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APPROXIMATION SOLVABILITY OF HAMMERSTEIN EQUATION

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To Professor Dan Pascali, at his 70's anniversary

Abstract

Let X be a real reflexive Banach space and X^* its dual space. Let $K : D(K) \subset X \to X^*$ be a linear operator and $F : D(F) \subset X^* \to X$ be a nonlinear one with $R(F) \subset D(K)$ and $f \in X^*$. We study the abstract equation of Hammerstein type $u + KFu = f$ and we present an approximation solvability method by using the class of perturbation of type (C).

1. Mappings of type (C)

In this paper, we introduce a class of mappings of type (C) which is fit for study of the approximation solvability of the equation $u + KFu = f$. Throughout this paper, X is a reflexive Banach space. "$\to$" and "$\rightharpoonup$" denote strong and weak convergence.

In the framework of monotone-like mapping, we introduce:

Definition 1. Let $A : D(A) \subset X \to X^*$. A is called *mapping of type (C)* if for any $\{u_n\} \subset D(A)$ such that $u_n \rightharpoonup u_0$ and $\overline{\lim\limits_{n\to\infty}} (A(u_n), u_n - u_0) \leq 0$,it follows that $A(u_n) \to A(u_0)$, as $n \to \infty$.

In order to determine the relationship of the class of the operators of type (C) with another mappings of monotone type we recall some definitions from [5], [7]:

Key Words: mapping of type (C), mapping of type (S_+), equation of Hammerstein type, regularizing equation.

1) A is called *mapping of type (S_+)* if for any sequence $\{u_n\} \subset D(A)$ converging weakly to u_0 in X, for which $\overline{\lim\limits_{n\to\infty}}\,(A(u_n) - A(u_0), u_n - u_0) \leq 0$ is in fact strongly convergent in X.

2) A is said to be *quasi-monotone* if each sequence $\{u_n\} \subset D(A)$ with $u_n \rightharpoonup u_0$ in X, it follows that $\overline{\lim\limits_{n\to\infty}}\,(A(u_n), u_n - u_0) \geq 0$.

3) A is called *angle-bounded* with the constant $a \geq 0$ if $|(A(u), v) - (A(v), u)| \leq 2a\,(A(u), u)^{\frac{1}{2}} \cdot (A(v), v)^{\frac{1}{2}}$, for all u and v in D(A). The angle-boundedness of A with $a = 0$ corresponds to the *symmetry* of A, i.e. $(A(u), v) = (A(v), u)\,, \forall u, v \in D(A)$.

Proposition 2. *If $A : D(A) \subset X \to X^*$ satisfies one of the following conditions:*

1.A is a continuous mapping of type (S_+),

2.A is a continuous angle-bounded mapping,

then A is a mapping of type (C).

To give some useful results we recall other definitions from [5]:

1) The map $J : X \to X^*$ given by $Ju = \left\{ f \in X \mid (f, u) = \|u\|^2 = \|f\|^2 \right\}$ is called the *normalized duality map* of X.

Without loss of generality we suppose further that is locally uniformly convex Banach space and J is single valued map.

2) Let $A : D(A) \subset X \to X^*$. A is called *mapping of type quasi-(S_+)* if for any $\varepsilon > 0$, $A + \varepsilon J$ are the mappings of type (S_+), where $J : X \to X^*$ is a normalized duality map.

A basic relation between quasi-monotone operators and mappings of type (S) due to Calvert and Webb ([5]) is:

Theorem 3. *The demicontinuous operator A is quasi-monotone if and only if $A + \varepsilon J \in (S)$ for each $\varepsilon > 0$.*

Proposition 4. *Let $\Omega \subset X$ be a weakly closed set, $\{u_n\} \subset \Omega$ such that $u_n \rightharpoonup u_0$ in X and $A : \Omega \to X^*$ be a mapping of type (C). Then A is a mapping of type quasi-(S_+).*

Proposition 5. *Let $\Omega \subset X$ be a closed set, $\{u_n\} \subset \Omega$, $u_n \rightharpoonup u_0$ in X and $A : \Omega \to X^*$ be a bounded mapping of type (S_+). If*

$$\overline{\lim_{m,n}}\,(A(u_n) - A(u_m), u_n - u_m) \leq 0,$$

then $u_n \to u_0$, as $n \to \infty$.

Proposition 6. *Let $\Omega \subset X$ be a weakly closed set, $\{u_n\} \subset \Omega$ and $u_n \rightharpoonup u_0$ in X. If $A : \Omega \to X^*$ is a bounded mapping of type (C) and*

$$\varlimsup_{m,n} (A(u_n) - A(u_m), u_n - u_m) \leq 0,$$

then $A(u_n) \to A(u_0)$, as $n \to \infty$.

Proposition 7. *Let $A : X \to X^*$ be a hemicontinuous monotone mapping of type (C) and $J : X \to X^*$ a normalized duality map. Then, for any $\varepsilon > 0$, $A + \varepsilon J$ is invertible and $(A + \varepsilon J)^{-1} : X^* \to X$ is a bounded continuous monotone mapping of type (S_+).*

2. Approximation solvability of Hammerstein equation

Let $K : D(K) \subset X \to X^*$, $F : D(F) \subset X^* \to X$, $R(F) \subset D(K)$ and $f \in X^*$.

The equation $\quad u + KFu = f \qquad (1)$

is called *equation of Hammerstein type.*

We consider the case of $D(K) = X$ and $D(F) = X^*$.

Definition 8. The equation

$$u + (K + \lambda J)(F + \varepsilon J^*)u = f \qquad (2)$$

is called a *regularizing equation* of equation (1), where λ, ε are arbitrary positive numbers, and $J : X \to X^*, J^* : X^* \to X$ are normalized duality maps.

Suppose that equation (2) is approximatively solvable, i.e. there is a sequence of monotonically increasing finite dimensional subspaces $\{X_n\} \subset X$, in which the equations have a solutions u_n such that $u_n \to u$, where u is a solution of the original equation.

Let $u + (K + \lambda_n J)(F + \varepsilon_m J^*) u = f \qquad (3)$

with $\lambda_n > 0, \varepsilon_m > 0, n, m \in \mathbb{N}$ and $\lambda_n \to 0$ $(n \to \infty)$, $\varepsilon_m \to 0$ $(m \to \infty)$. If the equation (3) has a solution $u_{\lambda_n \varepsilon_m}$ satisfying $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} u_{\lambda_n \varepsilon_m} = u$ and u is a solution of the equation (1), then the equation (1) is said to be regularizing approximatively solvable.

Theorem 9.: *Let $K : X \to X^*$ be continuous monotone mapping of type (S_+) and $K(0) = 0$. Suppose $N : X \to X^*$ is coercive with respect to $f \in X^*$, i.e. there is $r > 0$ such that*

$$(Nu - f, u) > 0$$

for all $u \in X$, with $\|u\| > r$. Then the equation:

$$Nu + Ku = f \qquad (4)$$

has at least a solution.

Theorem 10. *Let K and N satisfy the condition of Theorem 9.*

Let $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ *be an injective approximation scheme in the Petryshyn sense ([6]). Define an approximation equation of equation (4) by*

$$N_n u + K_n u = Q_n f$$

where $N_n = Q_n N P_n, K_n = Q_n K P_n$ *and* $u \in X_n$*. Then*

(i) $\forall n \in \mathbb{N}, \exists u_n \in X_n$*, such that* $N_n u_n + K_n u_n = Q_n f$

(ii) $\exists \{u_{n_k}\} \subset \{u_n\}$ *such that* $P_{n_k} u_{n_k} \to u$*,* $k \to \infty$ *and* u *is a solution of equation (4).*

Theorem 11. *Let* $K : X \to X^*$ *be a continuous monotone mapping,* $K(0) = 0$*,* $F : X^* \to X$ *a bounded and hemicontinuous monotone mapping of type (C),* $f \in X^*$ *and* $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ *an injective approximation scheme. Suppose that F is coercive, i.e.*

$$(u - f, Fu) > 0 \qquad (5)$$

for all $u \in X^*$, with $\|u\| \geq R > 0$. *Then, the equation of Hammerstein type (1) is regularizing approximately solvable.*

Proof. First, we prove that the equation (2) is approximatively solvable.

For any $\varepsilon > 0$ and $\lambda > 0$, we introduce the mappings $F_\varepsilon : X^* \to X$ with $F_\varepsilon = F + \varepsilon J^*$ and $K_\lambda : X \to X^*$ with $K_\lambda = K + \lambda J$. By the condition (5), for $\|u\| \geq \max(\|f\|, R)$ we have

$$(u - f, F_\varepsilon u) > 0 \qquad (6)$$

Let $v = F_\varepsilon u$. Since F is bounded, we have $\|u\| \to \infty$ as $\|v\| \to \infty$.

Conversely, since F_ε is coercive, we have $\|v\| \to \infty$ as $\|u\| \to \infty$. Thus, the condition (6) is equivalent to the following: $\exists r > 0$ such that for $v \in X, \|v\| \geq r$, we have

$$\left(F_\varepsilon^{-1} v - f, v\right) > 0$$

By the Proposition 7, the assumption of the theorem and the above discussion, the corresponding conditions of Theorems 9 and 10 are satisfied. Hence, the equation

$$F_\varepsilon^{-1} v + K_\lambda v = f \qquad (7)$$

is solvable in X. Furthermore, the equation (2) is also solvable in X^*. Equation (7) is approximatively solvable. Its approximation equation is

$$Q_n F_\varepsilon^{-1} P_n v + Q_n K_\lambda P_n v = Q_n f,\ v \in X_n \qquad (8)$$

By the definition of the mapping P_n, $R(P_n)$ is a closed subspace of X and $P_n : X_n \to R(P_n)$ is a bijection. Also, $R(Q_n)$ is a closed subspace of X_n^*, $Q_n : X^* \to R(Q_n) \subset X_n^*$ is a bijection.

Note that the equation (8) is solvable.

Thus the equation $u + Q_n K_\lambda F_\varepsilon Q_n^{-1} u = Q_n f$ has a solution in $R(Q_n)$.

By Theorem 10 and the continuity of F_ε^{-1}, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to u$ $(k \to \infty)$ and it can be proved easily that u is a solution of the equation (2).

Secondly,we are to show that $\exists \lambda_n > 0$, with $\lambda_n \to 0$ $(n \to \infty)$ and $\varepsilon_m > 0$, with $\varepsilon_m \to 0$ $(m \to \infty)$ such that solutions $\{u_{\lambda_n \varepsilon_m}\}$ of the equation (3) satisfy $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} u_{\lambda_n \varepsilon_m} = u$ and u is a solution of the equation (1)

Let $v_{\lambda\varepsilon} = F_\varepsilon u_{\lambda\varepsilon}$. Then, $\forall \varepsilon > 0$ fixed, $\{v_{\lambda\varepsilon}\}$ are bounded with respect to $\lambda > 0$. Otherwise, there would be a subsequence $\{v_{\lambda_n\varepsilon}\}$ such that $\|v_{\lambda_n\varepsilon}\| \to \infty$ $(n \to \infty)$. Thus, $\exists N_1 \in \mathbb{N}$, such that for $n > N_1$, $\|v_{\lambda_n\varepsilon}\| \geq r$. Since $F_\varepsilon^{-1} v_{\lambda_n\varepsilon} - f = -(K + \lambda_n J)\, v_{\lambda_n\varepsilon}$, we get a contradictory result:

$$0 < \left(F_\varepsilon^{-1} v_{\lambda_n\varepsilon} - f, v_{\lambda_n\varepsilon}\right) = -\left((K + \lambda_n J)\, v_{\lambda_n\varepsilon}, v_{\lambda_n\varepsilon}\right) \leq 0.$$

Since X is reflexive and $\{v_{\lambda\varepsilon}\}$ is bounded, there exists a subsequence $\{v_{\lambda_n\varepsilon}\}$ such that $v_{\lambda_n\varepsilon} \rightharpoonup v_\varepsilon$ $(n \to \infty, \lambda_n \to 0)$.

Let $A_1 = \{v_{\lambda_n\varepsilon}\}$. We choose arbitrarily $v_{\lambda_n\varepsilon} \in A_1, v_{\lambda_k\varepsilon} \in A_1$. Then $F_\varepsilon^{-1} v_{\lambda_n\varepsilon} + (K + \lambda_n J)\, v_{\lambda_n\varepsilon} = f$ and $F_\varepsilon^{-1} v_{\lambda_k\varepsilon} + (K + \lambda_k J)\, v_{\lambda_k\varepsilon} = f$. Thus $F_\varepsilon^{-1} v_{\lambda_n\varepsilon} - F_\varepsilon^{-1} v_{\lambda_k\varepsilon} + K v_{\lambda_n\varepsilon} - K v_{\lambda_k\varepsilon} + \lambda_n J v_{\lambda_n\varepsilon} - \lambda_k J v_{\lambda_k\varepsilon} = 0$. Since A_1 is bounded and K is monotone, we have

$$\overline{\lim_{n,k}} \left(F_\varepsilon^{-1} v_{\lambda_n\varepsilon} - F_\varepsilon^{-1} v_{\lambda_k\varepsilon}, v_{\lambda_n\varepsilon} - v_{\lambda_k\varepsilon}\right) \leq 0$$

By the Proposition 5, we have $v_{\lambda_n\varepsilon} \to v_\varepsilon$ $(n \to \infty)$. In addition, by the continuity of F_ε^{-1} and K, v_ε satisfies the equation

$$F_\varepsilon^{-1} v + K v = f$$

This equation is equivalent to the following equation:

$$u + K F_\varepsilon u = f \qquad (9)$$

Let $u_\varepsilon = F_\varepsilon^{-1} v_\varepsilon$. Then u_ε satisfies the equation (9) and $u_{\lambda_n\varepsilon} = F_\varepsilon^{-1} v_{\lambda_n\varepsilon} \to F_\varepsilon^{-1} v_\varepsilon = u_\varepsilon$ $(n \to \infty)$.

Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a solution set of the equation (9). Similarly, we can prove that $\{u_\varepsilon\}_{\varepsilon>0}$ are bounded. By the reflexivity of X^*, then $\exists \{u_{\varepsilon_m}\}$ such that $u_{\varepsilon_m} \rightharpoonup u$ $(m \to \infty, \varepsilon_m \to 0)$.

Let $A_2 = \{u_{\varepsilon_m}\}$. We consider arbitrarily $u_{\varepsilon_m} \in A_2$, $u_{\varepsilon_l} \in A_2$, then we have $u_{\varepsilon_m} + K F_{\varepsilon_m} u_{\varepsilon_m} = f$ and $u_{\varepsilon_l} + K F_{\varepsilon_l} u_{\varepsilon_l} = f$.

Thus $u_{\varepsilon_m} - u_{\varepsilon_l} + K F_{\varepsilon_m} u_{\varepsilon_m} - K F_{\varepsilon_l} u_{\varepsilon_l} = 0$.

By the boundedness of A_2 and the monotonicity of K, we have

$$\overline{\lim_{m,l}} \left(F u_{\varepsilon_m} - F u_{\varepsilon_l}, u_{\varepsilon_m} - u_{\varepsilon_l}\right) \leq 0.$$

By Proposition 7, we have $F u_{\varepsilon_m} \to F u$ $(m \to \infty)$. Finally, by the continuity of K and the equation (9), we have $u_{\varepsilon_m} \to u$ $(m \to \infty)$ and $u + K F u = f$.

Theorem 12. *Let $K : X \to X^*$ be a bounded and continuous monotone mapping of type (C), $K(0) = 0$, $F : X^* \to X$ a bounded and continuous monotone mapping and $f \in X^*$. Suppose that $\exists R > 0$, such that for $u \in X^*$, with $\|u\| \geq R$, we have the condition (5). Suppose also that $\Gamma = (\{X_n\}, \{X_n^*\}; \{P_n\}, \{Q_n\})$ is an injective approximation scheme. Then, the equation (1) is regularizing approximatively solvable.*

Proposition 13. *Let monotone mappings $K : X \to X^*$ and $F : X^* \to X$ be such that the equation (1) has a solution and either K or F satisfies one of the following conditions:*

1. *Either K or F is strictly monotone;*
2. *Either K or F is angle-bounded.*

Then the solution of the equation (1) is unique.

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