c]$, $[b^2,c^2]$, $\cdots$, $[b^2,c^{q-1}]$, $\cdots$, $[b^{q-1},c]$, $[b^{q-1},c^2]$, $\cdots$, $[b^{q-1},c^{q-1}]$, $[a,bc]$, $[a,b^2c]$, $\cdots$, $[a,b^{q-1}c]$, $[a,bc^2]$, $[a,b^2c^2]$, $\cdots$, $[a,b^{q-1}c^2]$, $\cdots$, $[a,bc^{q-1}]$, $[a,b^2c^{q-1}]$, $\cdots$, $[a,b^{q-1}c^{q-1}]$, $[a^2,bc]$, $[a^2,b^2c]$, $\cdots$,

$[a^2, b^{q-1}c]$, $[a^2, bc^2]$, $[a^2, b^2c^2]$, $\cdots$, $[a^2, b^{q-1}c^2]$, $\cdots$, $[a^2, bc^{q-1}]$, $[a^2, b^2c^{q-1}]$, $\cdots$, $[a^2, b^{q-1}c^{q-1}]$, $\cdots$, $[a^{(p/2)-1}, bc]$, $[a^{(p/2)-1}, b^2c]$, $\cdots$, $[a^{(p/2)-1}, b^{q-1}c]$, $[a^{(p/2)-1}, bc^2]$, $[a^{(p/2)-1}, b^2c^2]$, $\cdots$, $[a^{(p/2)-1}, b^{q-1}c^2]$, $\cdots$, $[a^{(p/2)-1}, bc^{q-1}]$, $[a^{(p/2)-1}, b^2c^{q-1}]$, $\cdots$, $[a^{(p/2)-1}, b^{q-1}c^{q-1}]$, $[ab, bc]$, $[ab, b^2c]$, $\cdots$, $[ab, b^{q-1}c]$, $[ab, bc^2]$, $[ab, b^2c^2]$, $\cdots$, $[ab, b^{q-1}c^2]$, $\cdots$, $[ab, bc^{q-1}]$, $[ab, b^2c^{q-1}]$, $\cdots$, $[ab, b^{q-1}c^{q-1}]$, $[a^2b, bc]$, $[a^2b, b^2c]$, $\cdots$, $[a^2b, b^{q-1}c]$, $[a^2b, bc^2]$, $[a^2b, b^2c^2]$,$\cdots$,$[a^2b, b^{q-1}c^2]$, $\cdots$, $[a^2b, bc^{q-1}]$, $[a^2b, b^2c^{q-1}]$,$\cdots$, $[a^2b, b^{q-1}c^{q-1}]$, $\cdots$,$[a^{(p/2)-1}b, bc]$,$[a^{(p/2)-1}b, b^2c]$,$\cdots$,$[a^{(p/2)-1}b, b^{q-1}c]$,$[a^{(p/2)-1}b, bc^2]$,$[a^{(p/2)-1}b, b^2c^2]$, $\cdots$,$[a^{(p/2)-1}b, b^{q-1}c^2]$,$\cdots$,$[a^{(p/2)-1}b, bc^{q-1}]$,$[a^{(p/2)-1}b, b^2c^{q-1}]$,$\cdots$,$[a^{(p/2)-1}b, b^{q-1}c^{q-1}]$. $\square$

Also, the signature of $\overline{H}''_{p,q}$ is $(\frac{q^2(p-1)-q(p+1)+2}{2}; \infty^{(q)})$.

**Example 2.** *Let $p = 4$ and $q = 5$. Then we have $\left|\overline{H}'_{4,5} : \overline{H}''_{4,5}\right| = 50$. We choose $\Sigma = \{I, a, b, b^2, b^3, b^4, c, c^2, c^3, c^4, ab, ab^2, ab^3, ab^4, ac, ac^2, ac^3, ac^4, bc, bc^2, bc^3, bc^4, b^2c, b^2c^2, b^2c^3, b^2c^4, b^3c, b^3c^2, b^3c^3, b^3c^4, b^4c, b^4c^2, b^4c^3, b^4c^4, abc, abc^2, abc^3, abc^4, ab^2c, ab^2c^2, ab^2c^3, ab^2c^4, ab^3c, ab^3c^2, ab^3c^3, ab^3c^4, ab^4c, ab^4c^2, ab^4c^3, ab^4c^4\ \}$ as a Schreier transversal for $\overline{H}''_{4,5}$. According to the Reidemeister-Schreier method, we get total 56 generators of $\overline{H}''_{4,5}$ as $[a,b]$, $[a,b^2]$, $[a,b^3]$, $[a,b^4]$, $[a,c]$, $[a,c^2]$, $[a,c^3]$, $[a,c^4]$, $[b,c]$, $[b,c^2]$, $[b,b^3]$, $[b,c^4]$, $[b^2,c]$, $[b^2,c^2]$, $[b^2,c^3]$, $[b^2,c^4]$, $[b^3,c]$, $[b^3,c^2]$, $[b^3,c^3]$, $[b^3,c^4]$, $[b^4,c]$, $[b^4,c^2]$, $[b^4,c^3]$, $[b^4,c^4]$, $[a,bc]$, $[a,b^2c]$, $[a,b^3c]$, $[a,b^4c]$, $[a,bc^2]$, $[a,b^2c^2]$, $[a,b^3c^2]$, $[a,b^4c^2]$, $[a,bc^3]$, $[a,b^2c^3]$, $[a,b^3c^3]$, $[a,b^4c^3]$, $[a,bc^4]$, $[a,b^2c^4]$, $[a,b^3c^4]$, $[a,b^4c^4]$, $[ab,bc]$,$[ab,b^2c]$,$[ab,b^3c]$,$[ab,b^4c]$,$[ab,bc^2]$, $[ab,b^2c^2]$, $[ab,b^3c^2]$, $[ab,b^4c^2]$, $[ab,bc^3]$, $[ab,b^2c^3]$, $[ab,b^3c^3]$, $[ab,b^4c^3]$, $[ab,bc^4]$, $[ab,b^2c^4]$, $[ab,b^3c^4]$, $[ab,b^4c^4]$. Also, the signature of $\overline{H}''_{4,5}$ is $(26; \infty^{(5)})$.*

**Corollary 3.** *Let $p$ and $q$ be integers such that $2 \le p \le q$, $p+q > 4$. In case $p$ and $q$ are both even numbers, $\left|\overline{H}_{p,q} : \overline{H}^{(n)}_{p,q}\right| = \infty$ for $n \ge 2$ and, in other cases of $p$ and $q$, $\left|\overline{H}_{p,q} : \overline{H}^{(n)}_{p,q}\right| = \infty$ for $n \ge 3$.*

**Corollary 4.** *If $p = q$ odd number, then the commutator subgroups $\overline{H}''_{q,q}$ coincides with the ones given in [15] for extended Hecke groups $\overline{H}_q$.*

## References

[1] I. N. Cangül and O. Bizim, *Commutator subgroups of Hecke groups*, Bull. Inst. Math. Acad. Sinica 30 (2002), no. 4, 253–259.

[2] K. Calta and T. A. Schmidt, *Infinitely many lattice surfaces with special pseudo-Anosov maps*, J. Mod. Dyn. 7, No. 2, 239-254 (2013).

[3] K. Calta and T. A. Schmidt, *Continued fractions for a class of triangle groups*, J. Aust. Math. Soc. 93, No. 1-2, 21-42 (2012).

[4] Y.H. He and J. Read, *Hecke Groups, Dessins d′Enfants and the Archimedean solids*, http://arxiv.org/pdf/1309.2326.

[5] E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, Math. Ann. **112** (1936), 664-699.

[6] S. Huang, *Generalized Hecke groups and Hecke polygons*, Ann. Acad. Sci. Fenn., Math. 24, No.1, 187-214 (1999).

[7] S. Huang, *Realizability of torsion free subgroups with prescribed signatures in Fuchsian groups*, Taiwanese J. Math. 13 (2009), no. 2A, 441–457.

[8] S. Ikikardes, R. Sahin and I. N. Cangül, *Principal congruence subgroups of the Hecke groups and related results*, Bull. Braz. Math. Soc. (N.S.) 40, No. 4, 479-494 (2009).

[9] G. A. Jones, *Combinatorial categories and permutation groups*, http://arxiv.org/pdf/1309.6119.

[10] R. S. Kulkarni, *A new proof and an extension of a theorem of Millington on the modular group*, Bull. Lond. Math. Soc. 17, 458-462 (1985).

[11] J. Lehner, *Uniqueness of a class of Fuchsian groups,* Illinois J. Math. 19 (1975), 308–315.

[12] D. Mayer, T. Mühlenbruch and F. Strömberg, *The transfer operator for the Hecke triangle groups*, Discrete Contin. Dyn. Syst. 32 (2012), no. 7, 2453-2484.

[13] J. Nielsen, *Kommutatorgruppen for det frie produkt af cykliske gruper*, Mat. Tidsskr. B (1948), 49-56.

[14] A. D. Pohl, *Odd and even Maass cusp forms for Hecke triangle groups, and the billiard flow*, http://arxiv.org/pdf/1303.0528.

[15] R. Sahin and O. Bizim, *Some subgroups of the extended Hecke groups* $\overline{H}(\lambda_q)$, Acta Math. Sci., Ser. B, Engl. Ed. **23**, No.4 (2003), 497-502.

[16] R. Sahin, O. Bizim and I. N. Cangül, *Commutator subgroups of the extended Hecke groups*, Czech. Math. J. **54**, No.1 (2004), 253-259.

[17] R. Sahin, S. Ikikardes and Ö. Koruoğlu, *Some normal subgroups of the extended Hecke groups* $\overline{H}(\lambda_p)$, Rocky Mountain J. Math. **36** (2006), no. 3, 1033–1048.

[18] R. Sahin and Ö. Koruoğlu, *Commutator subgroups of the power subgroups of some Hecke groups*, Ramanujan J. 24 (2011), no. 2, 151–159.

[19] R. Sahin and Ö. Koruoğlu, *Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$ II,* C. R. Math. Acad. Sci. Paris 349 (2011), no. 3-4, 127–130.

[20] T. A. Schmidt and M. Sheingorn, *Covering the Hecke triangle surfaces*, Ramanujan J. 1 (1997), no. 2, 155–163.

[21] D. Singerman, *Subgroups of Fuchsian groups and finite permutation groups*, Bull. Lond. Math. Soc. 2, 319-323 (1970).

[22] V. V. Tsanov, Valdemar, *Triangle groups, automorphic forms, and torus knots*, Enseign. Math. (2) 59 (2013), no. 1-2, 73–113.

Şule KAYMAK (SARICA),
Institute of Science, Department of Mathematics,
Balıkesir University,
Çağış Campus, 10145, Balıkesir, Turkey.
Email: sulekaymak0@gmail.com

Bilal DEMİR,
Necatibey Faculty of Education, Department of Mathematics,
Balıkesir University,
Soma Street, 10100, Balıkesir, Turkey.
Email: bdemir@balikesir.edu.tr

Özden Koruoğlu,
Necatibey Faculty of Education, Department of Mathematics,
Balıkesir University,
Soma Street, 10100, Balıkesir, Turkey.
Email: ozdenk@balikesir.edu.tr

Recep Şahin,
Faculty of Science and Arts, Department of Mathematics,
Balıkesir University,
10145 Çağış Campus, Balıkesir, Turkey
Email: rsahin@balikesir.edu.tr