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Diameter and girth of Torsion Graph

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Abstract

Let R be a commutative ring with identity. Let M be an R-module and $T(M)^*$ be the set of nonzero torsion elements. The set $T(M)^*$ makes up the vertices of the corresponding torsion graph, $\Gamma_R(M)$, with two distinct vertices $x, y \in T(M)^*$ forming an edge if $Ann(x) \cap Ann(y) \neq 0$. In this paper we study the case where the graph $\Gamma_R(M)$ is connected with $diam(\Gamma_R(M)) \leq 3$ and we investigate the relationship between the diameters of $\Gamma_R(M)$ and $\Gamma_R(R)$. Also we study girth of $\Gamma_R(M)$, it is shown that if $\Gamma_R(M)$ contains a cycle, then $gr(\Gamma_R(M)) = 3$.

1 INTRODUCTION

Let R be a commutative ring with identity and M a unitary R-module. The idea of associating a graph with the zero-divisors of a commutative ring was introduced by Beck in [10], where the author talked about the colorings of such graphs. He lets every elements of R is a vertex in the graph, and two vertices x, y are adjacent if and only if $xy = 0$. In [5], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are non-zero zero-divisors while $x - y$ is an edge whenever $xy = 0$. Anderson and Badawi also introduced and investigated total graph of commutative ring in [1, 2]. The zero-divisor graph of a commutative ring has been studied extensively by several authors [3, 4, 6, 9, 14, 15, 16]. The concept of zero-divisor graph has been extended to non-commutative rings by Redmond [17].

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Let $x \in M$. The residual of Rx by M denoted by $[x : M] = \{r \in R|\ rM \subseteq Rx\}$. The annihilator of an R-module M, denoted by $Ann_R(M) = [0 : M]$. If $m \in M$, then $Ann(m) = \{r \in R|rm = 0\}$. Let $T(M) = \{m \in M|Ann(m) = 0\}$. It is clear that if R is an integral domain then $T(M)$ is a submodule of M, which is called torsion submodule of M. If $T(M) = 0$ then the module M is said torsion-free, and it is called a torsion module if $T(M) = M$.

In this paper, we investigate the concept of torsion-graph for modules as a natural generalization of zero-divisor graph for rings. Here the torsion graph $\Gamma_R(M)$ of M is a simple graph whose vertices are non-zero torsion elements of M and two different elements x, y are adjacent if and only if $Ann(x) \cap Ann(y) \neq 0$. Thus $\Gamma_R(M)$ is an empty graph if and only if M is a torsion-free R-module. Clearly if R is a domain or $Ann(M) \neq 0$, then $\Gamma_R(M)$ is complete. This study helps to illuminate the structure of $T(M)$, for example, if M is a multiplication R-module, we show that M is finite if and only if $\Gamma_R(M)$ is finite.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol $|\Gamma_R(M)|$ to denote the number of vertices in graph $\Gamma_R(M)$. Also, a graph G is connected if there is a path between any two distinct vertices. The distance, $d(x, y)$ between connected vertices x, y is the length of the shortest path from x to y, ($d(x, y) = \infty$ if there is no such path). An isolated vertex is a vertex that has no edges incident to it. The diameter of G is the diameter of connected graph which is the supremum of the distance between vertices. The diameter is zero if the graph consist of a single vertex. The girth of G, denoted by $gr(G)$ is defined as the length of the shortest cycle in G, ($gr(G) = \infty$ if G contains no cycle). A complete graph is a simple graph whose vertices are pairwise adjacent, the complete graph with n vertices is denoted K_n.

A ring R is called reduced if $Nil(R) = 0$. A ring R is von Neumann regular if for each $a \in R$, there exists an element $b \in R$ such that $a = a^2b$. It is clear that every von Neumann regular ring is reduced.

One may address three major problem in this area: characterization of the resulting graphs, characterization of module with isomorphic graphs, and realization of the connection between the structures of a module and the corresponding graph, in this paper we focus on the third problem.

The organization of this paper is as follows:

In section 2, we study the torsion graph of finite multiplication module, we show that if the torsion graph of multiplication R-module M is finite(when $\Gamma_R(M)$ is not empty) then M is finite, specially if $\Gamma_R(M)$ has an isolated vertex, then $M \cong M_1 \oplus M_2$, in which M_1, M_2 are simple submodule of M.

In section 3, we show that $\Gamma_R(M)$ is connected with $diam(\Gamma_R(M)) \leq 3$ if one of the following hold:

(1) $\Gamma_R(R)$ is a complete graph.

(2) R be a von Neumann regular ring and $R \ncong Ann(x) \oplus Ann(y)$ for any

$x, y \in T(M)^*$.

(3) $Nil((R) \neq 0$.

In section 4, we study the girth of torsion graph for an R-module M. It is shown that if $\Gamma_R(M)$ contains a cycle, then $gr(\Gamma_R(M)) = 3$

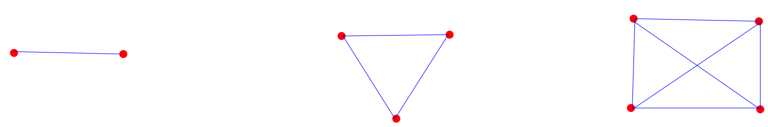
We follow standard notation and terminology from graph theory [12] and module theory [8].

2 Properties of torsion graph

This section is concerned with some basic and important results in the theory of torsion graphs over a module.

The following examples show that non-isomorphic modules may have the same torsion graph.

Example 2.1. Let $M = M_1 \oplus M_2$ be an R-module, where M_1 is a torsion-free module. So $T(M)^* = \{(0, m_2) \mid m_2 \in T(M_2)^*\}$. Below are the torsion graphs for some $\mathbb{Z}$-modules and ring R as R-modules.



$Z \oplus Z_3$ $R = Z_9$ $Z \oplus Z_4$ $Z_2 \oplus Z_2$ $Z \oplus Z_5$ $R = Z_{25}$

Lemma 2.2. *If R is an integral domain, then $\Gamma_R(M)$ is complete.*

Proof. Let R be an integral domain and $x, y \in T(M)^*$, so there is non-zero element $r, s \in R$ such that $rx = sy = 0$. Since R is an integral domain, $0 \neq rs \in Ann(x) \cap Ann(y)$. Thus $d(x, y) = 1$ and $\Gamma_R(M)$ is complete. $\square$

Before we go on discussing the other properties of $\Gamma_R(M)$, we give, the following theorem shows that for a multiplication R- module M, $\Gamma_R(M)$ is finite (except, when $\Gamma_R(M)$ is empty) if and only if M is finite.

Theorem 2.3. *Let M be an R-module with $Ann(x) = Ann([x : M]M)$ for all $x \in T(M)^*$. Then $\Gamma_R(M)$ is finite if and only if either M is finite or M is a torsion free R-module.*

Proof. Suppose that $\Gamma_R(M)$ is finite and nonempty. Let $x \in T(M)^*$, hence there is $0 \neq s \in [x : M]$. Let $N = [x : M]M$, so $0 \neq Ann(x) \subseteq Ann(n)$ for all $n \in N$, thus $N \subseteq T(M)^*$, therefore N is finite. Now if M is infinite, then there is a $n \in N$ with $H = \{m \in M \mid sm = n\}$ infinite, then for all distinct elements $m_1, m_2 \in H$, $s \in Ann(m_1 - m_2)$. So $m_1 - m_2 \in T(M)^*$, which is a contradiction, therefore M be finite. □

In the following example, it is shown that the condition $Ann(x) = Ann([x : M]M)$ for all $x \in T(M)^*$ in the above Theorem cannot be omitted.

Example 2.4. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_3$. Clearly M is not finite, but $V(\Gamma_R(M)) = \{(0, \bar{1}), (0, \bar{2})\}$ and so $\Gamma_R(M)$ is finite.

Corollary 2.5. *Let M be a multiplication R-module. Then $\Gamma_R(M)$ is finite if and only if either M is finite or M is a torsion free R-module.*

Theorem 2.6. *Let M be a multiplication R-module. If $\Gamma_R(M)$ has an isolated vertex, then $M = M_1 \oplus M_2$ is a faithful R-module, where M_1 and M_2 are two submodules of M such that M_1 has only two elements. Especially, if M is finite then M_2 is simple.*

Proof. Suppose that $x \in T(M)^*$ be an isolated vertex, so for all $y \in T(M)^*$ we have $Ann(x) \cap Ann(y) = 0$ and M is faithful. If $Rx \cap Ry = 0$, then there is vertex $z \in Rx \cap Ry$ that is adjacent to x, which is a contradiction. Thus $[x : M]y \in Rx \cap Ry = 0$. If $[x : M]x = 0$, then $[x : M] \in Ann(x) \cap Ann(y)$, which is a contradiction. Therefore $[x : M]x \neq 0$ and there is $\alpha \in [x : M]$ such that $\alpha x \neq 0$. Since x is an isolated vertex $Rx = \{0, x\}$, thus $\alpha x = x$. One can easily check that $M = Rx + Ann(x)M$. Now suppose that $w \in Rx \cap Ann(x)M$, thus $w = rx$ for some $r \in R$, hence $\alpha w = r\alpha x = rx = w$ and so $w = r\alpha x \in Ann(x)\alpha M = 0$. Therefore $M = M_1 \oplus M_2$, in which $|M_1| = |Rx| = 2$.

Now, suppose that M be a finite multiplication R-module. Since $M = M_1 \oplus M_2$, we have M_2 is finite and so M_2 is an Artinian R-module, Also by Theorem 2.2 and Corollary 2.9 [13], M_2 is cyclic, so $M_2 \cong \frac{R}{Ann(M_2)}$. Assume that

$$D(M_2) = \{m_2 \in M_2 | [m_2 : M][m_2' : M]M = 0\}.$$

We claim that $D(M_2) = 0$. If $D(M_2) \neq 0$, then there is a $0 \neq m_2 \in M_2$, such that

$$[m_2 : M][m_2' : M]M = 0$$

for some $0 \neq m_2' \in M_2$. Thus $\alpha m_2 = 0$ for some non-zero element $\alpha \in [m_2' : M]$. Also $\alpha x \in Rx \cap M_2 = 0$, so $\alpha(m_2 + x) = 0 = \alpha x$, which is a contradiction, consequently $D(M_2) = 0$. Now we show that $Ann(M_2)$ is prime ideal of R. Let $st \in Ann(M_2)$ for $s, t \in R$. So $stM_2 = 0$, hence

$$[sM_2 : M][tM_2 : M]M = 0.$$

Since $D(M_2) = 0$, we have $sM_2 = tM_2 = 0$. thus $Ann(M_2)$ is prime ideal of R. Hence $\frac{R}{Ann(M_2)}$ is a finite integral domain and so is a field, thus $Ann(M_2)$ is a maximal ideal of R. Therefore M_2 is a simple R-module. □

3 Diameter of torsion graph

In this section, we investigate the relationship between the diameter of $\Gamma_R(M)$ and $\Gamma_R(R)$. First, we study the case where $\Gamma_R(M)$ is connected with diameter ≤ 3.

Theorem 3.1. *Let M be an R-module. Then $\Gamma_R(M)$ is connected with $diam(\Gamma_R(M)) \leq 3$ if one of the following hold:*

(1) $\Gamma_R(R)$ is a complete graph.

(2) R be a von Neumann regular ring and $R \ncong Ann(x) \oplus Ann(y)$ for any $x, y \in T(M)^*$.

(3) $Nil((R) \neq 0$.

Proof. Let $x, y \in T(M)^*$ be two distinct elements. If $Ann(x) \cap Ann(y) \neq 0$ or $Ann(M) \neq 0$, then $d(x, y) = 1$. Therefore we suppose that M is faithful and $Ann(x) \cap Ann(y) = 0$. So there are two non-zero elements $s, t \in R$ such that $sx = ty = 0$ but $sy \neq 0, tx \neq 0$.

(1) Suppose that $\Gamma_R(R)$ is a complete graph, hence $Ann(s) \cap Ann(t) \neq 0$, so $x - tx - sy - y$ is a path of length 3. Hence $d(x, y) \leq 3$, thus $diam(\Gamma_R(M)) \leq 3$.

(2) Let R is a von Neumann regular ring. We know $s = u_1e_1$ and $t = u_2e_2$ for some non-zero idempotent elements e_1, e_2 and unit elements u_1, u_2 such that $(1 - e_1)(1 - e_2) \in Ann(s) \cap Ann(t)$. If $Ann(s) \cap Ann(t) = 0$, then $1 \in Rs + Rt \subseteq Ann(x) \cap Ann(y)$, hence $R \cong Ann(x) \oplus Ann(y)$, which is a contradiction. Therefore $Ann(s) \cap Ann(t) \neq 0$ and $x - tx - sy - y$ is a path of length 3, so $d(x, y) \leq 3$. Thus $diam(\Gamma_R(M)) \leq 3$.

(3) Let $0 \neq a \in Nil(R)$, so $a^n = 0$ and $a^{n-1} \neq 0$ for some $n \in \mathbb{N}$. Suppose that x, y are vertices of $\Gamma_R(M)$ such that $d(x, y) \neq 1$. If $ax = 0 = ay$ we have $d(x, y) \leq 2$. Let $ax = 0$ and $ay \neq 0$, so $a^{n-1} \in Ann(x) \cap Ann(y)$, hence $x - ay - y$ is a path of length 2 and $d(x, y) \leq 2$. If $ax \neq 0$ and $0 = ay$, then $x - ax - y$ is a path of length 2 and $d(x, y) \leq 2$. Also if $ax \neq 0$ and $ay \neq 0$, then $x - ax - ay - y$ is a path of length 3 and $d(x, y) \leq 3$. Therefore $diam(\Gamma_R(M)) \leq 3$.

□

The following example shows that $\Gamma_R(R)$ is complete in Theorem 3.1 (1) is crucial.

Example 3.2. Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_6$. Clearly $V(\Gamma_R(M)) = \{\bar{2}, \bar{3}, \bar{4}\}$ and vertex $\bar{3}$ is not adjacent to other vertices. This shows that $\Gamma_R(M)$ is not connected graph.

In the following example, it is shown that the condition $Nil(R) \neq 0$ in Theorem 3.1 (3) cannot be omitted.

Example 3.3. Let $R = \mathbb{Z}_6$ and $M = \mathbb{Z}_6 \oplus \mathbb{Z}_3$. Clearly

$$V(\Gamma_R(M)) = \{(0, \bar{1}), (0, \bar{2}), (\bar{2}, \bar{0}), (\bar{2}, \bar{1})(\bar{3}, \bar{0}), (\bar{4}, \bar{0}), (\bar{4}, \bar{1}), (\bar{4}, \bar{2}), (\bar{5}, \bar{0})\}.$$

It is easy to see that $(\bar{3}, \bar{0})$ is an isolated vertex and so $\Gamma_R(M)$ is not connected.

Corollary 3.4. *If $R = \mathbb{Z}_{p^n}$, where p is a prime number.*

$$Z(R)^* = \{\overline{p}, \overline{2p}, \dots, \overline{(p-1)p}, \overline{p^2}, \dots, \overline{(p-1)p^2}, \dots, \overline{p^{n-1}}, \dots, \overline{(p-1)p^{n-1}}\}.$$

Then $p^{n-1} \in Ann(x) \cap Ann(y)$, for every $x, y \in Z(R)^*$ and so $\Gamma_R(R)$ is a complete graph. Hence $\Gamma_R(M)$ is connected with $diam(\Gamma_R(M)) \leq 3$, for every R-module M.

Theorem 3.5. *Let M be a multiplication R-module and $Nil((R) \neq 0$. Then $\Gamma_R(M)$ is connected with $diam(\Gamma_R(M)) \leq 2$.*

Proof. Let $0 \neq a \in Nil(R)$, so $a^n = 0$ and $a^{n-1} \neq 0$ for some $n \in \mathbb{N}$. Suppose that x, y are vertices of $\Gamma_R(M)$ such that $d(x, y) \neq 1$. If $[x : M]y \neq 0$, then there is $0 \neq \alpha \in [x : M]$ such that $x - \alpha y - y$ is a path of length 2 and so $d(x, y) \leq 2$. If $[y : M]x \neq 0$, then similar to the above argument, we have $d(x, y) \leq 2$. If $ax = ay = 0$, then we have $d(x, y) \leq 2$. Let $ax = 0$ and $ay \neq 0$, so $a^{n-1} \in Ann(x) \cap Ann(y)$, hence $x - ay - y$ is a path of length 2. Therefore $diam(\Gamma_R(M)) \leq 2$. □

Theorem 3.6. *Let M be a multiplication R-module with $T(M) \neq M$. Then the following hold:*

(1) $\Gamma_R(M)$ is a complete graph if and only if $\Gamma_R(R)$ is a complete graph.

(2) If R be a Bézout ring, then $diam(\Gamma_R(R)) = diam(\Gamma_R(M))$.

Proof. (1) Let $\Gamma_R(M)$ be a complete graph and $Ann(m) = 0$ for some $m \in M$. Suppose that α, β are two vertices of $\Gamma_R(R)$. One can easily check that $\alpha m, \beta m \in T(M)^*$. Therefore $Ann(\alpha m) \cap Ann(\beta m) \neq 0$, so $r\alpha m = r\beta m = 0$ for some $0 \neq r \in R$. Hence $r\alpha = r\beta = 0$ and $d(\alpha, \beta) = 1$. Consequently $\Gamma_R(R)$ is a complete graph.

Now, let $\Gamma_R(R)$ be a complete graph, and $x, y \in T(M)^*$. So $Ann(x) \neq 0$ and $Ann(y) \neq 0$. Thus there are two non-zero elements $r, s \in R$ such that $rx = 0 = sy$. Hence $r[x : M] = 0 = s[y : M]$. So for all $\alpha \in [x : M]$ and $\beta \in [y : M]$ we have $r\alpha = 0 = s\beta$ and α, β are the vertices of $\Gamma_R(R)$. Therefore $0 \neq t \in Ann(\alpha) \cap Ann(\beta) \neq 0$. Let $x = \sum_{i=1}^{n} \alpha_i m_i$ and $y = \sum_{j=1}^{m} \beta_j m_j$, where $0 \neq \alpha_i \in [x : M], 0 \neq \beta_j \in [y : M]$. Hence $t \in Ann(x) \cap Ann(y)$ and $d(x, y) = 1$. Consequently $\Gamma_R(M)$ is a complete graph.

(2) Let R be a Bézout ring and M be a multiplication R-module. By (1), $diam(\Gamma_R(M)) = 1$ if and only if $diam(\Gamma_R(R)) = 1$. Suppose that $diam(\Gamma_R(R)) = 2$ and $x, y \in T(M)^*$ such that $d(x, y) \neq 1$. Let $x = \sum_{i=1}^{n} \alpha_i m_i$ and $y = \sum_{j=1}^{m} \beta_j m_j$, where $0 \neq \alpha_i \in [x : M], 0 \neq \beta_j \in [y : M]$. Since R is a Bézout ring, $\sum_{i=1}^{n} R\alpha_i = R\alpha$ and $\sum_{j=1}^{m} R\beta_j = R\beta$, for some $\alpha, \beta \in R$. Hence there exist $m, m_0 \in M$ such that $x = \alpha m$, $y = \beta m_0$. Thus $\alpha, \beta \in Z(R)^*$. If $d(\alpha, \beta) = 1$, then $d(x, y) = 1$, and so we have a contradiction. Thus $d(\alpha, \beta) = 2$, so there exists $\gamma \in Z(R)^*$ such that $\alpha - \gamma - \beta$ is a path of length 2 and there are non-zero elements $r, s \in R$ such that

$$r \in Ann(\alpha) \cap Ann(\gamma), s \in Ann(\gamma) \cap Ann(\beta)$$

Since $M \neq T(M)$, then there is $n \in M$ such that $\gamma n \in T(M)^*$. Therefore

$$r \in Ann(x) \cap Ann(\gamma n), s \in Ann(\gamma n) \cap Ann(y)$$

and $\alpha m = x - \gamma n - y = \beta m.$ is a path of length 2. So $d(x, y) = 2$ and $diam(\Gamma_R(M)) = 2$.

Suppose that $diam(\Gamma_R(M)) = 2$ and $\alpha, \beta \in Z(R)^*$ such that $d(\alpha, \beta) \neq 1$. Since $M \neq T(M)$, there is $n \in M$ such that $\alpha n \neq 0$ and $\beta n \neq 0$. Hence $\beta n \neq \alpha n \in T(M)^*$. One can easily check that $d(\alpha n, \beta n) \neq 1$. So $d(\alpha n, \beta n) = 2$, and there is $z = \gamma m \in T(M)^*$ such that $\alpha n - \gamma m - \beta n$, is a path of length 2. Thus $r\alpha n = 0 = rz$ for some $0 \neq r \in R$, so $r\gamma \in r[z : M] = 0$, hence $\alpha - \gamma - \beta$ is a path of length 2 and $d(\alpha, \beta) = 2$. Therefore $diam(\Gamma_R(R)) = 2$.

Now, by similar to above argument $diam(\Gamma_R(R)) = n$ if and only if $diam(\Gamma_R(M)) = n$. Consequently $diam(\Gamma_R(M)) = diam(\Gamma_R(R))$.

□

4 Girth of torsion graph

In this section we study the girth of torsion graph.

Theorem 4.1. *Let M be an R-module. If $\Gamma_R(M)$ contains a cycle, then $gr(\Gamma_R(M)) = 3$.*

Proof. Let $x-y-z-w-x$ be the shortest cycle of $T(M)$, so there are non-zero elements r, s such that $r \in Ann(x) \cap Ann(y)$ and $s \in Ann(y) \cap Ann(z)$. If $x + y = 0$, then $s \in Ann(x) \cap Ann(z)$ and so $x - y - z - x$ is a cycle, which is a contradiction. Hence suppose that $x + y \neq 0$, we know that $r \in Ann(x) \cap Ann(x + y)$ and $s \in Ann(x + y) \cap Ann(y)$. Thus $\Gamma_R(M)$ contains a cycle $x - x + y - y - x$ which is a contradiction. Consequently, $gr(\Gamma_R(M)) = 3$. □

As a result of Theorem4.1, we could say that the torsion graph of R-module M can not be an n-gon for $n \geq 4$.

Corollary 4.2. *Let M be an R-module. If $\Gamma_R(M)$ is a connected graph with $|\Gamma_R(M)| > 2$, then $\Gamma_R(M)$ contains a cycle and $gr(\Gamma_R(M)) = 3$*

Proof. Let $\Gamma_R(M)$ is a connected graph with $|\Gamma_R(M)| > 2$. Suppose that $x - y - z$ be the path in $\Gamma_R(M)$. By the same argument as in the proof of Theorem 4.1, and if $x + y = 0$, then $\Gamma_R(M)$ $x - y - z - x$ is a cycle, and if $x + y \neq 0$, we have $\Gamma_R(M)$ contains a cycle $x - x + y - y - x$. Consequently, $\Gamma_R(M)$ contains a cycle and so $gr(\Gamma_R(M)) = 3$. □

Theorem 4.3. *Let M be a faithful multiplication R-module. Then $gr(\Gamma_R(M)) = gr(\Gamma_R(R))$.*

Proof. Let M be a faithful multiplication R-module. We show that $\Gamma_R(M)$ contains a cycle if and only if $\Gamma_R(M)$ contains a cycle. Let $\Gamma_R(M)$ contains a cycle, by Theorem 4.1 $gr(\Gamma_R(M)) = 3$. So there are $x, y, z \in T(M)^*$ such that $x - y - z - x$ is a cycle. Hence $r \in Ann(x) \cap Ann(y)$, $s \in Ann(y) \cap Ann(z)$ and $t \in Ann(z) \cap Ann(x)$ for some $r, s, t \in R \setminus \{0\}$. Therefore for all $\alpha \in [x : M]$, $\beta \in [y : M]$ and $\gamma \in [z : M]$ we have $r \in Ann(\alpha) \cap Ann(\beta)$, $s \in Ann(\beta) \cap Ann(\gamma)$ and $t \in Ann(\gamma) \cap Ann(\alpha)$. Thus $\alpha - \beta - \gamma - \alpha$ is a cycle in $\Gamma_R(R)$. So $gr(\Gamma_R(R)) = 3$. Conversely, suppose that $\alpha - \beta - \gamma - \alpha$ is a cycle in $\Gamma_R(R)$. Since M is faithful, there are non-zero elements $m_1, m_2, m_3 \in M$ such that $\alpha m_1, \beta m_2, \gamma m_3 \in T(M)^*$. Therefore $\alpha m_1 - \beta m_2 - \gamma m_3 - \alpha m_1$ is a cycle in $\Gamma_R(M)$ and so $gr(\Gamma_R(M)) = 3$. □

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