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Properties on a subclass of univalent functions defined by using a multiplier transformation and Ruscheweyh derivative

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Abstract

In this paper we have introduced and studied the subclass $\mathcal{RI}(d,\alpha,\beta)$ of univalent functions defined by the linear operator $RI^{\gamma}_{n,\lambda,l}f(z)$ defined by using the Ruscheweyh derivative $R^n f(z)$ and multiplier transformation $I(n,\lambda,l)f(z)$, as $RI^{\gamma}_{n,\lambda,l}:\mathcal{A}\to\mathcal{A}$, $RI^{\gamma}_{n,\lambda,l}f(z)=(1-\gamma)R^n f(z)+\gamma I(n,\lambda,l)f(z)$, $z\in U$, where $\mathcal{A}_n=\{f\in\mathcal{H}(U): f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{RI}(d,\alpha,\beta)$.

1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U): f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$.

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Definition 1. *(Ruscheweyh [20]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z), \ ... \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*

$$R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!\,(j-1)!} a_j z^j, \ z \in U.$$

Definition 2. *For $f \in \mathcal{A}$, $n \in \mathbb{N}$, $\lambda, l \geq 0$, the operator $I(n,\lambda,l) f(z)$ is defined by the following infinite series*

$$I(n,\lambda,l) f(z) = z + \sum_{j=2}^{\infty} \left(\frac{\lambda (j-1) + l + 1}{l+1} \right)^n a_j z^j.$$

Remark 2. *It follows from the above definition that*

$$I(0,\lambda,l) f(z) = f(z),$$

$$(l+1) I(n+1,\lambda,l) f(z) = (l+1-\lambda) I(n,\lambda,l) f(z) + \lambda z (I(n,\lambda,l) f(z))',$$

$z \in U$.

Remark 3. *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^n = I(n,\lambda,0)$ was introduced and studied by Al-Oboudi [16], which is reduced to the Sălăgean differential operator [21] for $\lambda = 1$.*

Definition 3. *[7] Let $\gamma, \lambda, l \geq 0$, $n \in \mathbb{N}$. Denote by $RI_{n,\lambda,l}^{\gamma}$ the operator given by $RI_{n,\lambda,l}^{\gamma} : \mathcal{A} \to \mathcal{A}$, $RI_{n,\lambda,l}^{\gamma} f(z) = (1-\gamma) R^n f(z) + \gamma I(n,\lambda,l) f(z)$, $z \in U$.*

Remark 4. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$RI_{n,\lambda,l}^{\gamma} f(z) = z + \sum_{j=2}^{\infty} \left\{ \gamma \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$, $z \in U$.

This operator was studied also in [13], [14].

Remark 5. *For $\alpha = 0$, $RI_{m,\lambda,l}^{0} f(z) = R^m f(z)$, where $z \in U$ and for $\alpha = 1$, $RI_{m,\lambda,l}^{1} f(z) = I(m,\lambda,l) f(z)$, where $z \in U$, which was studied in [3], [4], [10], [9]. For $l = 0$, we obtain $RI_{m,\lambda,0}^{\alpha} f(z) = RD_{\lambda,\alpha}^{m} f(z)$ which was studied in [5], [6], [11], [12], [17], [18] and for $l = 0$ and $\lambda = 1$, we obtain $RI_{m,1,0}^{\alpha} f(z) = L_\alpha^m f(z)$ which was studied in [1], [2], [8], [15].*

We follow the works of A.R. Juma and H. Ziraz [19].

Definition 4. *Let the function* $f \in \mathcal{A}$. *Then* $f(z)$ *is said to be in the class* $\mathcal{RI}(d,\alpha,\beta)$ *if it satisfies the following criterion:*

$$|\frac{1}{d}(\frac{z(RI^{\gamma}_{n,\lambda,l}f(z))' + \alpha z^2 RI^{\gamma}_{n,\lambda,l}f(z))''}{(1-\alpha)RI^{\gamma}_{n,\lambda,l}f(z) + \alpha z(RI^{\gamma}_{n,\lambda,l}f(z))'} - 1)| < \beta, \tag{1}$$

where $d \in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{RI}(d,\alpha,\beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ, $0 \le \delta < 1$, for these functions.

2 Coefficient Inequality

Theorem 1. *Let the function* $f \in \mathcal{A}$. *Then* $f(z)$ *is said to be in the class* $\mathcal{RI}(d,\alpha,\beta)$ *if and only if*

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\cdot$$

$$\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j \le \beta|d|, \tag{2}$$

where $d \in \mathbb{C} - \{0\}$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $z \in U$.

Proof. Let $f(z) \in \mathcal{RI}(d,\alpha,\beta)$. Assume that inequality (2) holds true. Then we find that

$$|\frac{z(RI^{\gamma}_{n,\lambda,l}f(z))' + \alpha z^2 (RI^{\gamma}_{n,\lambda,l}f(z))''}{(1-\alpha)RI^{\gamma}_{n,\lambda,l}f(z) + \alpha z(RI^{\gamma}_{n,\lambda,l}f(z))'} - 1|$$

$$= |\frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^j}{z + \sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^j}|$$

$$\le \frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j |z|^{j-1}}$$

$$< \beta|d|.$$

Choosing values of z on real axis and letting $z \to 1^{-}$, we have

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j \le \beta|d|.$$

Conversely, assume that $f(z) \in \mathcal{RI}(d,\alpha,\beta)$, then we get the following inequality

$$Re\{\frac{z(RI_{n,\lambda,l}^{\gamma}f(z))'+\alpha z^2(RI_{n,\lambda,l}^{\gamma}f(z))''}{(1-\alpha)RI_{n,\lambda,l}^{\gamma}f(z)+\alpha z(RI_{n,\lambda,l}^{\gamma}f(z))'}-1|\} > -\beta|d|$$

$$Re\{\frac{z+\sum_{j=2}^{\infty}j(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}-1+\beta|d|\}>0$$

$$Re\frac{\beta|d|z+\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}>0.$$

Since $Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, the above inequality reduces to

$$\frac{\beta|d|r-\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j r^j}{r-\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j r^j}>0.$$

Letting $r \to 1^{-}$ and by the mean value theorem we have desired inequality (2).

This completes the proof of Theorem 1 □

Corollary 1. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RI}(d,\alpha,\beta)$. Then*

$$a_j \le \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad j \ge 2.$$

3 Distortion Theorems

Theorem 2. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RI}(d,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$r-\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^{n}+(1-\gamma)(n+1)\right]}r^2 \le |f(z)|$$

$$\leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r^2.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} z^2.$$

Proof. Given that $f(z) \in \mathcal{RI}(d, \alpha, \beta)$, from the equation (2) and since

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]$$

is non decreasing and positive for $j \geq 2$,then we have

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]\sum_{j=2}^{\infty} a_j \leq$$

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j$$
$$\leq \beta|d|,$$

which is equivalent to,

$$\sum_{j=2}^{\infty} a_j \leq \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]}. \tag{3}$$

Using (3), we obtain

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

$$|f(z)| \leq |z| + \sum_{j=2}^{\infty} a_j |z|^j \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} a_j$$

$$\leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r^2.$$

Similarly,

$$|f(z)| \geq r^2 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r^2.$$

This completes the proof of Theorem 2. □

Theorem 3. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RI}(d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$-\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r \le |f'(z)|$$

$$\le \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} z^2.$$

Proof. From (3)

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$$

$$|f'(z)| \le 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \le 1 + \sum_{j=2}^{\infty} j a_j r^{j-1}$$

$$\le 1 + \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r.$$

Similarly,

$$|f'(z)| \ge 1 - \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]} r.$$

This completes the proof of Theorem 3. □

4 Closure Theorems

Theorem 4. *Let the functions f_k, $k = 1, 2, ..., m$, defined by*

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j, \quad a_{j,k} \ge 0, \tag{4}$$

be in the class $\mathcal{RI}(d, \alpha, \beta)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{k=1}^{m} \mu_k f_k(z), \quad \mu_k \ge 0,$$

is also in the class $\mathcal{RI}(d,\alpha,\beta)$*, where*

$$\sum_{k=1}^{m}\mu_k = 1.$$

Proof. We can write

$$h(z) = \sum_{k=1}^{m}\mu_m z + \sum_{k=1}^{m}\sum_{j=2}^{\infty}\mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty}\sum_{k=1}^{m}\mu_k a_{j,k} z^j.$$

Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., m$, are in the class $\mathcal{RI}(d,\alpha,\beta)$, then from Theorem 1 we have

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\cdot$$

$$\left\{\gamma\left(\frac{1+\lambda\,(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_{j,k} \le \beta|d|.$$

Thus it is enough to prove that

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\cdot$$

$$\left\{\gamma\left(\frac{1+\lambda\,(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!\,(j-1)!}\right\}\left(\sum_{k=1}^{m}\mu_k a_{j,k}\right) =$$

$$\sum_{k=1}^{m}\mu_k\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\cdot$$

$$\left\{\gamma\left(\frac{1+\lambda\,(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_{j,k}$$

$$\le \sum_{k=1}^{m}\mu_k\beta|d| = \beta|d|.$$

Hence the proof is complete. □

Corollary 2. *Let the functions* f_k*,* $k = 1, 2$*, defined by (4) be in the class* $\mathcal{RI}(d,\alpha,\beta)$*. Then the function* $h(z)$ *defined by*

$$h(z) = (1-\zeta)f_1(z) + \zeta f_2(z), \quad 0 \le \zeta \le 1,$$

is also in the class $\mathcal{RI}(d,\alpha,\beta)$*.*

Theorem 5. *Let*

$$f_1(z) = z,$$

and

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j,$$

$j \geq 2$.
Then the function $f(z)$ is in the class $\mathcal{RI}(d,\alpha,\beta)$ if and only if it can be expressed in the form:

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z),$$

where $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

Proof. Assume that $f(z)$ can be expressed in the form

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) =$$

$$z + \sum_{j=2}^{\infty} \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \mu_j z^j.$$

Thus

$$\sum_{j=2}^{\infty} \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} \cdot$$

$$\frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \mu_j$$

$$= \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1.$$

Hence $f(z) \in \mathcal{RI}(d,\alpha,\beta)$.

Conversely, assume that $f(z) \in \mathcal{RI}(d,\alpha,\beta)$.

Setting

$$\mu_j = \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} a_j,$$

since

$$\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j.$$

Thus

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z).$$

Hence the proof is complete. □

Corollary 3. *The extreme points of the class* $\mathcal{RI}(d, \alpha, \beta)$ *are the functions*

$$f_1(z) = z,$$

and

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j,$$

$j \geq 2$.

5 Inclusion and Neighborhood Results

We define the δ- neighborhood of a function $f(z) \in \mathcal{A}$ by

$$N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta\}. \tag{5}$$

In particular, for $e(z) = z$

$$N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|b_j| \leq \delta\}. \tag{6}$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{RI}^{\xi}(d, \alpha, \beta)$ if there exists a function $h(z) \in \mathcal{RI}(d, \alpha, \beta)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1 - \xi, \quad z \in U, \quad 0 \leq \xi < 1. \tag{7}$$

Theorem 6. *If*

$$\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}$$

$$\geq \left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right], \quad j\geq 2$$

and

$$\delta=\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]},$$

then

$$\mathcal{R}\mathcal{I}(d,\alpha,\beta)\subset N_\delta(e).$$

Proof. Let $f\in\mathcal{R}\mathcal{I}(d,\alpha,\beta)$. Then in view of assertion (2) of Theorem 1 and the condition $\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} \geq \left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]$ for $j\geq 2$, we get

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]\sum_{j=2}^{\infty}a_j\leq$$

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j$$

$$\leq\beta|d|,$$

which implise

$$\sum_{j=2}^{\infty}a_j\leq\frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]}. \tag{8}$$

Applying assertion (2) of Theorem 1 in conjunction with (8), we obtain

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]\sum_{j=2}^{\infty}a_j\leq\beta|d|,$$

$$2(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]\sum_{j=2}^{\infty}a_j\leq 2\beta|d|$$

$$\sum_{j=2}^{\infty}ja_j\leq\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n+(1-\gamma)(n+1)\right]}=\delta,$$

by virtue of (5), we have $f\in N_\delta(e)$.

This completes the proof of the Theorem 6. □

Theorem 7. *If* $h \in \mathcal{RI}(d,\alpha,\beta)$ *and*

$$\xi = 1 - \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right] - \beta|d|}, \tag{9}$$

then

$$N_\delta(h) \subset \mathcal{RI}^\xi(d,\alpha,\beta).$$

Proof. Suppose that $f \in N_\delta(h)$, we then find from (5) that

$$\sum_{j=2}^{\infty} j|a_j - b_j| \le \delta,$$

which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty} |a_j - b_j| \le \frac{\delta}{2}. \tag{10}$$

Next, since $h \in \mathcal{RI}(d,\alpha,\beta)$ in the view of (8), we have

$$\sum_{j=2}^{\infty} b_j \le \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]}. \tag{11}$$

Using 10) and (11), we get

$$\left|\frac{f(z)}{h(z)} - 1\right| \le \frac{\sum_{j=2}^{\infty}|a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j} \le \frac{\delta}{2\left(1 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]}\right)}$$

$$\le \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right]}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(\frac{1+\lambda+l}{l+1}\right)^n + (1-\gamma)(n+1)\right] - \beta|d|} = 1 - \xi,$$

provided that ξ is given by (9), thus by condition (7), $f \in \mathcal{RI}^\xi(d,\alpha,\beta)$, where ξ is given by (9). □

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 8. *Let the function* $f \in \mathcal{A}$ *be in the class* $\mathcal{RI}(d,\alpha,\beta)$*. Then* f *is univalent starlike of order* δ, $0 \le \delta < 1$, *in* $|z| < r_1$, *where*

$$r_1 =$$

$$\inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j,$$

$j \geq 2$.

Proof. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1-\delta, \quad |z| < r_1.$$

Since

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} a_j z^{k-1}} | \right| \leq \frac{\sum_{j=2}^{\infty}(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j |z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty}(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j |z|^{j-1}} \leq 1-\delta.$$

It is equivalent to

$$\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1} \leq 1-\delta,$$

using Theorem 1, we obtain

$$|z| \leq \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. □

Theorem 9. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RI}(d, \alpha, \beta)$. Then f is univalent convex of order δ, $0 \leq \delta \leq 1$, in $|z| < r_2$, where*

$$r_2 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{k-p}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j,$$

$$j \geq 2. \tag{12}$$

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z))}\right| \leq 1-\delta, \qquad |z| < r_2.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} ja_j z^{j-1}}\right| \leq \frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j |z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j |z|^{j-1}} \leq 1-\delta,$$

$$\sum_{j=2}^{\infty} j(j-\delta)a_j |z|^{j-1} \leq 1-\delta,$$

using Theorem 1, we obtain

$$|z|^{j-1} \leq \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|},$$

or

$$|z| \leq \left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|}\right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. □

Theorem 10. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RI}(d,\alpha,\beta)$. Then f is univalent close-to-convex of order δ, $0 \leq \delta < 1$, in $|z| < r_3$, where*

$$r_3 = \inf_j \left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}\right\}^{\frac{1}{j-1}}.$$

The result is sharp for the function $f(z)$ given by (12).

Proof. It suffices to show that

$$|f'(z)-1| \leq 1-\delta, \quad |z| < r_3.$$

Then

$$|f'(z)-1| = \left|\sum_{j=2}^{\infty} ja_j z^{j-1}\right| \leq \sum_{j=2}^{\infty} ja_j |z|^{j-1}.$$

Thus $|f'(z)-1|\leq 1-\delta$ if $\sum_{j=2}^{\infty}\frac{ja_j}{1-\delta}|z|^{j-1}\leq 1$. Using Theorem 1, the above inequality holds true if

$$|z|^{j-1}\leq\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}$$

or

$$|z|\leq\left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}\right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. □

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