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On partial sums of normalized Mittag-Leffler functions

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Abstract

This article deals with the ratio of normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$ and its sequence of partial sums $(\mathbb{E}_{\alpha,\beta})_m(z)$. Several examples which illustrate the validity of our results are also given.

1 Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathcal{U}$.

Consider the function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},\ \alpha > 0,\ z \in \mathcal{U}, \tag{1.2}$$

where $\Gamma(s)$ denotes the familiar Gamma function. This function was introduced by Mittag-Leffler in 1903 [9] and is therefore known as the Mittag-Leffler function.

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Another function $E_{\alpha,\beta}(z)$, having similar properties to those of Mittag-Leffler function, was introduced by Wiman [20], [21] and is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},\ \alpha>0,\ \beta>0,\ z\in\mathcal{U}. \tag{1.3}$$

During the last years the interest in Mittag-Leffler type functions has considerably increased due to their vast potential of applications in applied problems such as fluid flow, electric networks, probability, statistical distribution theory etc. For a detailed account of properties, generalizations and applications of functions (1.2) - (1.3) one may refer to [6], [7], [12], [16].

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ were recently investigated by Bansal and Prajapat in [1]. Differential subordination results associated with generalized Mittag-Leffler function were also obtained in [14].

The function defined by (1.3) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$:

$$\mathbb{E}_{\alpha,\beta}(z) = \Gamma(\beta) z E_{\alpha,\beta}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} z^{n+1},\ \alpha>0,\ \beta>0,\ z\in\mathcal{U}. \tag{1.4}$$

Note that some special cases of $\mathbb{E}_{\alpha,\beta}(z)$ are:

$$\begin{cases} \mathbb{E}_{2,1}(z) = z\cosh\sqrt{z} \\ \mathbb{E}_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}) \\ \mathbb{E}_{2,3}(z) = 2[\cosh(\sqrt{z})-1] \\ \mathbb{E}_{2,4}(z) = 6[\sinh(\sqrt{z})-\sqrt{z}]/\sqrt{z}. \end{cases} \tag{1.5}$$

Recently, several results related to partial sums of special functions, such as Bessel [10], Struve [22], Lommel [2] and Wright functions [3] were obtained.

Motivated by the work of Bansal and Prajapat [1] and also by the above mentioned results, in this paper we investigate the ratio of normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$ defined by (1.4) to its sequence of partial sums

$$\begin{cases} (\mathbb{E}_{\alpha,\beta})_0(z) = z \\ (\mathbb{E}_{\alpha,\beta})_m(z) = z + \sum_{n=1}^{m} A_n z^{n+1},\ m\in\mathbb{N}=\{1,2,\ldots\}, \end{cases} \tag{1.6}$$

where

$$A_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)},\ \alpha>0,\ \beta>0,\ n\in\mathbb{N}.$$

We obtain lower bounds on ratios like

$$\Re\left\{\frac{\mathbb{E}_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\alpha,\beta})_m(z)}{\mathbb{E}_{\alpha,\beta}(z)}\right\},\ \Re\left\{\frac{\mathbb{E}'_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})'_m(z)}\right\},\ \Re\left\{\frac{(\mathbb{E}_{\alpha,\beta})'_m(z)}{\mathbb{E}'_{\alpha,\beta}(z)}\right\}.$$

Several examples will be also given.

Results concerning partial sums of analytic functions may be found in [4], [8], [11], [13], [17], [18], [19].

2 Main results

In order to obtain our results we need the following lemma.

Lemma 2.1. *Let $\alpha \geq 1$ and $\beta \geq 1$. Then the function $\mathbb{E}_{\alpha,\beta}(z)$ satisfies the next two inequalities:*

$$|\mathbb{E}_{\alpha,\beta}(z)| \leq \frac{\beta^2+\beta+1}{\beta^2},\ z \in \mathcal{U} \tag{2.1}$$

$$|\mathbb{E}'_{\alpha,\beta}(z)| \leq \frac{\beta^2+3\beta+2}{\beta^2},\ z \in \mathcal{U}. \tag{2.2}$$

Proof. Under the hypothesis we have $\Gamma(n+\beta) \leq \Gamma(\alpha n+\beta)$ and thus

$$\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} \leq \frac{\Gamma(\beta)}{\Gamma(n+\beta)} = \frac{1}{(\beta)_n},\ n \in \mathbb{N}, \tag{2.3}$$

where

$$(x)_n = \begin{cases} 1 & ,\, n=0 \\ x(x+1)\ldots(x+n-1) & ,n \in \mathbb{N} \end{cases}$$

is the well-known Pochhammer symbol.

Note that

$$(x)_n = x(x+1)_{n-1},\ \ n \in \mathbb{N} \tag{2.4}$$

and

$$(x)_n \geq x^n,\ \ n \in \mathbb{N}. \tag{2.5}$$

Making use of (2.3) - (2.5) and also of the well-known triangle inequality, for $z \in \mathcal{U}$, we obtain

$$|\mathbb{E}_{\alpha,\beta}(z)| = \left|z + \sum_{n=1}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}z^{n+1}\right| \leq 1 + \sum_{n=1}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} \leq 1 + \sum_{n=1}^{\infty}\frac{1}{(\beta)_n}$$

$$= 1 + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{(\beta+1)_{n-1}} \leq 1 + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{(\beta+1)^{n-1}} = 1 + \frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{1}{\beta+1}\right)^{n}$$
$$= \frac{\beta^2+\beta+1}{\beta^2}$$

and thus, inequality (2.1) is proved.

Using once more the triangle inequality, for $z \in \mathcal{U}$, we obtain

$$|\mathbb{E}'_{\alpha,\beta}(z)| = \left|1 + \sum_{n=1}^{\infty}\frac{(n+1)\Gamma(\beta)}{\Gamma(\alpha n+\beta)}z^n\right| \leq 1 + \sum_{n=1}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha n+\beta)} + \sum_{n=1}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}. \tag{2.6}$$

For $\beta \geq 1$ we have

$$\frac{n}{(\beta)_n} = \frac{n}{\beta(\beta+1)_{n-1}} = \frac{n}{\beta(\beta+1)_{n-2}(\beta+n-1)} \leq \frac{1}{\beta(\beta+1)_{n-2}}. \tag{2.7}$$

Taking into account inequalities (2.3) - (2.5) and (2.7), from (2.6), we obtain

$$|\mathbb{E}'_{\alpha,\beta}(z)| \leq 1 + \sum_{n=1}^{\infty}\frac{n}{(\beta)_n} + \sum_{n=1}^{\infty}\frac{1}{(\beta)_n}$$
$$\leq 1 + \frac{1}{\beta} + \frac{1}{\beta}\sum_{n=2}^{\infty}\frac{1}{(\beta+1)_{n-2}} + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{(\beta+1)_{n-1}}$$
$$\leq 1 + \frac{1}{\beta} + \frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{1}{\beta+1}\right)^{n} + \frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{1}{\beta+1}\right)^{n} = \frac{\beta^2+3\beta+2}{\beta^2}$$

and thus, inequality (2.2) is also proved. □

Let $w(z)$ be an analytic function in $\mathcal{U}$. In the sequel, we will frequently use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0,\ z \in \mathcal{U} \text{ if and only if } |w(z)| < 1,\ z \in \mathcal{U}.$$

Theorem 2.1. *Let* $\alpha \geq 1$ *and* $\beta \geq \frac{1+\sqrt{5}}{2}$. *Then*

$$\Re\left\{\frac{\mathbb{E}_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})_m(z)}\right\} \geq \frac{\beta^2-\beta-1}{\beta^2},\ z \in \mathcal{U} \tag{2.8}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\alpha,\beta})_m(z)}{\mathbb{E}_{\alpha,\beta}(z)}\right\} \geq \frac{\beta^2}{\beta^2+\beta+1},\ z \in \mathcal{U}. \tag{2.9}$$

Proof. From inequality (2.1) we get

$$1+\sum_{n=1}^{\infty} A_n \leq \frac{\beta^2+\beta+1}{\beta^2}, \quad \text{where} \quad A_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}, \quad n \in \mathbb{N}.$$

The last inequality is equivalent to

$$\frac{\beta^2}{\beta+1} \sum_{n=1}^{\infty} A_n \leq 1.$$

In order to prove the inequality (2.8), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \frac{\beta^2}{\beta+1} \frac{\mathbb{E}_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})_m(z)} - \frac{\beta^2-\beta-1}{\beta+1}$$

or

$$\frac{1+w(z)}{1-w(z)} = \frac{1+\sum_{n=1}^{m} A_n z^n + \frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}. \tag{2.10}$$

From (2.10), we obtain

$$w(z) = \frac{\frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n z^n}{2+2\sum_{n=1}^{m} A_n z^n + \frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| < \frac{\frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n}{2-2\sum_{n=1}^{m} A_n - \frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n}.$$

The inequality $|w(z)| < 1$ holds true if and only if

$$\frac{2\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n \leq 2 - 2\sum_{n=1}^{m} A_n$$

which is equivalent to

$$\sum_{n=1}^{m} A_n + \frac{\beta^2}{\beta+1} \sum_{n=m+1}^{\infty} A_n \leq 1. \tag{2.11}$$

To prove (2.11), it suffices to show that its left-hand side is bounded above by

$$\frac{\beta^2}{\beta+1}\sum_{n=1}^{\infty} A_n$$

which is equivalent to

$$\frac{\beta^2-\beta-1}{\beta+1}\sum_{n=1}^{m} A_n \geq 0.$$

The last inequality holds true for $\beta \geq \dfrac{1+\sqrt{5}}{2}$.

We use the same method to prove inequality (2.9). Consider the function $w(z)$ given by

$$\frac{1+w(z)}{1-w(z)} = \frac{\beta^2+\beta+1}{\beta+1}\frac{(\mathbb{E}_{\alpha,\beta})_m(z)}{\mathbb{E}_{\alpha,\beta}(z)} - \frac{\beta^2}{\beta+1}.$$

From the last equality we obtain

$$w(z) = \frac{-\dfrac{\beta^2+\beta+1}{\beta+1}\displaystyle\sum_{n=m+1}^{\infty} A_n z^n}{2+2\displaystyle\sum_{n=1}^{m} A_n z^n - \dfrac{\beta^2-\beta-1}{\beta+1}\displaystyle\sum_{n=m+1}^{\infty} A_n z^n}$$

and

$$|w(z)| < \frac{\dfrac{\beta^2+\beta+1}{\beta+1}\displaystyle\sum_{n=m+1}^{\infty} A_n}{2-2\displaystyle\sum_{n=1}^{m} A_n - \dfrac{\beta^2-\beta-1}{\beta+1}\displaystyle\sum_{n=m+1}^{\infty} A_n}.$$

Then, $|w(z)| < 1$ if and only if

$$\frac{\beta^2}{\beta+1}\sum_{n=m+1}^{\infty} A_n + \sum_{n=1}^{m} A_n \leq 1. \tag{2.12}$$

Since the left-hand side of (2.12) is bounded above by

$$\frac{\beta^2}{\beta+1}\sum_{n=1}^{\infty} A_n$$

we have that the inequality (2.9) holds true. Now, the proof of our theorem is completed. □

In the next theorem we consider ratios involving derivatives.

Theorem 2.2. *Let $\alpha \geq 1$ and let $\beta \geq \dfrac{3+\sqrt{17}}{2}$. Then*

$$\Re\left\{\frac{\mathbb{E}'_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})'_m(z)}\right\} \geq \frac{\beta^2-3\beta-2}{\beta^2},\ z\in\mathcal{U} \tag{2.13}$$

and

$$\Re\left\{\frac{(\mathbb{E}_{\alpha,\beta})'_m(z)}{\mathbb{E}'_{\alpha,\beta}(z)}\right\} \geq \frac{\beta^2}{\beta^2+3\beta+2},\ z\in\mathcal{U}. \tag{2.14}$$

Proof. From (2.2) we have

$$1+\sum_{n=1}^{\infty}(n+1)A_n \leq \frac{\beta^2+3\beta+2}{\beta^2},\quad \text{where}\ \ A_n=\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)},\ n\in\mathbb{N}.$$

The above inequality is equivalent to

$$\frac{\beta^2}{3\beta+2}\sum_{n=1}^{\infty}(n+1)A_n \leq 1.$$

To prove (2.13), define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\beta^2}{3\beta+2}\frac{\mathbb{E}'_{\alpha,\beta}(z)}{(\mathbb{E}_{\alpha,\beta})'_m(z)} - \frac{\beta^2-3\beta-2}{3\beta+2}$$

which gives

$$w(z)=\frac{\dfrac{\beta^2}{3\beta+2}\displaystyle\sum_{n=m+1}^{\infty}(n+1)A_nz^n}{2+2\displaystyle\sum_{n=1}^{m}(n+1)A_nz^n+\dfrac{\beta^2}{3\beta+2}\displaystyle\sum_{n=m+1}^{\infty}(n+1)A_nz^n}$$

and

$$|w(z)| < \frac{\dfrac{\beta^2}{3\beta+2}\displaystyle\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\displaystyle\sum_{n=1}^{m}(n+1)A_n-\dfrac{\beta^2}{3\beta+2}\displaystyle\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The condition $|w(z)|<1$ holds true if and only if

$$\sum_{n=1}^{m}(n+1)A_n+\frac{\beta^2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{2.15}$$

The left-hand side of (2.15) is bounded above by

$$\frac{\beta^2}{3\beta+2}\sum_{n=1}^{\infty}(n+1)A_n \quad \text{if} \quad \frac{\beta^2-3\beta-2}{3\beta+2}\sum_{n=1}^{m}(n+1)A_n \geq 0$$

which holds true for $\beta \geq \dfrac{3+\sqrt{17}}{2}$.

The proof of (2.14) follows the same pattern. Consider the function $w(z)$ given by

$$\frac{1+w(z)}{1-w(z)} = \frac{\beta^2+3\beta+2}{3\beta+2}\frac{(\mathbb{E}_{\alpha,\beta})'_m(z)}{\mathbb{E}'_{\alpha,\beta}(z)} - \frac{\beta^2}{3\beta+2}$$

$$= \frac{1+\sum_{n=1}^{m}(n+1)A_n z^n - \frac{\beta^2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n z^n}{1+\sum_{n=1}^{\infty}(n+1)A_n z^n}. \tag{2.16}$$

From (2.16), we can write

$$w(z) = \frac{-\frac{\beta^2+3\beta+2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n z^n}{2+2\sum_{n=1}^{m}(n+1)A_n z^n - \frac{\beta^2-3\beta-2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n z^n}$$

and

$$|w(z)| < \frac{\frac{\beta^2+3\beta+2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n}{2-2\sum_{n=1}^{m}(n+1)A_n - \frac{\beta^2-3\beta-2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n}.$$

The last inequality implies that $|w(z)| < 1$ if and only if

$$\frac{2\beta^2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 2-2\sum_{n=1}^{m}(n+1)A_n$$

or equivalently

$$\sum_{n=1}^{m}(n+1)A_n + \frac{\beta^2}{3\beta+2}\sum_{n=m+1}^{\infty}(n+1)A_n \leq 1. \tag{2.17}$$

It remains to show that the left-hand side of (2.17) is bounded above by

$$\frac{\beta^2}{3\beta+2}\sum_{n=1}^{\infty}(n+1)A_n.$$

This is equivalent to

$$\frac{\beta^2-3\beta-2}{3\beta+2}\sum_{n=1}^{m}(n+1)A_n \geq 0 \ \text{ which holds true for } \ \beta \geq \frac{3+\sqrt{17}}{2}.$$

Now, the proof of our theorem is completed. □

3 Examples

In this section we give several examples which illustrate our theorems.

A result involving the functions $\mathbb{E}_{2,2}(z)$ and $\mathbb{E}_{2,3}(z)$, defined by (1.5), can be obtained from Theorem 2.1 by taking $m = 0, \alpha = 2, \beta = 2$ and $m = 0, \alpha = 2, \beta = 3$, respectively.

Corollary 3.1. *The following inequalities hold true:*

$$\Re\left\{\frac{\sinh(\sqrt{z})}{\sqrt{z}}\right\} \geq \frac{1}{4} = 0,25 \ , \ \ \Re\left\{\frac{\sqrt{z}}{\sinh(\sqrt{z})}\right\} \geq \frac{4}{7} \approx 0,57$$

and

$$\Re\left\{\frac{\cosh(\sqrt{z})-1}{z}\right\} \geq \frac{5}{18} \approx 0,28 \ , \ \ \Re\left\{\frac{z}{\cosh(\sqrt{z})-1}\right\} \geq \frac{18}{13} \approx 1,38.$$

Setting $m = 0, \alpha = 2$ and $\beta = 4$ in Theorem 2.1 and Theorem 2.2 respectively, we obtain the next result involving the function $\mathbb{E}_{2,4}(z)$, defined by (1.5), and its derivative.

Corollary 3.2. *The following inequalities hold true:*

$$\Re\left\{\frac{\sinh(\sqrt{z})-\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{11}{96} \approx 0,11 \ , \ \ \Re\left\{\frac{z\sqrt{z}}{\sinh(\sqrt{z})-\sqrt{z}}\right\} \geq \frac{32}{7} \approx 4,57$$

and

$$\Re\left\{\frac{\sqrt{z}\cosh(\sqrt{z})-\sinh(\sqrt{z})}{z\sqrt{z}}\right\} \geq \frac{1}{24} \approx 0,04 \ ,$$

$$\Re\left\{\frac{z\sqrt{z}}{\sqrt{z}\cosh(\sqrt{z})-\sinh(\sqrt{z})}\right\} \geq \frac{8}{5} = 1,6.$$

Remark 3.1. *If we consider $m = 0$ in inequality (2.13), we obtain $\Re\left\{\mathbb{E}'_{\alpha,\beta}(z)\right\} > 0$. In view of Noshiro-Warschawski Theorem (see [5]), we have that the normalized Mittag-Leffler function is univalent in $\mathcal{U}$ for $\alpha \geq 1$ and $\beta \geq \dfrac{3+\sqrt{17}}{2}$.*

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