An. Şt. Univ. Ovidius Constanţa Vol. 13(1), 2005, 79–88

SOME EQUIVALENT GEOMETRICAL RESULTS WITH EKELAND'S VARIATIONAL PRINCIPLE

Georgiana Goga

To Professor Dan Pascali, at his 70's anniversary

Abstract

The equivalence between Ekeland's Variational Principle, The Drop Theorem, The Petal Theorem is proved and The Drop Theorem is generalized to Localy Convex Spaces

Mathematics Subject Classification : 47H10; 47H15; 46A22; 46B20; 46B99.

1. Introduction and notations.

Ekeland's variational principle is a very useful tool for nonlinear analysis, being a classical mean for investigations of many nonlinear problems in various areas in mathematics (see, for instance, [1] , [4] , [5] , [6], for surveys).

In this paper, we show that Ekeland's variational principle, or more precisely, a slightly amended form of it, is a consequence of a beautiful geometrical result known as the drop theorem [2], used in various situation (see [3] , [6] , [7] , [8]). We also point out that in turn the drop theorem is a consequence of the Ekeland's variational principle. As well as a consequence of this principle we introduce the petal theorem. All this results are presented in Section 2 and the relationships between them are proved in Section 3. In the end of this article we establish a generalization of the drop theorem in locally convex spaces(see [10] , [11] , [12] , [14]).

In what follows, the *petal* $P_\gamma(a,b)$ associated with $\gamma \in \,]0,\infty[$ and the points a,b in a metric space (X,d) is the set

Key Words: Ekeland's variational principle; Drop theorem; Petal theorem.

$$P_\gamma(a,b) = \{x \in X : \gamma d(x,a) + d(x,b) \le d(a,b)\}.$$

If (X,d) is a metric space, A and B sets in X, c a point in X and $r > 0$, then $B(c,r)$ denotes the closed r-ball centered at c, $d(c,A) = \inf\{d(x,c) : x \in A\}$ (the distance of c from A), $d(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$ (the distance of the sets A and B) and $diam(A) = \sup\{d(x,y) : x,y \in A\}$ (the diameter of A). If E is a normed vector space (n.v.s.) the *drop* $D(x,B)$ associated with a point $x \in E$ and a convex subset B of E is the convex hull of $\{x\} \cup B$:

$$D(x,B) = \{x + t(y-x) : y \in B, t \in [0,1]\}.$$

We note that $P_\gamma(a,b) \subset P_\delta(a,b)$ if $\delta \le \gamma$. Moreover, when B is the $r-$ball with center b in the n.v.s. E and $\gamma \le (t+r)^{-1}(t-r)$ with $t := d(a,b) > r$ then, by convexity, we obtain $D(a,B) \subset P_\gamma(a,b)$.

For any subset A of the metric space (X,d), we take the function $d_A = d|_{A\times A}$ .Then (A, d_A) is a metric space and d_A represents the induced metric on A by the metric d of X. We call that A is complete, if the metric space (A, d_A) is complete.

2. The statements

We start with the classic statement of Ekeland's variational principle (EVP) and then we give our altered statement and its proof, because this form of EVP will be crucial for our purposes.

Theorem 2.1 (basic EVP). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper $(f(x) \ne \infty)$, lower semicontinuous (l.s.c.) function which is bounded from below. Then there exists $x_0 \in X$ such that $f(x_0) < f(x) + d(x_0,x)$ for $x \in X, x \ne x_0$.*

Theorem 2.2 (altered EVP). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper l.s.c. function which is bounded from below. Then for every $y \in X$ and every $\varepsilon > 0$, there exists $x_0 \in X$ such that*

i) $f(x_0) < f(x) + \varepsilon d(x,x_0)$, for $x \ne x_0$;

ii) $f(x_0) \le f(y) - \varepsilon d(x_0,y)$.

Proof. Define the sets

$$X_\varepsilon = \{z \in X;\ f(z) \le f(y) - \varepsilon d(z,y).\}$$

and

$$M(x) = \{z \in X_\varepsilon;\ f(x) \ge f(z) + \varepsilon d(z,x)\}.$$

Then X_ε is a nonempty complete metric space and $f : X_\varepsilon \to \mathbb{R} \cup \{\infty\}$ is also proper l.s.c. Thus for every $x \in X_\varepsilon$, $M(x)$ is nonempty and closed. Moreover $z \in M(x)$ implies $M(z) \subseteq M(x)$. Indeed, if $t \in M(z)$, we have $f(z) \geq f(t) + \varepsilon d(t,z)$.On the other hand, $z \in M(x)$ implies $f(x) \geq f(z) + \varepsilon d(z,x)$.From the last two inequalities, we have $f(x) \geq f(t) + \varepsilon d(t,x)$, that is $t \in M(x)$.

Let x_0 be given. We construct a set $(x_n)_{n\geq 1}$ as follows: choose $x_1 \in X_\varepsilon$ with $f(x_1) < \infty$ and then find $x_{n+1} \in M(x_n)$ such that

$$f(x_{n+1}) < \inf_{u \in M(x_n)} f(u) + \tfrac{1}{n}.$$

Obviously, $M(x_{n+1}) \subseteq M(x_n)$ and, for every $z \in M(x_{n+1})$, we have

$$\varepsilon d(z, x_{n+1}) \leq f(x_{n+1}) - f(z) \leq \inf_{u \in M(x_n)} f(u) - f(z) + \tfrac{1}{n} \leq \tfrac{1}{n}.$$

So $diam M(x_n) \to 0$ as $n \to \infty$ and, since X is complete , $\bigcap_{n\geq 1} M(x_n) = \{x_0\}$.As $x_0 \in M(x_n)$ implies $M(x_0) \subseteq M(x_n)$ for $n \geq 1$, we have $M(x_0) = \{x_0\}$.Therefore $f(x_0) < f(x) + \varepsilon d(x, x_0)$, for $x \neq x_0$.The same inequality holds on $X \diagdown X_\varepsilon$ since, for $z \notin X_\varepsilon$, we have $f(y) - \varepsilon d(z,y) < f(z)$ and this, together with the fact that $x_0 \in X_\varepsilon$, implies that

$$f(x_0) \leq f(y) - \varepsilon d(x_0, y) \leq f(y) - \varepsilon d(z,y) + \varepsilon d(z, x_0) < f(z) + \varepsilon d(z, x_0).$$

Although varied forms of Ekeland's variational principle have been presented by their author in [5], our slightly altered form does not appear there.The Theorem 2.2 is useful in the following form.

Theorem 2.3 (Ekeland's usual variational theorem). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper l.s.c. function which is bounded from below. Then, for every $\varepsilon, \delta > 0$ and $\overline{x} \in X$ with $f(\overline{x}) \leq \inf_{x\in X} f(x) + \varepsilon$, there exists $x_0 \in X$ such that*

i) $f(\overline{x}) \leq f(x_0)$;
ii) $d(\overline{x}, x_0) \leq \delta$;
iii) $f(x) > f(\overline{x}) - \frac{\varepsilon}{\delta} d(\overline{x}, x)$, *for* $x \neq \overline{x}$.

Theorem 2.4. (the drop theorem). *Let X be a Banach space, C be a nonempty and closed subset of X, z_0* be *a point in $X \backslash C$, $\rho > 0$ and $0 < r < R = d(z_0, C) < \rho$. Then there exists $a \in C$ such that $\|a - z_0\| \leq \rho$ and $D(B(z_0, r), a) \cap C = \{a\}$, where $B(z_0, r)$ is the closed ball with center z_0 and radius r.*

Proof. By a translation we may assume that $z_0 = 0$.Let $K = B(0, \rho) \cap C$, which is a closed subset of X, and consequently a complete metric space with a distance induced naturally by the norm of X. Define the following functional $f : K \to \mathbb{R}$ by $f(x) = \frac{\rho+r}{R-r}\|x\|$.By the Ekeland's variational principle, given $\varepsilon = 1$, there exists $a \in K$ such that

(1) $$f(a) < f(x) + \|x-a\|.$$

Such an element a satisfies the first requirement of conclusion. We claim now that $D(B(z_0,r),a) \cap C = \{a\}$. Suppose by contradiction that $x \neq a, x \in D(B(z_0,r),a) \cap C$. So

(2) $$x \in C \text{ and } x = (1-t)a + tv,$$

for some $v \in B(0,r)$ and $0 \leq t \leq 1$. Clearly $0 < t < 1$. From (2) we obtain

$$\|x\| \leq (1-t)\|a\| + t\|v\|,$$

which gives

(3) $$t(R-r) \leq t(\|a\| - \|v\|) \leq \|a\| - \|x\|.$$

It follows from (1) and (2) that

$$\tfrac{\rho+r}{R-r}\|a\| < \tfrac{\rho+r}{R-r}\|x\| + \|x-a\| = \tfrac{\rho+r}{R-r}\|x\| + t\|a-v\|.$$

Using (3) to estimate t in the above inequality and estimating $\|a-v\| \leq \rho + r$, we obtain $\|a\| < \|x\| + (\|a\| - \|x\|)$, which is impossible. Therefore, we have $D(B(z_0,r),a) \cap C = \{a\}$.

The above theorem is due to Danes, who gave in [2] a different proof from the above one, using the following result of Krasnoselskii and Zabreiko: "Let X be a Banach space and let $x, y \in X$ be given points such that $0 < r < \rho < \|x-y\|$. Then

$$diam\left[D(B(x,r),y) \backslash B(x,\rho)\right] \leq \tfrac{2[\|x-y\|+r]}{\|x-y\|-r}(\|x-y\| - \rho)."$$

Theorem 2.5(the generalized drop theorem). *Let $(X, \|.\|)$ be a Banach space, $C \subseteq X$ a nonempty closed subset and $B \subseteq X$ a nonempty bounded closed and convex subset of X. If $d(C,B) > 0$, then given $x_0 \in C$ there exists $a \in C \cap D(x_0,B)$ such that $C \cap D(a,B) = \{a\}$.*

Here it is an another geometrical result known as *"the petal theorem"*.

Theorem 2.6 (the flower petal theorem). *Let A a complete subset of a metric space (X,d). Let $x_0 \in A$ and let*

$$b \in X \backslash A,\ r \leq d(b,A),\ s = d(b,x_0).$$

Then for every $\gamma > 0$ there exists $a \in A \cap P_\gamma(x_0,b)$ (so that in particular $d(a,x_0) \leq \gamma^{-1}(s-r)$) such that $P_\gamma(a,b) \cap A = \{a\}$.

3. The implications

Proposition 3.1. *The altered Ekeland's variational principle (Theorem 2.2) implies the generalized drop theorem (Theorem 2.5).*

Proof. Set $Y = C \cap D(x_0, B)$ and define $f : Y \to \mathbb{R}_+$ by $f(x) = d(x, B)$. Then Y is a complete metric space and f is continuous on Y. Take $\varepsilon > 0$ such that $(1-\varepsilon)\, d(Y, B) > \varepsilon \cdot diamB$. By Theorem 2.2 there exists $a \in Y$ such that

$$f(a) < f(x) + \varepsilon \|x - a\|, x \neq a,$$

i.e.

$$d(a, B) < d(x, B) + \varepsilon \|x - a\|, x \neq a, x \in Y.$$

We show that $C \cap D(a, B) = \{a\}$ or equivalently $Y \cap D(a, B) = \{a\}$ since $D(a, B) \subseteq D(x_0, B)$. Assume that this is not true. Then there exists $t \in (0, 1)$ and $u \in B$ such that $x = ta + (1-t)\, u \in Y$. The convexity of B implies

$$d(x, B) + \varepsilon \|x - a\| \leq td(a, B) + (1-t)\, d(u, B) + \varepsilon (1-t) \|a - u\| \leq$$
$$\leq td(a, B) + \varepsilon (1-t) [d(a, B) + diamB].$$

Hence we have that

$$d(a, B) < td(a, B) + \varepsilon (1-t) [d(a, B) + diamB],$$

which leads to the contradiction $(1-\varepsilon)\, d(a, B) < \varepsilon \cdot diamB$.

Proposition 3.2. *The altered Ekeland's variational principle (Theorem 2.2) implies the petal theorem (Theorem 2.6).*

Proof. Let A with the induced metric by the metric of X and we define $f : A \to \mathbb{R}$ by $f(x) = d(x, b)$. Function f is continuous and bounded below by r. Using the Theorem 2.2 we can find $a \in A$ such that

(1) $$f(a) < f(x) + \gamma d(a, x) \text{ for each } x \in A, x \neq a,$$

(2) $$f(a) \leq f(x_0) + \gamma d(a, x_0).$$

Then (1) shows that for each $x \in A \setminus \{a\}$ we have $x \notin P_\gamma(a, b)$, while (2) implies $\gamma d(a, x_0) \leq s - d(a, b) \leq s - r$.

In what follows we prove that these theorems and Ekeland's variational principle are equivalent.

Proposition 3.3. *The petal Theorem 2.6 implies the generalized drop Theorem 2.5.*

Proof. It knows that if $C \subset X$ is complete, then C is closed. We denote by B the ball with center b and radius r. In Theorem 2.6 we take $A = C \cap D(x_0, B)$, $\gamma = \frac{d-r}{d+r}$, with $d = d(b, C)$. As $t := d(a, b) \geq d$ we have $\frac{d-r}{d+r} \leq \frac{t-r}{t+r}$, hence $D(a, B) \subset P_\gamma(a, b)$ as $t > r$. As $a \in D(x_0, B)$ we have $D(a, B) \subset D(x_0, B)$, hence

$$D(a, B) \cap C \subset D(a, B) \cap (C \cap D(x_0, B)) \subset P_\gamma(a, b) \cap A = \{a\}.$$

The following lemma will be used to prove that Theorem 2.5 implies Theorem 2.2.

Lemma 3.4. *Let E be a n.v.s. and let $B((0, h), r)$ be the closed ball with center $(0, h)$ and radius $r \in]0, h[$ in $E \times \mathbb{R}$ endowed with the norm $\|(x, r)\| = \max(\|x\|, r)$. Then the cone $K := \mathbb{R}_+ B$ generated by B is given by*

$$K = \left\{(x, t) \in E \times \mathbb{R}; t \geq r^{-1}(h - r)\|x\|\right\}.$$

Proof. Let us note that K has a base

$$S = \{(x, h - r); \|x\| \leq r\}.$$

Then $S \subset K$ and, for each $(x, t) \in B$, we can write

$$(x, t) = \left(t^{-1}(h - r)x, h - r\right)$$

with $\left\|t^{-1}(h - r)x\right\| \leq r$ as $t \in [h - r, h + r]$, $\|x\| \leq r$.As $(x, t) \in S$ if $t = h - r$ and $t \geq r^{-1}(h - r)\|x\|$, by homogeneity the last relation characterizes K.

Proposition 3.5. *The generalized drop Theorem 2.5 implies the Ekeland's variational principle (Theorem 2.2).*

Proof. Let $M \subset E$ a subset of n.v.s. E. Replacing d by $d' = \min(\delta, d)$, with $\delta := \gamma^{-1}(f(x_0) - \inf f(M) + 1)$ we may suppose d is bounded on M. In fact (1) is true when d replaced by d' and as $\gamma^{-1}(f(x_0) - f(a)) < \delta$,then (2) is satisfied when d replaced by d'. Then (M, d) can be isometrically embedded into a n.v.s. E and we may suppose that M is a complete subspace of a n.v.s. E.

In Theorem 2.5 we set $X = E \times \mathbb{R}$, endowed with the norm $\|(x, t)\| = \max(\|x\|, |t|)$.Taking $g = -f$ and replacing f by $x \longmapsto f(x_0 + x) - f(x_0)$, if necessary, we suppose that $x_0 = 0, f(x_0) = g(x_0) = 0$.Setting $m = \sup\{g(x); x \in M\}$,we take $r > \gamma^{-1}m$, $h = \gamma r + r > m + r$. Let $B := B((0, h), r) = B(0, r) \times [h - r, h + r]$ and let $K := \mathbb{R}_+ B$.For any $(x, t) \in B$ we have $t \geq h - r > m$, hence (x, t) does not belong to the hypograph C of g :

$$C = \{(x,t) \in M \times \mathbb{R}; t \leq g(x)\}.$$

The drop theorem yields some $(a,\alpha) \in C \cap D((0,0),B)$ such that $C \cap D((a,\alpha),B) = \{(a,\alpha)\}$ .As $(a,\alpha) \in D((0,0),B) = [0,1]B$, we have $a \in B(0,r)$ and $(a,h) \in B \subset D((0,0),B)$. By convexity, we also have $(a,t) \in D((0,0),B)$ for $t \in [\alpha,h]$; thus $\alpha < g(a)$ is impossible and $\alpha = g(a)$. The Lemma 3.4 ensures that $g(a) \geq r^{-1}(h-r)\|a\| = \gamma d(a,x_0)$, that is (ii) holds true.

Now let $(x,t) \in (a,\alpha)+K$ with $x \in M, x \neq a, t \leq m$. As $t-\alpha \geq \gamma \|x-a\| > 0$, by the Lemma 3.4, we obtain

$$(x-a, t-\alpha) = s(z, h-r-\alpha)$$

with

$$z = s^{-1}(x-a), \quad s = \tfrac{t-\alpha}{h-r-\alpha} \in]0,1[$$

as $h-r-\alpha \geq h-r-m > 0$ and $t-\alpha \leq m-\alpha < h-r-\alpha$.As K is a convex cone we have

$$(a+z, h-r) = (a,\alpha) + (z, h-r-\alpha) \in K$$

hence

$$(a+z, h-r) \in K \cap (E \times \{h-r\}) = S \subset B.$$

Using the convexity of $D((a,\alpha),B)$ we get

$$(x,t) = (a,\alpha) + s((a+z,h-r)-(a,\alpha)) \in D((a,\alpha),B).$$

It follows that $(x,t) \notin C$.As for each $x \in M$ we have $g(x) \leq m$ we get $(x,g(x)) \notin (a,\alpha)+K$ for $x \in M, x \neq a$ or $g(x)-g(a) < \gamma\|x-a\|$.

4. Danes' drop theorem in locally convex spaces.

Now we give a generalization of Danes' Theorem 2.5 to locally convex spaces by substituting "sequentially closed bounded convex set C " in the space for "the closed bounded convex set B" of the Banach space and " A is strongly Minkowski separated from C " for "A is a positive distance from B".

We say that two nonempty subsets A and B of a locally convex space E are *Minkowski separated* (respectively, *strongly Minkowski separated*) if there exist a continuous Minkowski gauge p on E and a point x_0 in E such that either $p(x) > p(y)$ for all $x \in A_{x_0} \equiv A + x_0$ and $y \in B_{x_0} \equiv B + x_0$ or $p(x) <$

$p(y)$ for all $x \in A_{x_0}$ and $y \in B_{x_0}$ (respectively, either $\inf\{p(x); x \in A_{x_0}\} > \sup\{p(y); y \in B_{x_0}\}$ or $\sup\{p(x); x \in A_{x_0}\} < \inf\{p(y); y \in B_{x_0}\}$).

We replace the Minkowski gauge p by a continuous linear functional in the above definition to obtain the common concept of separation sets. Clearly, two separated (respectively, strongly separated) sets A and B are Minkowski separation (respectively, strongly Minkowski separation) sets, if either of them is bounded.

Proposition 4.1. *Suppose that A is a convex set in a normed linear space E, which is a positive distance from $B \subset E$, if either A or B is bounded. Then A and B are strongly Minkowski separated.*

Proof. Let $d = d(A,B) > 0$ and let $S = \left\{x \in E; d(A,x) \leq \frac{d}{2}\right\}$. Then $S \neq \emptyset$ and $d(S,B) > 0$.Without loss of generality we assume that $0 \in int\, S$ and let P_S be the Minkowski gauge of S. It suffices to show

$$R = \inf\{P_S(x) - P_S(y); x \in B, y \in S\} > 0.$$

Note $d(S,B) \geq \frac{d}{2}$.Suppose, to the contrary, that $R = 0$.Then we can choose sequences $\{x_n\}, \{y_n\}$ from B and S, respectively, such that $P_S(x_n) - P_S(y_n) \to 0$.Since $P_S(x_n) > 1, P_S(y_n) < 1$, we get $P_S(x_n) \to 1$. Let $k_n = P_S(x_n)^{-1}$; then $P_S(k_n x_n) = 1, k_n x_n \in S$.

i) If B is bounded, then $\|x_n - k_n x_n\| = (1 - k_n)\|x_n\| \to 0$, but this contradicts the hypothesis that $d(S,B) \geq \frac{d}{2}$.

ii)If A is bounded, then S is bounded also. Thus, there exists a positive constant $k \geq 1$ such that $k^{-1}P_S(x) \leq \|x\| \leq kP_S(x), \forall x \in E$.Therefore $\{x_n\}$ is a bounded sequence. Hence $\frac{d}{2} \leq d(S,B) \leq \|x_n - k_n x_n\| \to 0$, which is a contradiction.

Theorem 4.2.(Danes' drop theorem in l.s.c.) *Let C be a sequentially closed bounded convex set in a sequentially complete locally convex space (E,τ). For every sequentially closed set A , which is strongly Minkowski separated from C , there exists $z \in A$ such that $D(z,C) \cap A = \{z\}$, where $D(z,C) = co(C \cup z)$.*

Proof. Without loss of generality we assume that $0 \in C$. Fix an element $u_0 \in A$.Let $G = \overline{co(C \cup -C \cup \pm u_0)}$ and $E_1 = span\, G$. Let p be the Minkowski gauge by G; then p is a norm on E_1.

First, we show that (E_1, p) is a Banach space. It suffices to show that the unit ball G of $E_1$is complete with respect to p.Suppose that $\{x_n\}$ is a τ- Cauchy sequence, since G is bounded and p is generated by G, and which implies $\tau < \tau_p$ on E_1, where τ_p denotes the topology generated by the norm p. Since C is τ-sequentially complete, x_n must be τ-convergent to some point $x_0 \in G$. Given a positive number $\varepsilon > 0$, there is an integer k such that $p(x_m - x_n) > \varepsilon$ whenever $m, n \geq k$, or equivalently, $x_m - x_n \in \varepsilon G$, whenever $m, n \geq k$, because G is τ-sequentially closed , $x_m - x_0 \in \varepsilon G$, that is $p(x_m - x_0) \leq \varepsilon$ for all $m \geq k$.Therefore the sequence $\{x_n\}$ converges to x_0 with respect to the norm topology τ_p. Thus, G is complete relatively to p.

Since C is bounded and convex, A is strongly Minkowski separated from C.The Proposition 4.1 implies that there exists a point $x_0 \in E$ and a τ-continuous Minkowski gauge $p_1$on E_1 such that

$$p_1(x) \leq \alpha < \alpha + \varepsilon \leq p_1(y),$$

whenever $x \in C + x_0, y \in A + x_0$ for some fixed $\alpha, \varepsilon > 0$ .Without loss of generality we can assume that $x_0 = 0$ and write $(\varepsilon \leq) \, d = \inf\{p_1(y) - p_1(x); x \in C, y \in A\}$. Since C is closed, bounded and convex relatively to the norm p_1 on E_1 , then $A \cap D(u_0, C)$ is also nonempty , closed and bounded. Define the function $f : E_1 \to \mathbb{R} \cup \{\infty\}$ by

$$f(x) = \begin{cases} p_1(x), x \in A \cap D(u_0, C) \\ \infty \quad , \qquad \text{otherwise}; \end{cases} .$$

Then f is a norm (p)-lower- semicontinuous proper function on E_1 since p_1 is τ-continuous on E_1 .Choose $\lambda > 0$ such that$diam\ D(u_0, C) < \frac{d}{\lambda}$, where the $diam\ D(u_0, C)$ is in norm p. Use Ekeland's variational principle to obtain a point $z \in D(u_0, C) \cap A$ such that

$$f(x) + \lambda p(x - z) > f(z), \text{ for all } x \neq z \text{ in } E_1 .$$

We claim that $D(z, C) \cap A = \{z\}$. Suppose that $y \in D(z, C) \cap A$, with $y \neq z$. Then there exists $0 < \mu < 1$ and $v \in C$ such that $y = (1 - \mu) z + \mu v$, so that $p_1(y) \leq (1 - \mu) p_1(z) + \mu p_1(v)$ and $\mu d \leq \mu [p_1(z) - p_1(v)] \leq p_1(z) - p_1(y)$ .Hence

$$\begin{gathered} p_1(z) = f(z) < f(x) + \lambda p(x - z) = f(x) + \lambda p(\mu(v - z)) = \\ p_1(y) + \lambda\mu(p(v - z)) \leq \\ \leq p_1(y) + \lambda\mu diam \\ D(u_0, C) \leq p_1(y) + \mu d \leq p_1(y) + (p_1(z) - p_1(y)) = p_1(z), \end{gathered}$$

a contradiction.

References

[1] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, A Wiley-Interscience Publ., John Wiley and Sons, 1984.

[2] J. Danes, *Equivalence of some geometric and related results of nonlinear functional analysis,* Comment. Math. Univ. Carolin., **26** (1985), 443 - 454.

[3] J. Danes, *A geometric theorem useful in nonlinear functional analysis,* Boll. Un. Math. Ital., **6** (1972), 369 - 375.

[4] I. Ekeland, *On the variational principle,* J. Math. Anal. Applic.,**47** (1974), 325 - 353.

[5] I. Ekeland, *Nonconvex minimization problems,* Bull. Am. Math. Soc., **1** (1979), 443 - 474.

[6] D. G. de Figueredo, *Lectures on the Ekeland Principle with Application and Detour,* Publ. Tata Inst. 81**,** 6 + 96 pp., Springer-Verlag, Berlin, 1989.

[7] P. G. Georgiev, *The strong Ekeland variational principle, the strong drop theorem and applications,* J. Math. Anal. Appl., **131** (1988), 1 - 21.

[8] P. G. Georgiev, D. N. Kutzarova, A. Maaden, *On the smooth drop property,* Nonlin. Anal., **26** (1996), 595 - 602.

[9] S. Hu, N. S. Papageorgiou, *Handbook of multivalued analysis,* vol.1, 419, Kluwer Ac. Publ., Dordrecht, 1997.

[10] D. N. Kutzarova, *On drop property of convex sets in Banach spaces,* Constructive Theory of Functions ' 87 , Sofia (1988), 283 -287.

[11] D. N. Kutzarova, S. Rolewicz, *On drop property for convex sets,* Arch. Math.(Basel), **57** (1991), 385 - 394.

[12] C. Lixin, Z. Yunchi, Z. Fong, *Danes ' Drop Theorem in locally convex spaces,* Proc. Am. Math. Soc., **124** (1996), 3699-3702.

[13] J. P. Penot, *The Drop Theorem, The Petal Theorem and Ekeland Variational Principle,* Nonlin. Anal., **10** (1986), 813 - 822.

[14] R. R. Phelps, *Convex functions, monoton operators and differentiability,* Lecture Notes in Math., vol. 1364, Springer-Verlag 1989.

"Ovidius" University of Constanta
Department of Mathematics and Informatics,
900527 Constanta, Bd. Mamaia 124
Romania