DOI: 10.2478/auom-2014-0003
An. Şt. Univ. Ovidius Constanţa **Vol. 22(1),2014, 35–40**

SOMETHING ABOUT h - MEASURES OF SETS IN PLANE

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Abstract

In this article we estimate the Hausdorff h-measures of the graphs of some functions, for different measure functions.

1 Preliminaries

The fractal properties of some sets are characterised by different types of dimensions, as ruler, Box, information etc., that are difficult to be calculated. The Hausdorff h - measure that generalises the Hausdorff measure, from which the Hausdorff dimension arises is of big importance in problems in which the equality between the p-module and p-capacity of a set must be proved.

In this article, which continues the works [1]–[4], we present some results concerning the Hausdorff h - measure of some sets in plane.

Definition 1.1. Consider the the Euclidean n - dimensional space $\mathbf{R}^n$, $E \subset \mathbf{R}^n$ and denote by $d(E)$ the diameter of E.

If $r_0 > 0$ is a fixed number, a continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim\limits_{r \to 0} h(r) = 0$ is called a measure function.

If $0 < \beta < \infty$ and h is a measure function, then, the Hausdorff h-measure of E is defined by:

$$H_h(E) = \lim_{\beta \to 0} \inf \left\{ \sum_i h(d(U_i)) : E \subseteq \bigcup_i U_i : 0 < d(U_i) < \beta \right\},$$

Key Words: Hausdorff h-measure, measure function, graph
2010 Mathematics Subject Classification: Primary 28A78, 28A80
Received: April, 2013.
Accepted: August, 2013.

where U_i is open.

Remark. If in the previous definition the covering of the set E is made with balls, a new spherical measure, denoted by H'_h is obtained.

The relation between the two measures is: $H_h(E) \leq H'_h(E)$.

Definition 1.2. Let $\delta > 0$ and $f : D(\subset \mathbf{R}) \to \overline{\mathbf{R}}$. f is said to be a δ - class Lipschitz function if there is $M > 0$ such as:

$$|f(x+\alpha) - f(x)| \leq M\,|\alpha|^{\delta}, \forall x \in D, \forall \alpha \in \mathbf{R},\ x+\alpha \in D.$$

f is said to be a Lipschitz function if $\delta = 1$.

Definition 1.3. $\varphi_1, \varphi_2 : D(\subset \mathbf{R}) \to (0, +\infty)$ are similar and we denote by: $\varphi_1 \sim \varphi_2$, if there exists $Q > 0$, such as: $\frac{1}{Q}\varphi_1(x) \leq \varphi_2(x) \leq Q\varphi_1(x), \forall x \in D$.

If $f : I \to \mathbf{R}$ is a function defined on the interval I and $[t_1, t_2] \subset I$, denote by $\Gamma(f)$, the graph of function f and $R_f(t_1, t_2) = \sup\limits_{t_1 \leq u,v \leq t_2} |f(t) - f(u)|$.

Proposition 1.5. [5] Let f be a continuous function on $[0,1]\,, 0 < \beta < 1$ and m be the least integer number greater than or equal to $1/\beta$. If N_β is the least number of squares of the $\beta-$ mesh that intersect $\Gamma(f)$, then:

$$\beta^{-1}\sum_{j=0}^{m-1} R_f\,[j\beta, (j+1)\beta] \leq N_\beta \leq 2m + \beta^{-1}\sum_{j=0}^{m-1} R_f\,[j\beta, (j+1)\beta]\,.$$

2 Results

Theorem 2.1. *Let $\delta > 0$ and $f : [0,1] \to \overline{\mathbf{R}}$ be a δ- class Lipschitz function. If h is a measure function such that $h(t) \sim e^{-t}t^p, p \geq 2$, then $H_h(\Gamma(f)) < \infty$. The assertion remains true if $p \geq 1$ and $\delta \geq 1$.*

Proof. The first part of the proof follows a ideea from [6] and [2].

First, we suppose that $M+1$ in Definition 1.3, so that to any x corresponds an interval $(x-k, x+k)$ such that, for any $x+\alpha$ in this interval:

$$|f(x+\alpha) - f(x)| \leq |\alpha|^{\delta}.$$

Since $[0,1]$ is a compact set, there exists a finite set of overlapping intervals covering (0, 1):

$$(0, k_0), (x_1 - k_1, x_1 + k_1), ..., (x_{n-1} - k_{n-1}, x_{n-1} + k_{n-1}), (1 - k_n, 1).$$

If c_i are arbitrary points, satisfying:

$$c_1 \in (0, x_1), c_i \in (x_{i-1}, x_i), i = 2, ..., n-1, c_n \in (x_{n-1}, 1)$$

$$c_i \in (x_{i-1} - k_{i-1}, x_{i-1} + k_{i-1}) \bigcap (x_i - k_i, x_i + k_i), i = 2, ..., n-1,$$

then

$$0 < c_1 < x_1 < c_2 < x_2 < ... < x_{n-1} < c_n < 1.$$

The oscillation of f in the interval (c_{i-1}, c_i) is less than $2\,(c_i - c_{i-1})^{\delta}$ and thus the part of the curve corresponding to the interval (c_{i-1}, c_i) can be enclosed in a rectangle of height $2\ (c_i - c_{i-1})^{\delta}$ and of base $c_i - c_{i-1}$, and consequently in $\left[2\,(c_i - c_{i-1})^{\delta-1}\right] + 1$ squares of side $c_i - c_{i-1}$ or in the same number of circles of radius $\frac{c_i - c_{i-1}}{\sqrt{2}}$ circumscribed about each of these squares.

The integer part of the number x was denoted by $[x]$.

Given an arbitrary $r \in \left(0, \frac{1}{2}\right)$, we can always assume: $c_i - c_{i-1} < r$, $i = 2, 3, ..., n$.

Let us denote by C_r the set of all the above circles and let us consider

$$\sum_{C_r} h(2r) = \sum_{C_r} \left\{ \frac{h(2r)}{(2r)^p e^{-2r}} \cdot (2r)^p e^{-2r} \right\} \tag{1}$$

$$r \in \left(0, \frac{1}{2}\right) \Rightarrow e^{-2r} \in \left(\frac{1}{e}, 1\right) \tag{2}$$

We have to estimate $\sum_{C_r} (2r)^p$. The sum of the terms corresponding to the interval (c_{i-1}, c_i) is:

$$S = \left\{ \left[2\,(c_i - c_{i-1})^{\delta-1}\right] + 1 \right\} \left\{ (c_i - c_{i-1})\,\sqrt{2} \right\}^p \Leftrightarrow$$

$$S = 2^{\frac{p}{2}}\,(c_i - c_{i-1})^p \left\{ \left[2\,(c_i - c_{i-1})^{\delta-1}\right] + 1 \right\}.$$

$$S \le 2^{\frac{p}{2}}\,(c_i - c_{i-1})^p \left\{ 2\,(c_i - c_{i-1})^{\delta-1} + 1 \right\} \Rightarrow$$

$$S \le 2^{\frac{p}{2}+1}\,(c_i - c_{i-1})^{p+\delta-1} + 2^{\frac{p}{2}}\,(c_i - c_{i-1})^p \tag{3}$$

$$p \ge 2, \delta \ge 0 \Rightarrow p + \delta - 1 \ge 1 \Rightarrow$$

$$c_i - c_{i-1} < 1 \Rightarrow (c_i - c_{i-1})^{p+\delta-1}, (c_i - c_{i-1})^p < c_i - c_{i-1}. \tag{4}$$

From (3) and (4) it results:

$$S \le 2^{\frac{p}{2}+1}(c_i - c_{i-1}) + 2^{\frac{p}{2}}(c_i - c_{i-1}) = 3 \cdot 2^{\frac{p}{2}}(c_i - c_{i-1}) \Rightarrow$$

$$\sum_{C_r} (2r)^p \le \sum_{i=2}^{n} 3 \cdot 2^{\frac{p}{2}}\,(c_i - c_{i-1}) \le 3 \cdot 2^{\frac{p}{2}} \Leftrightarrow$$

$$\sum_{C_r} 2r)^p \le 3 \cdot 2^{\frac{p}{2}} \tag{5}$$

Using Definition 1.3 and the relations $(1),(2)$ and (5), we obtain:

$$\sum C_r h(2r) = \sum_{C_r} \left\{ \frac{h(2r)}{(2r)^p} \cdot (2r)^p \right\} < Q \sum_{C_r} (2r)^p \leq 3 \cdot 2^{\frac{p}{2}} \cdot Q,$$

where $Q > 0$ and $r \in \left(0, \frac{1}{2}\right)$, small enough.

Then $H'_h(\Gamma(f)) < +\infty$ and by (1), $H_h(\Gamma(f)) < +\infty$.

If $M \neq 1$, then

$$\sum_{C_r} h(2r) \leq 3 \cdot 2^{\frac{p}{2}} \cdot Q \cdot M \Rightarrow H'_h(\Gamma(f)) < +\infty \Rightarrow H_h(\Gamma(f)) < +\infty.$$

If $p \geq 1$ and $\delta > 1$, then $(c_i - c_{i-1})^{p+\delta-1} < c_i - c_{i-1}$ and the proof is analogous.

Remark. In the hypotheses of the previous theorem, if $P(t) = t^p, T(t) = -t$, then $P'(t) + P(t)T'(t) = t^{p-1}(p-t)$, which is not always positive, so the case treated differs from that studied in [3].

Theorem 2.2 If $f : [0,1] \to \mathbf{R}$ is a δ-class Lipschitz function, $\delta > 0$ and h is a measure function such as $h(t) \sim e^t t^p$, $p \geq 2$, then $H_h(\Gamma(f)) = 0$.

The assertion remains true if $\delta \geq 1$ and $p > 1$.

Proof. Denoting by $N'_\beta(\Gamma(f))$ the number of β - mesh squares that cover $\Gamma(f)$ and by $N_\beta(\Gamma(f))$ the smallest number of discs of diameters at most β that cover $\Gamma(f)$, it results that [1]:

$$N_\beta(\Gamma(f)) \leq N'_{\frac{\beta}{\sqrt{2}}}(\Gamma(f)) < \frac{3}{\beta} + M'\beta^{\delta-2}, \tag{6}$$

with $M' = M/\sqrt{2}^{\delta-2}$.

By hypotheses, $\Gamma(f)$ is a compact set. Therefore, if $\beta > 0$, for every cover of $\Gamma(f)$ with open discs $U_i, i \in \mathbf{N}^*$, with diameters $d(U_i) \leq \beta$, there is a finite number of discs, n_β, that covers $\Gamma(f)$.

$$H'_h(\Gamma(f)) = \lim_{\beta\to 0} \inf \left\{ \sum_i h(d(U_i)) : E \subseteq \bigcup_i U_i : 0 < d(U_i)) \leq \beta \right\} =$$

$$= \lim_{\beta\to 0} \inf \left\{ \sum_{i=1}^{n_\beta} h(d(U_i)) \right\} \leq \lim_{\beta\to 0} \inf \{h(\beta) n_\beta\} \Rightarrow$$

$$H'_h(\Gamma(f)) \leq \lim_{\beta\to 0} \{N_\beta(\Gamma(f))h(\beta)\} = \lim_{\beta\to 0} \left\{ \frac{h(\beta)}{\beta^p e^\beta} \cdot N_\beta(\Gamma(f))\beta^p e^\beta \right\} \Rightarrow$$

$$H_h'(\Gamma(f)) \leq \lim_{\beta\to 0}\{N_\beta(\Gamma(f))h(\beta)\} = \lim_{\beta\to 0}\left\{\frac{h(\beta)}{\beta^p e^\beta}\cdot N'_{\frac{\beta}{\sqrt{2}}}(\Gamma(f))\beta^p e^\beta\right\}.$$

Since $h(t) \sim e^t t^p$, there is $Q > 0$ such as $h(t) < Qe^t t^p$. By (1) and the hypothesis concerning p and δ, it result that:

$$H_h'(\Gamma(f)) \leq Q \lim_{\beta\to 0} e^\beta \cdot \lim_{\beta\to 0}\left\{M'\beta^{p+\delta-2} + 3\beta^{p-1}\right\} = 0.$$

Using the remark from Introduction, we obtain:

$$H_h'(\Gamma(f)) = 0 \Rightarrow H_h(\Gamma(f)) = 0.$$

□

Theorem 2.3 *Let $\delta \geq 1$ and $f : [0,1] \to \overline{\mathbf{R}}$ be a $\delta-$ class Lipschitz function. Suppose that h is a measure function such that: $h(t) \sim P(t)e^{T(t)}, t \geq 0$, where P and T are the polynomials:*

$$P(t) = \sum_{j=1}^{p} a_j t^j, p \geq 1, a_1 \neq 0, T(t) = \sum_{j=0}^{m} b_j t^j,$$

with positive coefficients. Then $0 < H_h(\Gamma(f)) < \infty$.

Proof. In [2] it was proved that in the hypotheses of the theorem, $H_h'(\Gamma(f)) < \infty$. Using the theorem 1 [6], we obtain that $H_h'(\Gamma(f)) > 0$.

Acknowledgement: The publication of this article was partially supported by the grant PN-II-ID-WE-2012-4-169 of the Workshop "A new approach in theoretical and applied methods in algebra and analysis".

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