DOI: 10.2478/auom-2018-0007
An. Şt. Univ. Ovidius Constanţa **Vol. 26(1),2018, 111–123**

Some Extensions of Generalized Morphic Rings and EM-rings

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Abstract

Let R be a commutative ring with unity. The main objective of this article is to study the relationships between PP-rings, generalized morphic rings and EM-rings. Although PP-rings are included in the later rings, the converse is not in general true. We put necessary and sufficient conditions to ensure the converse using idealization and polynomial rings

1 Introduction

All rings are assumed to be commutative with unity 1. Let $Z(R)$ be the set of all zero divisors in R, and let $reg(R) = R \setminus Z(R)$.

A ring R is called a morphic ring if for each $a \in R$, there exists $b \in R$ such that $Ann(a) = bR$ and $Ann(b) = aR$. It is known that for reduced commutative rings, morphic rings are equivalent to von Neumann regular rings. A ring R is called generalized morphic ring if $Ann(a)$ is principal for each $a \in R$, for more details, see [10], [12], [13] and [14]. It is clear that the class of generalized morphic rings includes a wide range of rings such as integral domains, principal ideal rings, von Neumann regular rings, PP-rings, etc. If for each polynomial $f(x) \in Z(R[x])$ there exists $c_f \in R$ and $f_1(x) \in reg(R[x])$ such that $f(x) = c_f f_1(x)$, then R is called an EM-ring. Note that in this case $Ann_{R[x]}(f) = Ann_{R[x]}(c_f)$, which simplifies working and characterizing

Key Words: polynomial ring; annihilating content; EM-ring; generalized morphic ring; PP-ring
2010 Mathematics Subject Classification: 13F20, 13F25, 13E05.
Received:31.01.2017
Accepted:23.05.2017

zero-divisors in $R[x]$. These rings were defined and characterized in [2], and it was shown there that this class includes a wide range of rings.

It is shown in [2] that if R is a Noetherian ring, then R is generalized morphic if and only if it is an EM-ring. In fact the Noetherian condition is not necessary as will be shown later on.

Recall that if R is a ring, and M is an $R-$module, then the idealization $R(+)M$ is the set of all ordered pairs $(r, m) \in R \times M$, equipped with addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. It is well-known that $R(+)R \simeq R[x] \big/ (x^2)$. For the general case, we consider the ring $R[x] \big/ (x^{n+1})$, where $n \in \mathbb{N}$. In this case we set $R[x] \big/ (x^{n+1}) = \{\sum\limits_{i=0}^{n} a_i X^i : a_i \in R, X = x + (x^{n+1})\}$.

A ring R is called a PP-ring if every principal ideal of R is a projective $R-$module. It is well known that R is a PP-ring if and only if for each $a \in R, Ann(a)$ is generated by an idempotent. A ring R is called a PF-ring if every principal ideal of R is a flat $R-$module. It is well known that R is a PF-ring if and only if for each $a \in R, Ann(a)$ is pure, i.e. for each $b \in Ann(a)$, there exists $c \in Ann(a)$ such that $b = bc$.

It is clear that a PP-ring is generalized morphic ring, and it was shown in [2] that a PP-ring is also an EM-ring, while $\mathbb{Z}_4$ is generalized morphic EM-ring that is not PP-ring.

In this article we will characterize when some extensions of a generalized morphic ring are generalized morphic. To be more precise; we will characterize when the polynomial ring, the ring $R[x] \big/ (x^{n+1})$ and the idealization of a generalized morphic ring is generalized morphic. We show that the later two rings are generalized morphic if and only if their base ring R is a PP-ring.

We will characterize when the idealization of an EM-ring is an EM-ring. We will also continue the investigation of the polynomial rings of EM-rings we started in [2].

The following two lemmas will be used frequently in the following work.

Lemma 1.1. *Let R be a reduced ring. If $(a, x), (b, y) \in R(+)R$ such that $(a, x)(b, y) = (0, 0)$, then $ab = ay = bx = 0$.*

Proof. We have $(0, 0) = (ab, ay + bx)$, and so,

$$ab = 0,$$

$$ay + bx = 0,$$

$$0 = a(ay + bx) = a^2y + abx = a^2y = 0.$$

Thus, $(ay)^2 = 0$, and since R is reduced we have $ay = 0$, whence $bx = 0$. □

Lemma 1.2. *Let R be a ring, and let $S = \{(a_1, b_1), ..., (a_n, b_n)\} \subseteq R(+)R$. Then $Ann(S) \neq \{(0,0)\}$ if and only if $Ann(a_1, ..., a_n) \neq \{0\}$.*

Proof. Assume that $(a, b) \neq (0, 0)$ and $(a, b)(a_i, b_i) = (0, 0)$ for all i. Then $aa_i = 0$ for all i. If $a = 0$, then $b \neq 0$ and $ba_i = 0$ for all i. Thus $Ann(a_1, ..., a_n) \neq \{0\}$.

Now, if $a \neq 0$ and $aa_i = 0$ for all i, then $(0, a)(a_i, b_i) = (0, 0)$ for all i. Thus $Ann(S) \neq \{(0, 0)\}$. □

Recall that for any ring R, the set $Min(R)$ is the set of all minimal prime ideals of R, equipped with the hull kernel topology, and for any set I of $R, V(I) = \{P \in Min(R) : I \subseteq P\}$, and $Supp(I) = V(Ann(I))$. An ideal I of R is called a $z^0 - ideal$, if whenever $V(a) \subseteq V(b)$, with $a \in I$, we have $b \in I$.

2 Generalized Morphic Rings

In this section we will relate reduced generalized morphic rings to complemented rings, and characterize when the polynomial ring of a generalized morphic ring is generalized morphic, and characterize generalized morphic rings using their minimal prime ideals.

A ring R is called complemented if for each $a \in R$, there exists $b \in R$ such that $ab = 0$ and $a + b \in reg(R)$. A reduced ring R is complemented if and only if for each $a \in R$ there exists $b \in R$ such that $Ann(Ann(a)) = Ann(b)$. For more properties of complemented reduced rings, see Theorem 2.2 and Proposition 2.5 in [5], and Theorem 4.5 in [9].

It is clear that if R is a reduced generalized morphic ring, then for any $a \in R$, there exists $b \in R$ such that $Ann(a) = bR$, and so, $Ann(Ann(a)) = Ann(b)$. Thus R is a complemented ring.

For a complemented ring that is not generalized morphic, see Example 5.8 in [7] together with Theorem 1.3 in [8] and Theorem 2.2 below.

Recall that a ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero.

We now characterize the case at which the polynomial ring of a generalized morphic ring is generalized morphic.

Theorem 2.1. *If $R[x]$ is a generalized morphic ring, then R is generalized morphic. If R is Armendariz, then the converse is also true.*

Proof. Assume $R[x]$ is generalized morphic, and let $a \in Z^*(R)$. Then $Ann_{R[x]}(a) = f(x)R[x]$, where $f(x) = \sum_{i=0}^{n} a_i x^i$. Let j be the least index such that $a_j \neq 0$. Then $aa_j = 0$, and so, $a_jR \subseteq Ann_R(a)$. Let $b \in Ann_R(a) \setminus \{0\}$. Then $b = f(x)g(x)$ for some $g(x) = \sum_{i=0}^{m} b_i x^i \in R[x]$. Then $b = a_0b_0 \in a_0R$, and $a_0 \neq 0$. Thus $j = 0$ and $Ann_R(a) = a_0R$ is principal, and hence R is generalized morphic.

For the converse assume R is Armendariz generalized morphic and let $f(x) = \sum_{i=0}^{n} a_i x^i \in Z(R[x])$. Then there exists $a \in R$ such that $aa_i = 0$ for all i. Thus $\{0\} \neq Ann_R(a_0, a_1, ..., a_n)$. Since R is generalized morphic, there exists $b \in R$ such that $Ann_R(a_0, a_1, ..., a_n) = bR$, see Theorem 5 in [12]. Thus $bR[x] \subseteq Ann_{R[x]}(f)$. If $g(x) = \sum_{i=0}^{m} c_i x^i \in Ann(f)$, then $c_ia_j = 0$ for all i and j, since R is Armendariz, and so, $c_i \in bR$ for each i and $g(x) \in bR[x]$. Thus $Ann_{R[x]}(f) = bR[x]$ is principal and $R[x]$ is generalized morphic. $\square$

Question: While there are non-commutative generalized morphic rings that are non-Armendariz, is it necessary for a commutative generalized morphic ring to be Armendariz?

Next, we will characterize generalized morphic reduced rings using minimal prime ideals, and the concept of z^0-ideals, borrowed from the rings of continuous functions.

Theorem 2.2. *Let R be a reduced ring. Then R is a generalized morphic ring if and only if for each $a \in R$ there exists $b \in R$ such that $Supp(a) = V(b)$ and bR is a z^0-ideal.*

Proof. Assume R is a generalized morphic ring, and let $a \in R$. Then $Ann(a) = bR$ for some $b \in R$. So we have $Supp(a) = V(Ann(a)) = V(bR) = V(b)$. Moreover, if $V(br) \subseteq V(c)$, then $V(b) \subseteq V(br) \subseteq V(c)$, and so, for each $P \in Min(R)$, if $b \in P$, then $c \in P$ and hence, $ac \in P$. If $b \notin P$, then $a \in Ann(b) \subseteq P$, and so, $ac \in P$. Thus, $ac \in \bigcap_{P \in Min(R)} P = \{0\}$, since R is reduced. Therefore, $c \in Ann(a) = bR$, and bR is a z^0-ideal.

Conversely, assume $a, b \in R$ such that $Supp(a) = V(b)$ and bR is a z^0-ideal. Let $P \in Min(R)$. If $a \in P$, then $ab \in P$. If $a \notin P$, then $Ann(a) \subseteq P$, and so, $P \in Supp(a) = V(b)$. Hence, $b \in P$, and so, $ab \in P$. Thus, $ab \in \bigcap_{P \in Min(R)} P = \{0\}$, which implies that $bR \subseteq Ann(a)$. If $c \in Ann(a)$, then we have $V(b) = Supp(a) \subseteq V(c)$, and so, $c \in bR$, being a z^0-ideal. Hence $Ann(a) = bR$, and R is a generalized morphic ring. $\square$

3 When is $R[x]\,/(x^{n+1})$ Generalized Morphic ring?

In this section we characterize the case at which the idealization of a generalized morphic ring or more generally, the ring $R[x]\,/(x^{n+1})$ is generalized morphic.

Theorem 3.1. *Let R be a ring, M an $R-$module and let $S = R(+)M$. If S is generalized morphic ring, then R is generalized morphic ring.*

Proof. Let $a \in Z(R^*)$. Then $Ann((a,0)) = (r,m)S$, and hence $(0,0) = (a,0)(r,m) = (ar,am)$. So, $ar = 0$, and thus, $rR \subseteq Ann(a)$. Now, if $x \in Ann(a)$, then $(x,0)(a,0) = (xa,0) = (0,0)$.

But in this case, we must have $(x,0) = (r,m)(t,s) = (rt, rs+tm)$, for some $(t,s) \in S$. So, $x \in rR$. Therefore, $Ann(a) = rR$, and hence, R is generalized morphic ring. □

The converse of the above Theorem needs not be true, since $\mathbb{Z}_4$ is a generalized morphic ring, while $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not.

Now, the question is, for what rings R, the converse of this Theorem must be true. In the following, we will give the answer. But first we recall the following proposition which was proved in [12].

Proposition 3.2. *Let R be a reduced ring. Then the following are equivalent:*
(1) The ring R is morphic.
(2) The ring $R[x]\,/(x^{n+1})$ is morphic for each $n \in \mathbb{N}$.
(3) The ring $R(+)R$ is morphic.
(4) The ring R is von Neumann regular ring.

In the following, we will prove an analogue result for the equivalence of PP-rings and generalized morphic idealization.

Lemma 3.3. *Let R be a reduced ring, and let $f = \sum_{i=0}^{n} a_i X^i \in Z(R[x]\,/(x^{n+1}))\setminus \{0\}$, $g = \sum_{i=0}^{n} b_i X^i \in Ann(f)$. Then $b_i \in Ann(a_0, a_1, ..., a_{n-i})$ for $i = 0, 1, 2, ..., n$.*

Proof. Since $fg = 0$, we have $\sum_{i=0}^{j} a_i b_{j-i} = 0$, for $j = 0, 1, 2, ..., n$. Thus, $a_0 b_0 = 0$, and if $b_0 \in Ann(a_0, a_1, ..., a_j)$, $j < n$, then multiplying the equation $a_0 b_{j+1} + a_1 b_j + ... + a_j b_1 + a_{j+1} b_0 = 0$ by b_0 yields $a_{j+1} b_0^2 = 0$, and since R is reduced we have $a_{j+1} b_0 = 0$, i.e. $b_0 \in Ann(a_0, a_1, ..., a_{j+1})$. Hence $b_0 \in Ann(a_0, a_1, ..., a_n)$. Now, assume that $b_i \in Ann(a_0, a_1, ..., a_{n-i})$, for $i = 0, 1, ..., j < n$, then the equation $a_0 b_{j+1} + a_1 b_j + ... + a_j b_1 + a_{j+1} b_0 = 0$

reduces to $a_0b_{j+1} = 0$. So, assume that $a_kb_{j+1} = 0$, for $k = 0, 1, ..., l < n-j-1$, then the equation $\sum\limits_{s+k=l+1+j+1} a_kb_s = 0$, reduces to $a_{l+1}b_{j+1} = 0$, and so we have $b_{j+1} \in Ann(a_0, a_1, ..., a_{n-j-1})$. □

Theorem 3.4. *The following are equivalent for a ring R :*
(1) The ring R is a PP-ring.
(2) The ring $R[x]/(x^{n+1})$ is generalized morphic for each $n \in \mathbb{N}$.
(3) The ring $S = R(+)R$ is generalized morphic.
(4) The ring R is generalized morphic PF-ring.
(5) The ring R is complemented PF-ring.

Proof. (1)⇒ (2) Assume R is a PP-ring, and $f = \sum\limits_{i=0}^{n} a_iX^i \in Z(R[x]/(x^{n+1})) \setminus \{0\}$, $g = \sum\limits_{i=0}^{n} b_iX^i \in Ann(f)$. Then it follows by Lemma 3.3 that $b_i \in Ann(a_0, a_1, ..., a_{n-i}) = e_iR$, where $e_i^2 = e_i$ for $i = 0, 1, 2, ..., n$, and in this case we would have $e_ie_j = e_i$, whenever $i \leq j$. Let $e = \sum\limits_{i=0}^{n} e_iX^i$. Then it is clear that $e \in Ann(f)$. Let $K_0 = b_0 - b_0X, K_1 = b_1(1-e_0) - b_1X + 2b_1e_0X - b_0e_0X^2$. Then it is clear that $b_iX^i = eK_i$ for $i = 0, 1$.

Now, for $1 < m \leq n$, let $T_m = b_m(1 - e_{m-1}) - b_mX + 2b_me_{m-1}X - b_me_{m-1}X^2$. Then routine computations yields $eT_m = b_mX^m + \sum\limits_{j=0}^{m-2} b_me_jX^{j+1} - \sum\limits_{j=0}^{m-2} b_me_jX^{j+2}$. Let $G_{m,i} = -b_me_{m-1-i}X^i + 2b_me_{m-1-i}X^{i+1} - b_me_{m-1-i}X^{i+2}$, for all $1 \leq i \leq m-1$. Then $eG_{m,i} = -b_me_{m-1-i}X^{m-1} + b_me_{m-1-i}X^m - \sum\limits_{j=0}^{m-2-i} b_me_jX^{j+i} + 2\sum\limits_{j=0}^{m-2-i} b_me_jX^{j+i+1} - \sum\limits_{j=0}^{m-2-i} b_me_jX^{j+i+2}$, for $1 \leq i \leq m-2$, and $eG_{m,m-1} = -b_me_0X^{m-1} + b_me_0X^m$. Let $k_{m,r} = T_m + \sum\limits_{i=1}^{r} G_{m,i}$. Using finite induction, one can show that $ek_{m,r} = b_mX^m + \sum\limits_{j=0}^{m-2-r} b_me_jX^{j+r+1} - \sum\limits_{j=0}^{m-2-r} b_me_jX^{j+r+2}$, for $1 \leq r \leq m-2$, and $eK_{m,m-2} = b_mX^m + b_me_0X^{m-1} - b_me_0X^m$. Now, let $K_m = (T_m + \sum\limits_{i=1}^{m-2} G_{m,i} + G_{m,m-1})$, for $1 < m \leq n$. Then $eK_m = b_mX^m$, and so, $g = e\sum\limits_{m=0}^{n} K_m$. Thus, $Ann(f) = (e)$, and $R[x]/(x^{n+1})$ is generalized morphic.

(2)⇒ (3) Clear, since $R(+)R$ is isomorphic to $R[x]/(x^2)$.

(3)$\Rightarrow$ (1) Assume that S is generalized morphic, and let $a \in Z(R) \setminus \{0\}$. Then $(0,a) \in Z(S) \setminus \{(0,0)\}$, and so, $Ann(0,a) = (x,y)S$. It is clear that $xR \subseteq Ann(a)$, and if $b \in Ann(a)$, then $(b,0)(0,a) = (0,0)$, and hence, $(b,0) = (x,y)(z,w)$. Thus $b = xz \in xR$, and therefore $Ann(a) = xR$. But $(0,1)(0,a) = (0,0)$, and so, $(0,1) = (x,y)(\alpha,\beta)$. Thus we have:

$$0 = x\alpha,$$

$$1 = x\beta + y\alpha,$$

which yields that

$$x = x^2\beta,$$

and hence, $x\beta = (x\beta)^2$, and $Ann(a) = (x\beta)R$. Thus R is a PP-ring.
(1)$\Leftrightarrow$ (4) See Corollary 3.12 in [14].
(1)$\Leftrightarrow$ (5) See Proposition 2.7 in [11]. □

Example 3.5. *Let F be a field. Then $R = F[x,y]/(xy)$ is a reduced complemented ring that is not a PP-ring, see Remark 2 in [3], and Theorem 4.5 in [9]. One can see easily that R is a generalized morphic ring, while $R[x]/(x^{n+1})$ is not for any $n \in \mathbb{N}$.*

It is immediate that if R is a PF-ring that is not a PP-ring, then R and $R(+)M$ are not generalized morphic for any $R-$module M.
Since PP-rings are always reduced, we conclude the following easily.

Corollary 3.6. *If $R[x]/(x^{n+1})$ is generalized morphic, then R is reduced.*

4 Polynomial rings of EM-rings

In [1], the concept of the annihilating content of a polynomial $f(x)$ was introduced to be a constant c_f such that $f(x) = c_f f_1(x)$ with $f_1(x)$ is not a zero-divisor, and in [2], we called a ring R to be an EM-ring if every zero-divisor polynomial in $R[x]$ has an annihilating content. Many properties of this ring were investigated, and many open problems were posed. We now study the polynomial ring of an EM-ring.

Theorem 4.1. *If R is an EM-ring, then $R[x]$ is an EM-ring. If R is a reduced, then the converse is also true.*

Proof. Assume R is an EM-ring. To show that $R[x]$ is an EM-ring, we will follow the proof of the result in the unpublished article [2]. Let $f(x,y) = \sum_{i=0}^{n} f_i(x)y^i$ be zero-divisor in $R[x,y] = (R[x])[y]$. Then there exists nonzero $h(x)$ such that $hf_i = 0$ for all i. Define

$$g(x) = f_0 + f_1 x^{deg(f_0)+1} + f_2 x^{deg(f_0)+deg(f_1)+2} + ... + f_n x^{\sum_{i=1}^{n-1} \deg(f_i)+n}$$

Since $hg = 0$, there exists $c_g \in Z(R)$ and nonzero-divisor $g_1 = \sum_{i=1}^{m} b_i x^i$ such that $g = c_g g_1$. So, $\cap Ann(b_i) = \{0\}$, and $f_0 = c_g \sum_{i=0}^{\deg(f_0)} b_i x^i = c_g h_0(x), f_1 = c_g \sum_{i=0}^{\deg(f_1)} b_{i+\deg(f_0)+1} x^i = c_g h_1$, and so on. Hence, $f(x,y) = c_g \sum_{i=0}^{n} h_i(x)y^i$. If $\sum_{i=0}^{n} h_i(x)y^i$ is a zero-divisor, then there exists nonzero $k(x)$ such that $k(x)h_i(x) = 0$ for each i. Define

$$l(x) = \sum_{i=0}^{n} h_i(x)\ x^{\sum_{j<i} \deg(f_j)\ +1}$$

and so, $k(x)l(x) = 0$, and therefore there exists a nonzero $c \in R$ such that $ch_i(x) = 0$, and so, $cb_i = 0$ for all i, a contradiction, since $\cap Ann(b_i) = \{0\}$.Thus $\sum_{i=0}^{n} h_i(x)y^i$ is nonzero-divisor, and $R[x]$ is an EM-ring.

Assume now that R is a reduced ring, and $R[x]$ is an EM-ring. Let $f(x) = \sum_{i=0}^{l} a_i x^i \in Z(R[x]) \setminus \{0\}$. Then $g(y) = \sum_{i=0}^{l} a_i y^i \in Z((R[x])[y]) \setminus \{0\}$, and so, there exists $h(x) = \sum_{i=0}^{m} h_i x^i \in R[x]$ such that $g(y) = h(x) \sum_{i=0}^{l} k_i(x)y^i$, with $\bigcap Ann(k_i(x)) = \{0\}$. Assume that $k_i(x) = \sum_{j=0}^{n_i} k_{i,j} x^j$, which implies that $\bigcap Ann(k_{i,j}) = \{0\}$. Note that $a_i = h(x)k_i(x) = h_0 k_0$. But $h(x)k_i(x) = \sum_{k=0}^{m+n_i} c_k x^k$, with $c_k = \sum_{j=0}^{k} h_j k_{i,k-j}$. Now we have:

$$0 = c_{m+n_i} = h_m k_{i,n_i}$$

$$0 = c_{m+n_i-1} = h_m k_{i,n_i-1} + h_{m-1} k_{i,n_i},$$

which implies that $0 = h_m^2 k_{i,n_i-1}$, and so, $0 = h_m k_{i,n_i-1}$, since R is reduced.

$$0 = c_{m+n_i-2} = h_m k_{i,n_i-2} + h_{m-1} k_{i,n_i-1} + h_{m-1} k_{i,n_i},$$

which implies that $0 = h_m^2 k_{i,n_i-2}$, and so, $0 = h_m k_{i,n_i-2}$

Now, assume we have $h_m k_{i,s} = 0$, for $s = n_i, n_i - 1, ..., j+1$. Thus we have

$$0 = c_{m+j} = h_m k_{i,j} + h_{m-1} k_{i,j+1} + ... h_j k_{i,m},$$

which implies that $0 = h_m^2 k_{i,j}$, and so, $0 = h_m k_{i,j}$, this shows that $h_m k_{i,s} = 0$, for $s = 0, 1, 2, ..., n_i$.

Thus, $h(x)k_i(x) = (h(x) - h_m x^m)k_i(x)$.

Continue to get $h(x) = h_0 k_i(x)$, which implies that $h_0 k_{i,j} = 0$ for all $j \in \{1, 2, ..., n_j\}$, and $i \in \{1, 2, ..., l\}$

Now define $w(x) = \sum\limits_{i=0}^{n} k_{i,0} x^i + x^{n_0+1} \sum\limits_{j=1}^{n_0} k_{0,j} x^j + x^{n_0+n_1+2} \sum\limits_{j=1}^{n_1} k_{1,j} x^j + ... + x^{n_0+n_1+...+n_{l-1}+l} \sum\limits_{j=1}^{n_l} k_{l,j} x^j$. Then $Ann(w) = \{0\}$, and $f(x) = h_0 w(x)$. Hence, R is an EM-ring. □

Question: Is the above result true for nonreduced rings?

5 Idealization of EM-rings

It was shown in [2] that if R is a Noetherian ring, then R is an EM-ring if and only if it is a generalized morphic ring, and an example was given for an EM-ring that is not generalized morphic, but the precise relation between the two concepts was not accomplished. In the following, we will give a partial answer.

We now investigate the idealization of EM-rings, and relate it to generalized morphic rings.

Theorem 5.1. *Assume R is a ring such that $S = R(+)R$ is an EM-ring, then R is an EM-ring.*

Proof. Let $f(x) = \sum\limits_{i=0}^{n} a_i x^i \in Z(R[x]) \setminus \{0\}$. Then there exists $a \in R \setminus \{0\}$ such that $aa_i = 0$ for each i. Let $g(x) = \sum\limits_{i=0}^{n} (a_i, 0)x^i \in S[x]$. Then $(a, 0)(a_i, 0) = (0, 0)$ for each i, and so, $g(x) \in Z(S[x]) \setminus \{(0, 0)\}$. Thus there exists $(r, m) \in S$ such that $g(x) = (r, m) \sum\limits_{i=0}^{k} (r_i, m_i)x^i$, with $\bigcap\limits_{i=0}^{k} Ann(r_i, m_i) = \{(0, 0)\}, n \leq k$. Hence, we have $\bigcap\limits_{i=0}^{k} Ann(r_i) = \{0\}$, and $f(x) = r \sum\limits_{i=0}^{k} r_i x^i$. Thus, R is an EM-ring. □

The converse of the above Theorem needs not be true, since $\mathbb{Z}_4$ is an EM-ring, while $\mathbb{Z}_4(+)\mathbb{Z}_4$ is not.

In [2], we showed that if R is a PP-ring, then it is an EM-ring. We now give a more precise result.

Theorem 5.2. *A ring R is a PP-ring if and only if $S = R(+)R$ is an EM-ring.*

Proof. Assume that R is a PP-ring, and $f(x) = \sum_{i=0}^{n}(a_i, b_i)x^i \in Z(S[x]) \setminus \{(0,0)\}$. Since R is a PP-ring, we can write $a_i = u_i r_i$, and $b_i = v_i s_i$, where u_i and v_i are idempotents, r_i and s_i are regular elements for each i, see [4, Lemma 2]. Define the idempotents u, v and e as follows:

$$1 - u = \prod_{i=0}^{n}(1 - u_i),$$

$$1 - v = \prod_{i=0}^{n}(1 - v_i),$$

$$1 - e = (1 - u)(1 - v).$$

Note that $(a_i, 0) = (u, e-u)(a_i, 0)$ and $(0, b_i) = (u, e-u)((1-u)(b_i + 1 - e), b_i)$, and so, $\sum_{i=0}^{n}(a_i, b_i)x^i = (u, e-u)\sum_{i=0}^{n}(a_i + (1-u)(b_i + 1 - e), b_i)x^i$. Now, let I be the ideal in R generated by the elements $a_i + (1-u)(b_i+1-e)$. Then $a_i = u_i(a_i + (1-u)(b_i + 1 - e)) \in I$ for each i. Also, $(1-u)(b_i + 1 - e) = a_i + (1-u)(b_i + 1 - e) - a_i \in I$ for each i, which implies that $(1-u)b_i = e(1-u)(b_i + 1 - e) \in I$, since $eb_i = b_i$ for each i. Therefore, we have $1 - e = (1-e)(1-u) \in I$. Now, if $\alpha \in Ann(I)$, then $0 = \alpha a_i = \alpha u_i r_i$, and so, $\alpha u_i = 0$ for each i, which implies that $\alpha u = 0$, and so, $0 = \alpha(1-u)b_i = \alpha b_i$ for each i. Thus, $\alpha v_i = 0$ for each i. Hence we have $\alpha u = 0 = \alpha v$, and so, $\alpha e = 0$. But we have also $\alpha(1-e) = 0$, which implies that $\alpha = 0$, i.e. $Ann(I) = \{0\}$, and so it follows by Lemma 1.2 that $\sum_{i=0}^{n}(a_i + (1-u)(b_i + 1 - e), b_i)x^i \in reg(S[x])$. Thus S is an EM-ring.

Now assume that S is an EM-ring, $b \in Z(R)\setminus\{0\}$ and let $a \in Ann(b)\setminus\{0\}$. Then $f(x) = (0,1)+(b,0)x \in Z(S[x])\setminus\{(0,0)\}$, since it is annihilated by $(0,a)$. Thus $f(x) = (\alpha, \beta)\sum_{i=0}^{n}(n_i, m_i)x^i$, with $\bigcap_i Ann(n_i) = \{0\}$. Thus, we have:

$$0 = \alpha n_0,$$

$$1 = \alpha m_0 + \beta n_0,$$

$$b = \alpha n_1,$$

$$0 = \alpha n_i \text{ for all } i > 1.$$

But $b = b(\alpha m_0 + \beta n_0) = b\alpha m_0 + \alpha n_1 \beta n_0 = b(\alpha m_0)$. Also note that $\alpha m_0 = (\alpha m_0)^2 + \alpha m_0 \beta n_0 = (\alpha m_0)^2$. Thus, $Ann(\alpha m_0) \subseteq Ann(b)$. Now let $d \in Ann(b)$. Then we have:

$$0 = (dm_0)0 = (dm_0)\alpha n_0 = (d\alpha m_0)n_0,$$

$$0 = (dm_0)b = (dm_0)\alpha n_1 = (d\alpha m_0)n_1,$$

$$0 = (dm_0)0 = (dm_0)\alpha n_i = (d\alpha m_0)n_i \text{ for all } i > 1,$$

which implies that $d\alpha m_0 \in \bigcap_i Ann(n_i) = \{0\}$. Hence, $Ann(b) = Ann(\alpha m_0) = (1 - \alpha m_0)R$ is generated by an idempotent, and so, R is a PP-ring. □

Using Theorems 3.4 and 5.2, one can deduce the following:

Corollary 5.3. *For any ring R, we have $R(+)R$ is an EM-ring if and only if it is generalized morphic.*

Example 5.4. *The space $X = \beta\mathbb{N} \setminus \mathbb{N}$ is an F-space that is not a basically disconnected space nor complemented, see [6, 6W and 14.27], and so, $C(X)$ is a reduced Bézout ring that is not a PP-ring. Thus $C(X)(+)C(X)$ is not an EM-ring. Also we have $C(X)$ is an EM-ring which is not generalized morphic.*

Questions: It is still an open problem to characterize the relation between EM-rings and generalized morphic rings. Although they are not equivalent, we saw that $R(+)R$ is an EM-ring if and only if it is generalized morphic, even if R was not Noetherian. We also don't know yet what sufficient conditions must be add to an EM-ring to become a PP-ring. It is not difficult to show that if $R[x]\big/(x^{n+1})$ is an EM-ring, then R is a PP-ring. We are still working for the other direction.

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