DOI: 10.1515/auom-2017-0045
An. Şt. Univ. Ovidius Constanţa **Vol. 25(3),2017, 199–214**

Bounds on the Third Order Hankel Determinant for Certain Subclasses of Analytic Functions

S.P. Vijayalakshmi, T.V. Sudharsan, Daniel Breaz and K.G.Subramanian

Abstract

Let A be the class of analytic functions $f(z)$ in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series expansion about the origin given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \Delta$. The focus of this paper is on deriving upper bounds for the third order Hankel determinant $H_3(1)$ for two new subclasses of A.

1 Introduction

Let A be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is respectively said to be with bounded turning, starlike or convex if and only if for $z \in \Delta$, $Ref'(z) > 0,\ Re\frac{zf'(z)}{f(z)} > 0$ or $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. The classes of these functions are respectively denoted by R, S^* and C. For $n \geq 0$ and $q \geq 1$,

Key Words: Analytic functions, Hankel determinant, Functions with positive real part.
2010 Mathematics Subject Classification: Primary 30C45; Secondary 30C50.
Received: 09.09.2016
Accepted: 24.10.2016

the q^{th} Hankel determinant is defined as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \cdots \\ \vdots & & \vdots & \\ a_{n+q-1} & \cdots & & a_{n+2(q-1)} \end{vmatrix} \tag{2}$$

This determinant has been considered by several authors (see, for example, [1, 2, 3, 10, 17, 18, 22]). In fact Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. In particular, upper bounds for the second Hankel determinant were obtained by several authors [6, 9, 12, 20, 21] for different classes of analytic functions. Upper bound on the third Hankel determinant for different classes of functions has been studied recently [1, 2, 3, 10, 16, 22]. In the present investigation, the focus is on the third order Hankel determinant $H_3(1)$ for the classes R_α^β and S_α^β in Δ defined as follows:

Definition 1.1. *Let f be given by (1). Then $f \in R_\alpha^\beta$ if and only if for any $z \in \Delta, 0 \le \beta < 1, 0 \le \alpha \le 1$,*

$$Re\{f'(z) + \alpha z f''(z)\} > \beta. \tag{3}$$

The choice $\alpha = 0$, $\beta = 0$ yields $Re\ f'(z) > 0$, $z \in \Delta$, defining the class R of bounded turning [15] while the choice $\alpha = 0$, yields $Re\ f'(z) > \beta$ [5].

Definition 1.2. *Let f be given by (1). Then $f \in S_\alpha^\beta$ if and only if for any $z \in \Delta, 0 \le \beta < 1, 0 \le \alpha \le 1$,*

$$Re\left\{\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)}\right\} > \beta.$$

The choice $\alpha = 0$, $\beta = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > 0$, $z \in \Delta$, defining the class S^* of starlike functions [19] and the choice of $\alpha = 0$ yields $Re\ \frac{zf'(z)}{f(z)} > \beta$, $z \in \Delta$, defining the class $S^*(\beta)$ starlike functions of order β [19]. Setting $n = 1$ in (2), $H_3(1)$ is given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

and for $f \in A$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

Using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{4}$$

In obtaining an upper bound for $|H_3(1)|$, the approach used is to first determine upper bounds for the functionals $|a_2a_3 - a_4|$,$|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$. Furthermore techniques employed in [13, 14] are useful in establishing the results (see, for example [6, 9, 12, 21]).

2 Preliminary Results

Some preliminary results required in the following sections are now listed.
Let P denote the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{5}$$

which are regular in Δ and satisfy $Re\ p(z) > 0$, $z \in \Delta$. Throughout this paper, we assume that $p(z)$ is given by (5) and $f(z)$ is given by (1). To prove the main results, the following known Lemmas are required.

Lemma 2.1. *[4] Let $p \in P$. Then $|c_k| \leq 2$, $k = 1, 2, \ldots$ and the inequality is sharp.*

Lemma 2.2. *[13, 14] Let $p \in P$. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{6}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \tag{7}$$

for some x, y such that $|x| \leq 1$ and $|y| \leq 1$.

3 Main Results

For functions $f \in R_\alpha^\beta$, Lemma 3.1- Lemma 3.3 give the upper bounds for the three functionals mentioned earlier while Theorem 3.1 presents an estimate for $|H_3(1)|$.

Lemma 3.1. *Let $f \in R_\alpha^\beta$. Then*

$$|a_2a_3 - a_4| \leq \frac{(1 - \beta)}{2(1 + 3\alpha)} \tag{8}$$

Proof. Let $f \in R_\alpha^\beta$. Then there exists a p such that

$$f'(z) + \alpha z f''(z) = (1-\beta)p(z) + \beta,\ p(0) = 1,\ Re\ p(z) > 0.$$

Equating the coefficients, we find that

$$a_2 = \frac{c_1(1-\beta)}{2(1+\alpha)}, a_3 = \frac{c_2(1-\beta)}{3(1+2\alpha)}, a_4 = \frac{c_3(1-\beta)}{4(1+3\alpha)}, a_5 = \frac{c_4(1-\beta)}{5(1+4\alpha)}.$$

The functional $|a_2a_3 - a_4|$ is given by

$$|a_2a_3 - a_4| = \left| \frac{c_1c_2(1-\beta)^2}{6(1+\alpha)(1+2\alpha)} - \frac{c_3(1-\beta)}{4(1+3\alpha)} \right|. \tag{9}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$\begin{aligned} &|a_2a_3 - a_4| = \\ &\frac{(1-\beta)}{48(1+\alpha)(1+2\alpha)(1+3\alpha)} \left|4(1+3\alpha)(1-\beta)c_1(c_1^2 + x(4-c_1^2)) \right.\\ &\left. -3(1+\alpha)(1+2\alpha)(c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2))\right| \end{aligned}$$

$$\begin{aligned} =&A(\alpha,\beta)\left|c_1^3(2a-3b) - 2c_1x(4-c_1^2)(3b-a) - 3bx^2c_1(4-c_1^2)\right.\\ &\left.-2y \times 3b(1-|x|^2)(4-c_1^2)\right| \end{aligned}$$

where $A(\alpha,\beta) = \frac{(1-\beta)}{48(1+\alpha)(1+2\alpha)(1+3\alpha)}, a = 2(1+3\alpha)(1-\beta), b = (1+\alpha)(1+2\alpha)$. Suppose now that $c_1 = c$. Since $|c| = |c_1| \le 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \le 1$, we get

$$\begin{aligned} |a_2a_3 - a_4| \le&A(\alpha,\beta)\left\{c^3|2a-3b| + 2c\rho(4-c^2)(3b-a) + 3b\rho^2c(4-c^2)\right.\\ &\left.+2 \times 3b(1-\rho^2)(4-c^2)\right\} \end{aligned}$$

$$\begin{aligned} =&A(\alpha,\beta)\left\{c^3|2a-3b| + 2c\rho(4-c^2)(3b-a) + 3b\rho^2(4-c^2)(c-2)\right.\\ &\left.+6b(4-c^2)\right\} = F(\rho). \end{aligned}$$

Next we maximize the function $F(\rho)$.

$$F'(\rho) =A(\alpha,\beta)\left\{2c(4-c^2)(3b-a) + 6b\rho(4-c^2)(c-2)\right\} \tag{10}$$

$F'(\rho) = 0$ implies $\rho = \frac{c(3b-a)}{3b(2-c)}$. Set $\rho^* = \frac{c(3b-a)}{3b(2-c)}$. Now $0 \le \rho^* \le 1$. Also we have $F''(\rho) = A(\alpha,\beta)\{6b(4-c^2)(c-2)\} < 0$, for $c < 2$. Thus ρ^* is the only value in $[0,1]$ at which $F(\rho)$ attains a maximum. Hence $F(\rho) \leq F(\rho^*)$.Thus

$$\begin{aligned} F(\rho) \leq\ & A(\alpha,\beta)\left\{c^3|2a-3b| + \frac{c^2}{3b}(3b-a)^2(2+c) + 6b(4-c^2)\right\} \\ =\ & A(\alpha,\beta)\left\{c^3\{|2a-3b| + \frac{(3b-a)^2}{3b}\} - c^2\{[6b - \frac{2(3b-a)^2}{3b}]\} + 24b\right\} \end{aligned}$$

$$F(\rho) \leq A(\alpha,\beta)\left\{c^3\gamma - c^2\delta + 24b\right\} = G(c),$$

where $\gamma = |2a-3b| + \frac{(3b-a)^2}{3b}, \delta = [6b - \frac{2(3b-a)^2}{3b}]$. $G'(c) = 0$ implies $c = 0$ and at $c = 0$, $G''(c) < 0$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = \rho^*$ and $c = 0$. Hence $|a_2a_3 - a_4| \leq \frac{(1-\beta)}{2(1+3\alpha)}$. □

Corollary 3.1. *Choosing $\alpha = 0$, $\beta = 0$ in (8), we get $|a_2a_3 - a_4| \leq \frac{1}{2}$. This result coincides with the corresponding result in [3].*

Lemma 3.2. *Let $f \in R_\alpha^\beta$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}\frac{(1-\beta)^2}{(1+2\alpha)^2} \tag{11}$$

Proof. Let $f \in R_\alpha^\beta$. In a manner similar to the proof of Lemma 3.1, we can derive

$$|a_2a_4 - a_3^2| = \left|\frac{c_1c_3(1-\beta)^2}{8(1+\alpha)(1+3\alpha)} - \frac{c_2^2(1-\beta)^2}{9(1+2\alpha)^2}\right| \tag{12}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$\begin{aligned} = & \left|\frac{c_1(1-\beta)^2}{32(1+\alpha)(1+3\alpha)}[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2)\right. \\ & \left. +2y(1-|x|^2)(4-c_1^2)] - \frac{(1-\beta)^2}{36(1+2\alpha)^2}[c_1^2 + x(4-c_1^2)]^2\right|. \end{aligned}$$

$$\begin{aligned} = & \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\left|9c_1(1+2\alpha)^2[c_1^3 + 2xc_1(4-c_1^2)\right. \\ & -x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)] \\ & \left.-8(1+\alpha)(1+3\alpha)[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)]\right|. \end{aligned}$$

Let $N = \frac{(1-\beta)^2}{288(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$,$a = 9(1+2\alpha)^2$, $b = 8(1+\alpha)(1+3\alpha)$ and $a - b = 9(1+2\alpha)^2 - 8(1+\alpha)(1+3\alpha) = 1 + 12\alpha^2 + 4\alpha \geq 0$, since $\alpha \geq 0$.

$$\begin{aligned}|a_2a_4 - a_3^2| &= N\left|ac_1[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right.\\ &\left.\quad -b[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)]\right|\\ &= N\left|c_1^4(a-b) + 2xc_1^2(4-c_1^2)(a-b) - x^2(4-c_1^2)[ac_1^2 + b(4-c_1^2)]\right.\\ &\left.\quad +2yac_1(1-|x|^2)(4-c_1^2)\right|.\end{aligned}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq N\left\{c^4(a-b) + 2\rho c^2(4-c^2)(a-b)\right.\\ &\left.\quad +\rho^2(4-c^2)[c^2(a-b) - 2ac + 4b] + 2ac(4-c^2)\right\}\\ &= N\left\{c^4(a-b) + 2\rho c^2(4-c^2)(a-b)\right.\\ &\left.\quad +\rho^2(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)}) + 2ac(4-c^2)\right\} = F(\rho).\end{aligned}$$

Differentiating $F(\rho)$, we get

$$F'(\rho) = N[2c^2(4-c^2)(a-b) + 2\rho(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)})] \geq 0,$$

since $a - b > 0$, $2b/(a-b) > 2$ so that $c - 2b/(a-b) < c - 2 < 0$ and $(c-2)(c - \frac{2b}{(a-b)}) > 0$ for all $c \in [0,2]$. This implies that $F(\rho)$ is an increasing function of ρ on the closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is,

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq N\left\{c^4(a-b) + 2c^2(4-c^2)(a-b)\right.\\ &\left.\quad +(4-c^2)(a-b)(c-2)(c - \frac{2b}{(a-b)}) + 2ac(4-c^2)\right\}\\ &= N\left\{-2c^4(a-b) - 4c^2(4b-3a) + 16b\right\} = G(c).\end{aligned}$$

$G'(c) = 0$ implies $c = 0$ so that at $c = 0$,$G''(c) < 0$. Therefore $c = 0$ is a point of maximum for $G(c)$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence, $|a_2a_4 - a_3^2| \leq \frac{4}{9}\frac{(1-\beta)^2}{(1+2\alpha)^2}$. □

Corollary 3.2. *Choosing $\alpha = 0$, $\beta = 0$ in (11), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. This result coincides with [7].*

Corollary 3.3. *Choosing $\alpha = 0$ in (11), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}(1-\beta)^2$. This result coincides with [11].*

Lemma 3.3. *Let $f \in R_\alpha^\beta$. Then for $1/3 \leq \beta < 1$,*

$$|a_3 - a_2^2| \leq \frac{2(1-\beta)}{3(1+2\alpha)} \tag{13}$$

Proof. Let $f \in R_\alpha^\beta$. Then by proceeding as in Lemma 3.1, we have

$$|(a_3 - a_2^2)| = \left| \frac{c_2(1-\beta)}{3(1+2\alpha)} - \frac{c_1^2(1-\beta)^2}{4(1+\alpha)^2} \right| \tag{14}$$

$$|(a_3 - a_2^2)| = \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2} \left| 4(1+\alpha)^2 c_2 - 3(1+2\alpha)(1-\beta)c_1^2 \right|.$$

Substituting for c_2 from (6) of Lemma 2.2, we obtain

$$\begin{aligned} &|(a_3 - a_2^2)| \\ &= \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2} \left| 2(1+\alpha)^2[c_1^2 + x(4-c_1^2)] - 3(1+2\alpha)(1-\beta)c_1^2 \right| \\ &= M|k_1c_1^2 + k_1x(4-c_1^2) - k_2c_1^2|, \end{aligned}$$

where $M = \frac{(1-\beta)}{12(1+2\alpha)(1+\alpha)^2}, k_1 = 2(1+\alpha)^2, k_2 = 3(1+2\alpha)(1-\beta)$. Set $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \leq 1$, we get

$$|a_3 - a_2^2| \leq M[c^2|k_1 - k_2| + k_1\rho(4-c^2)] = F(\rho).$$

Differentiating $F(\rho)$, we get $F'(\rho) = M[k_1(4-c^2)] \geq 0$, implying that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is,

$$|(a_3 - a_2^2)| \leq M[c^2|k_1 - k_2| + k_1(4-c^2)] = G(c).$$

By hypothesis, $\beta \geq 1/3$ and hence $k_1 - k_2 = 2\alpha^2 - 2\alpha - 1 + 3\beta(1+2\alpha) \geq 2\alpha^2$. Hence $G(c) = M[4k_1 - c^2k_2]$, $G'(c) = -2Mk_2c$ and $G''(c) = -2Mk_2$. Since $c \in [0,2]$, it follows that G(c) attains the maximum at $c = 0$. Thus, the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|a_3 - a_2^2| \leq M[4k_1] = \frac{2(1-\beta)}{3(1+2\alpha)}$. $\square$

Corollary 3.4. *Choosing $\alpha = 0$ in (13), we get $|a_3 - a_2^2| \le \frac{2}{3}(1-\beta)$. This result coincides with [10], for $1/3 \le \beta < 1$.*

Remark 3.1. *Let $f \in R_\alpha^\beta$. By Lemma 2.1, we have*

$$|a_3| = \left|\frac{c_2(1-\beta)}{3(1+2\alpha)}\right| \le \frac{2(1-\beta)}{3(1+2\alpha)},$$
$$|a_4| = \left|\frac{c_3(1-\beta)}{4(1+3\alpha)}\right| \le \frac{(1-\beta)}{2(1+3\alpha)},$$
$$|a_5| = \left|\frac{c_4(1-\beta)}{5(1+4\alpha)}\right| \le \frac{2(1-\beta)}{5(1+4\alpha)}.$$

Using the above results, the upper bound for $|H_3(1)|$, $f \in R_\alpha^\beta$ is immediately obtained.

Theorem 3.1. *Let $f \in R_\alpha^\beta$. Then for $1/3 \le \beta < 1$,*

$$|H_3(1)| \le \frac{8(1-\beta)^3}{27(1+2\alpha)^3} + \frac{(1-\beta)^2}{4(1+3\alpha)^2} + \frac{4(1-\beta)^2}{15(1+2\alpha)(1+4\alpha)}.$$

In the following results, with similar approach and technique, an upper bound for $|H_3(1)|$ is attained for $f \in S_\alpha^\beta$. As before, we first derive estimates for the functionals $|a_2a_3 - a_4|$, $|a_2a_4 - a_3^2|$ and $|a_3 - a_2^2|$. Their estimates are given in Lemmas 3.4, 3.5, and 3.6.

Lemma 3.4. *Let $f \in S_\alpha^\beta$. Then*

$$|a_2a_3 - a_4| \le \frac{2(1-\beta)}{3(1+4\alpha)} \tag{15}$$

Proof. Let $f \in S_\alpha^\beta$. Then there exists a $p \in P$ such that

$$zf'(z) + \alpha z^2 f''(z) = [(1-\beta)p(z) + \beta]f(z),$$

for some $z \in \Delta$. Equating the coefficients, we have

$$a_2 = \frac{c_1(1-\beta)}{1+2\alpha}, \quad a_3 = \frac{c_2(1-\beta)}{2(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{2(1+2\alpha)(1+3\alpha)},$$

$$a_4 = \frac{c_3(1-\beta)}{3(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)},$$

and

$$\begin{aligned} a_5 &= \frac{c_1^4(1-\beta)^4}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_2^2(1-\beta)^2}{8(1+3\alpha)(1+5\alpha)} \\ &+ \frac{c_1^2c_2(1-\beta)^3(20\alpha+6)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_1c_3(1-\beta)^2(4+14\alpha)}{12(1+2\alpha)(1+4\alpha)(1+5\alpha)} \\ &+ \frac{c_4(1-\beta)}{4(1+5\alpha)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} |a_2a_3 - a_4| = \frac{(1-\beta)}{6(1+2\alpha)^2(1+3\alpha)(1+4\alpha)} &\big|c_1c_2(1-\beta)4\alpha(1+2\alpha) \\ +2c_1^3(1-\beta)^2(1+5\alpha) - 2c_3(1+2\alpha)^2(1+3\alpha)\big| \end{aligned} \tag{16}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we have

$$\begin{aligned} &|a_2a_3 - a_4| \\ &= B(\alpha,\beta)\left|\frac{4\alpha(1+2\alpha)(1-\beta)c_1}{2}(c_1^2 + x(4-c_1^2))\right. \\ &\quad +c_1^3 2(1-\beta)^2(1+5\alpha) - \frac{2(1+2\alpha)^2(1+3\alpha)}{4}[c_1^3 + 2xc_1(4-c_1^2) \\ &\quad \left. -x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right| \\ &= B(\alpha,\beta)\left|r_1c_1^3 + r_2c_1x(4-c_1^2) + \frac{r_3}{4}x^2c_1(4-c_1^2) - \frac{r_3}{2}y(1-|x|^2)(4-c_1^2)\right|, \end{aligned}$$

where

$$\begin{aligned} B(\alpha,\beta) &= \frac{(1-\beta)}{6(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}, \\ r_1 &= 2\alpha(1+2\alpha)(1-\beta) + 2(1-\beta)^2(1+5\alpha) - \frac{(1+2\alpha)^2(1+3\alpha)}{2}, \\ r_2 &= 2\alpha(1+2\alpha)(1-\beta) - (1+2\alpha)^2(1+3\alpha), r_3 = (1+2\alpha)^2(1+3\alpha). \end{aligned}$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \le 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying the triangle inequality with $\rho = |x| \le 1$, we get,

$$\begin{aligned} &|a_2a_3 - a_4| \\ &\le \beta(\alpha,\beta)\{|r_1|c^3 + |r_2|\rho c(4-c^2) + \frac{r_3}{2}\rho^2c(4-c^2) + r_3(4-c^2) - r_3\rho^2(4-c^2)\} \\ &= \beta(\alpha,\beta)\{|r_1|c^3 + |r_2|\rho c(4-c^2) + \frac{r_3}{2}\rho^2(c-2)(4-c^2) + r_3(4-c^2)\} = F(\rho). \end{aligned}$$

Next we maximize the function $F(\rho)$. Differentiating $F(\rho)$, we get

$$F'(\rho) = B(\alpha,\beta)[|r_2|c(4-c^2) + r_3\rho(4-c^2)(c-2)].$$

$F'(\rho) = 0$ implies $\rho = \frac{|r_2|c}{r_3(2-c)}$. Set $\rho^* = \frac{|r_2|c}{r_3(2-c)}$. Now, $0 \le \rho^* \le 1$. Also we have $F''(\rho) = B(\alpha,\beta)r_3(4-c^2)(c-2) \le 0$. Thus ρ^* is the only value in $[0,1]$ at which $F(\rho)$ attains maximum. Hence $F(\rho) \le F(\rho^*)$.Thus

$$\begin{aligned} F(\rho) &\le B(\alpha,\beta)[|r_1|c^3 + \frac{r_2^2c^2(2+c)}{2r_3} + 4r_3 - r_3c^2] \\ &= B(\alpha,\beta)[c^3\gamma - c^2\delta + 4r_3] = G(c), \end{aligned}$$

where $\gamma = |r_1| + \frac{r_2^2}{2r_3}$, $\delta = r_3 - \frac{r_2^2}{r_3} \ge 0$, $G'(c) = 0$ implies $c = 0$ and at $c = 0$, $G''(c) < 0$. Therefore $c = 0$ is a point of maximum of $G(c)$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = \rho^*$ and $c = 0$. Hence $|a_2a_3 - a_4| \le \frac{2(1-\beta)}{3(1+4\alpha)}$. □

Corollary 3.5. *Choosing $\alpha = 0$, $\beta = 0$ in (15), we get $|a_2a_3 - a_4| \le \frac{2}{3}$.*

Corollary 3.6. *Choosing $\alpha = 0$, in (15), we get*

$$|a_2a_3 - a_4| \le \frac{2(1-\beta)}{3}.$$

Lemma 3.5. *Let $f \in S_\alpha^\beta$. Then*

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+3\alpha)^2} \tag{17}$$

Proof. Let $f \in S_\alpha^\beta$. Then by proceeding as in Lemma 3.4, we have

$$\begin{aligned} &|a_2a_4 - a_3^2| = \\ &\left| \frac{c_1c_3(1-\beta)^2}{3(1+2\alpha)(1+4\alpha)} - \frac{c_2^2(1-\beta)^2}{4(1+3\alpha)^2} - \frac{c_1^4(1-\beta)^4(1+6\alpha)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right. \\ &\left. - \frac{c_1^2c_2(1-\beta)^3(2\alpha)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right| \end{aligned} \tag{18}$$

$$\begin{aligned} &= \frac{(1-\beta)^2}{48(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \left| 16(1+2\alpha)(1+3\alpha)^2c_1c_3 \right. \\ &\left. -12c_2^2(1+2\alpha)^2(1+4\alpha) - 4c_1^4(1-\beta)^2(1+6\alpha) - 4(1-\beta)2\alpha c_1^2c_2 \right|. \end{aligned}$$

Substituting for c_2 and c_3 from (6) and (7) of Lemma 2.2, we obtain

$$|a_2a_4 - a_3^2| = M\left|k_1c_1[c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2)]\right.$$
$$\left. -k_2[c_1^4 + x^2(4-c_1^2)^2 + 2xc_1^2(4-c_1^2)] - k_3c_1^4 - k_4c_1^2[c_1^2 + x(4-c_1^2)\right|,$$

where $M = \frac{(1-\beta)^2}{48(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}$,
$k_1 = 4(1+2\alpha)(1+3\alpha)^2$,$k_2 = 3(1+2\alpha)^2(1+4\alpha)$, $k_3 = 4(1-\beta)^2(1+6\alpha)$ and $k_4 = 8\alpha(1-\beta)$.

$$|a_2a_4 - a_3^2| =$$
$$M\left|c_1^4[k_1-k_2-k_3-k_4] + xc_1^2(4-c_1^2)[2k_1-2k_2-k_4] - x^2c_1^2(4-c_1^2)k_1\right.$$
$$\left. -x^2(4-c_1^2)^2k_2 + 2yc_1k_1(1-|x|^2)(4-c_1^2)\right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \le 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying triangle inequality with $\rho = |x| \le 1$, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| \le {} & M\left\{c^4|k_1-k_2-k_3-k_4| + \rho c^2(4-c^2)|2k_1-2k_2-k_4|\right. \\ & \left. +\rho^2(4-c^2)(c^2(k_1-k_2) - 2ck_1 + 4k_2) + 2ck_1(4-c^2)\right\} \\ = {} & M\left\{c^4|k_1-k_2-k_3-k_4| + \rho c^2(4-c^2)|2k_1-2k_2-k_4|\right. \\ & \left. +\rho^2(4-c^2)(k_1-k_2)(c-2)(c-\frac{2k_2}{k_1-k_2}) + 2ck_1(4-c^2)\right\} = F(\rho).\end{aligned}$$

Differentiating $F(\rho)$, we get

$$\begin{aligned}F'(\rho) = {} & M[c^2(4-c^2)|2k_1-2k_2-k_4| \\ & + 2\rho(4-c^2)(k_1-k_2)(c-2)(c-\frac{2k_2}{(k_1-k_2)}) \ge 0,\end{aligned}$$

since $2k_2/(k_1-k_2) > 2$ so that $c - 2k_2/(k_1-k_2) < c-2 < 0$ and $k_1 - k_2 = (1+2\alpha)(36\alpha^2+12\alpha+1) > 0$ as $\alpha > 0$, and so $(c-2)(c-\frac{2k_2}{(k_1-k_2)} > 0$ for all $c \in [0,2]$. This implies that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \le F(1)$ for all $\rho \in [0,1]$. That is,

$$\begin{aligned}& F(\rho) \\ & \le M\left\{c^4|k_1-k_2-k_3-k_4| + (4-c^2)[c^2|2k_1-2k_2-k_4| + (c^2(k_1-k_2)+4k_2)]\right\} \\ & = M\left\{[c^4[|k_1-k_2-k_3-k_4| - (|2k_1-2k_2-k_4| - (k_1-k_2))]\right. \\ & \left. -c^2[4k_2 - 4(|2k_1-2k_2-k_4| - 4(k_1-k_2))] + 16k_2\right\} = G(c).\end{aligned}$$

$G'(c) = 0$ implies $c = 0$ so that at $c = 0, G''(c) < 0$. Therefore $c = 0$ is a point of maximum for $G(c)$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+3\alpha)^2}$. □

Corollary 3.7. *Choosing* $\alpha = 0$, $\beta = 0$ *in* (17), *we get* $|a_2a_4 - a_3^2| \leq 1$. *This result coincides with [8].*

Corollary 3.8. *Choosing* $\alpha = 0$, *in* (17), *we get* $|a_2a_4 - a_3^2| \leq (1-\beta)^2$.

Lemma 3.6. *Let* $f \in S_\alpha^\beta$. *Then for* $1/2 \leq \beta < 1$,

$$|a_3 - a_2^2| \leq \frac{1-\beta}{1+3\alpha} \tag{19}$$

Proof. Let $f \in S_\alpha^\beta$. Then by proceeding as in Lemma 3.4, we have

$$|a_3 - a_2^2| = \left|\frac{c_2(1-\beta)}{2(1+3\alpha)} - \frac{c_1^2(1-\beta)^2(1+4\alpha)}{2(1+2\alpha)^2(1+3\alpha)}\right| \tag{20}$$

Substituting for c_2 from Lemma 2.2 we obtain

$$\begin{aligned}|a_3 - a_2^2| &= \left|\frac{(1-\beta)}{2(1+3\alpha)}\frac{1}{2}[c_1^2 + x(4-c_1^2)] - \frac{c_1^2(1-\beta)^2(1+4\alpha)}{2(1+3\alpha)(1+2\alpha)^2}\right| \\ &= \frac{(1-\beta)}{4(1+2\alpha)^2(1+3\alpha)}\left|[c_1^2 + x(4-c_1^2)](1+2\alpha)^2 - 2c_1^2(1-\beta)(1+4\alpha)\right| \\ &= M\left|b_1[c_1^2 + x(4-c_1^2)] - b_2c_1^2\right|,\end{aligned}$$

where $M = \frac{(1-\beta)}{4(1+2\alpha)^2(1+3\alpha)}, b_1 = (1+2\alpha)^2, b_2 = 2(1-\beta)(1+4\alpha)$. Therefore

$$|a_3 - a_2^2| = M\left|b_1c_1^2 + b_1x(4-c_1^2) - b_2c_1^2\right| = M\left|(b_1-b_2)c_1^2 + b_1x(4-c_1^2)\right|.$$

Suppose now that $c_1 = c$. Since $|c| = |c_1| \leq 2$, using the Lemma 2.1, we may assume without restriction that $c \in [0,2]$ and on applying triangle inequality with $\rho = |x| \leq 1$, we obtain

$$|a_3 - a_2^2| \leq M[c^2|b_1 - b_2| + b_1\rho(4-c^2)] = F(\rho).$$

Differentiating $F(\rho)$, we get $F'(\rho) = Mb_1(4-c^2) > 0$, implying that $F(\rho)$ is an increasing function of ρ on a closed interval [0,1]. Hence $F(\rho) \leq F(1)$ for all $\rho \in [0,1]$. That is

$$|(a_3 - a_2^2)| \leq M[c^2|b_1 - b_2| + b_1(4-c^2)] = G(c).$$

By hypothesis, $\beta \geq 1/2$ and hence $b_1 - b_2 = 4\alpha^2 - 4\alpha - 1 + 2\beta(1+4\alpha) \geq 4\alpha^2$. Hence $G(c) = M[4b_1 - b_2c^2]$, $G'(c) = -2b_2Mc$ and $G''(c) = -2b_2M$. Since $c \in [0,2]$, it follows that $G(c)$ attains a maximum at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Hence $|(a_3 - a_2^2)| \leq \frac{(1-\beta)}{(1+3\alpha)}$. □

Corollary 3.9. *Choosing $\alpha = 0$ in (19),we get $|a_3 - a_2^2| \leq (1-\beta)$.*

Using Lemma 2.1,the following estimates can be deduced.

Remark 3.2. *Let $f \in S_\alpha^\beta$. By Lemma 2.1, we have*

$$\begin{aligned}
|a_3| &= \left|\frac{c_2(1-\beta)}{2(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{2(1+2\alpha)(1+3\alpha)}\right|, \\
&\leq \frac{(1-\beta)(3+2\alpha-2\beta)}{(1+2\alpha)(1+3\alpha)}, \\
|a_4| &= \left|\frac{c_3(1-\beta)}{3(1+4\alpha)} + \frac{c_1c_2(3+8\alpha)(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)}\right|, \\
&\leq \frac{(1-\beta)[12+12\alpha^2+4\beta^2-16\alpha\beta-14\beta+26\alpha]}{3(1+2\alpha)(1+3\alpha)(1+4\alpha)} \quad \textit{and} \\
|a_5| &= \left|\frac{c_1^4(1-\beta)^4}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}\right. \\
&\quad + \frac{c_2^2(1-\beta)^2}{8(1+3\alpha)(1+5\alpha)} + \frac{c_1^2c_2(1-\beta)^3(20\alpha+6)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)} \\
&\quad \left.+ \frac{c_1c_3(1-\beta)^2(4+14\alpha)}{12(1+2\alpha)(1+4\alpha)(1+5\alpha)} + \frac{c_4(1-\beta)}{4(1+5\alpha)}\right| \\
&\leq \frac{(1-\beta)\left\{\begin{array}{r} 120+288\alpha^2-16\beta^3+744\alpha^2+548\alpha-188\beta \\ +96\beta^2-600\alpha\beta+144\alpha^2\beta+160\alpha\beta^2 \end{array}\right\}}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

Finally, using the above results, an upper bound for $|H_3(1)|$, $f \in S_\alpha^\beta$ is immediately obtained. 3.2.

Theorem 3.2. *Let $f \in S_\alpha^\beta$. Then for $1/2 \leq \beta < 1$,*

$$\begin{aligned}
|H_3(1)| &\leq \frac{(1-\beta)^3(3+2\alpha-2\beta)}{(1+2\alpha)(1+3\alpha)^3} \\
&\quad + \frac{2(1-\beta)^2[12+12\alpha^2+4\beta^2-16\alpha\beta-14\beta+26\alpha]}{9(1+2\alpha)(1+3\alpha)(1+4\alpha)^2} \\
&\quad + \frac{(1-\beta)^2\left\{\begin{array}{r} 120+288\alpha^2-16\beta^3+744\alpha^2+548\alpha-188\beta \\ +96\beta^2-600\alpha\beta+144\alpha^2\beta+160\alpha\beta^2 \end{array}\right\}}{24(1+2\alpha)(1+3\alpha)^2(1+4\alpha)(1+5\alpha)}.
\end{aligned}$$

Remark 3.3. *The determination of the sharp estimates for $|H_3(1)|$ for functions belonging to the classes R_α^β and S_α^β remain to be explored.*

References

[1] K.O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequality Theory and Applications, S.S. Dragomir and J.Y. Cho, (Eds.), Nova Science Publishers, 6 (2010) 1–7.

[2] R. Bucur, D. Breaz and L. Georgescu, Third Hankel determinant for a class of analytic functions with respect to symmetric points, Acta Universitatis Apulensis, 42 (2015) 79-86.

[3] D. Banasal, S. Maharana and J. K. Prajapat, Third Order Hankel determinant for certain univalent functions, J. Korean Math.Soc. 52(6) (2015) 1139-1148.

[4] P.L. Duren, Univalent functions, Springer Verlag, New York Inc., 1983.

[5] R.M. Goel and B.S. Mehrok, A class of univalent functions, J. Austral. Math. Soc. (Series A) 35 (1983) 1–17.

[6] T. Hayami and S. Owa, Generalized Hankel determinant for certain classes, Int. Journal of Math. Analysis, 4(52) (2010) 2473–2585.

[7] A. Janteng, S. Abdulhalim and M. Darus, Coefficient inequality for a function whose derivative has positive real part, J. Inequalities Pure and Applied Math. 7(2) Art. 50 (2006) 1-5.

[8] A. Janteng, S. Abdulhalim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal., 1(13) (2007) 619-625.

[9] A. Janteng, S.A. Halim and M. Darus, Estimate on the second Hankel functional for functions whose derivative has a positive real part, Journal of Quality Measurement and Analysis, 4(1) (2008) 189–195.

[10] D. V. Krishna, B. Venkateswarlu, T. RamReddy, Third Hankel determinant for bounded turning functions of order alpha, J. Nigerian Mathematical Society, 34 (2015) 121-127.

[11] D. V. Krishna, T. RamReddy, Coefficient inequality for a function whose derivative has a positive real part of order α, Mathematica Bohemica 140(1) (2015) 43-52.

[12] N. Kharudin, A. Akbarally, D. Mohamad and S.C. Soh, The second Hankel determinant for the class of close to convex functions, European J. Scientific Research, 66(3) (2011) 421–427.

[13] R.J. Libera and E.J. Zlotkiewiez, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc., 85(2) (1982) 225–230.

[14] R.J. Libera and E.J. Zlotkiewiez, Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer. Math. Soc., 87(2) (1983) 251–257.

[15] T.H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc., 104 (1962) 532–537.

[16] A.K. Mishra, J.K. Prajapat and S. Maharana, Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points, Cogent Mathematics, 3: 1160557 (2016).

[17] J.W. Noonan and D.K. Thomas. On the second Hankel determinant of areally mean p-valent functions. Trans. Amer. Math. Soc.,223(2)(1976) 337-346

[18] K.I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures et Appl., 28(c) (1983) 731–739.

[19] M.S. Robertson, On the theory of univalent functions, Annals of Math., 37 (1936) 374–408.

[20] C. Selvaraj and N. Vasanthi, Coefficient bounds for certain subclasses of close-to-convex functions, Int. J. Math. Analysis, 4(37) (2010) 1807–1814.

[21] G. Shanmugam, B. A. Stephen and K.G. Subramanian, Second Hankel determinant for certain classes of analytic functions, Bonfring Int. J. Data Mining, 2(2) (2012) 57–60.

[22] T.V. Sudharsan, S.P. Vijayalakshmi, B.A. Stephen, Third Hankel determinant for a subclass of analytic univalent functions, Malaya J. Matematik, 2(4)(2014) 438-444.

S.P. Vijayalakshmi
Department of Mathematics,
Ethiraj College, Chennai - 600008 India
Email:vijishreekanth@gmail.com

T.V. Sudharsan
Department of Mathematics,
SIVET College, Chennai - 600073 India
Email:tvsudharsan@rediffmail.com

Daniel Breaz
Department of Mathematics, "1 Decembrie 1918"
University of Alba-Iulia,
Alba-Iulia, Romania
email: dbreaz@uab.ro

K.G.Subramanian
Department of Mathematics,
Madras Christian College,
Chennai 600059 India
Email:kgsmani1948@gmail.com