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A NOTE ON ISOMORPHIC COMMUTATIVE GROUP ALGEBRAS OVER CERTAIN RINGS

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Abstract

Suppose G is an abelian group and R is a commutative ring with 1 of $char(R) \neq 0$. It is proved that if G is R-favorable torsion and RH and RG are R-isomorphic group algebras for some group H, then H is R-favorable torsion abelian if and only if either $inv(R) = \emptyset$ or $inv(R) \neq \emptyset$ and R is an ND-ring. This strengthens results due to W. Ullery (Comm. Algebra, 1986), (Rocky Mtn. J. Math., 1992) and (Comment. Math. Univ. Carolinae, 1995) and shows that in some instances the condition on H being a priory assumed as R-favorable may be removed.

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Let G be a multiplicatively written abelian group with p-component G_p for some prime number p and let R be a commutative ring with identity and arbitrary characteristic. Throughout the rest of this brief article, RG denotes the group ring viewed as an R-algebra of G over R, and $V(RG)$ is the group of all normalized invertible elements (often called normed units) in RG.

Before stating the main assertion motivating this paper, we need a few additional notations and definitions.

Following [Ma], we set $inv(R) = \{p|p$ is a unit in $R\}$, $zd(R) = \{p|p$ is a zero divisor in $R\}$, $supp(G) = \{p|G_p \neq 1\}$ and $G_R = \coprod_{p\in inv(R)} G_p$. Notice that the set $inv(R)$ can be equivalently restated as $inv(R) = \{p|p.1_R$ is a unit of $R\}$, while this is not the case for $zd(R)$ because of the following arguments: The

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fact that $p.1_R$ is a zero divisor of R implies that p has such a property in R, that is there exists $0 \neq r \in R$ with $pr = 0$, whereas the converse implication does not hold in general. For example, if F is a field of $char(F) = p > 0$, we have $p.1_F = 0$ whence it is not a zero divisor, but $pf = 0$ for each $0 \neq f \in F$ and so p is a prime zero divisor. Nevertheless, these two claims are equivalent for rings of characteristic distinct from p.

Definition. ([U], [Ulle]). The abelian group G is called *R-favorable* when $G_R = 1$, or equivalently when $supp(G) \cap inv(R) = \emptyset$.

Definition. The commutative ring R with identity is said to be *indecomposable* if it has no nontrivial idempotents.

Definition. ([U], [Ulle]). The commutative unitary ring R is termed as an *ND-ring* (= nicely decomposing) if R can be properly decomposed in the following manner: $R = R_1 \times \cdots \times R_n$ for some natural number n such that there is an index i: $1 \leq i \leq n$ with $inv(R) = inv(R_i)$.

By using this definition and some other crucial facts, Ullery obtains the following necessary and sufficient condition for a commutative unitary ring to be an ND-ring, namely:

Criterion. ([U]). *The commutative ring R with identity is an ND-ring if and only if there is a homomorphism $R \to K$ for some indecomposable commutative ring K with identity so that $inv(R) = inv(K)$.*

The major purpose of the present short note is to check whether or not the property of G being R-favorable torsion can be inherited by RG and, if yes, over what rings this remains realized. We settle below this matter in the affirmative by finding a criterion for any commutative unitary ring equipped with nonzero characteristic. We terminate the exploration with a problem concerning the case of rings of zero characteristic.

Before giving the main result and its proof, a few technicalities are in order (e.g. [Ma], [May] and [U], [Ulle] for nomenclature), namely:

Proposition. (May, 1976). *Let R be an indecomposable ring and let $supp(G) \cap inv(R) = \emptyset$. If $p \in inv(R)$, then $V_p(RG) = 1$.*

It is well-known that any ring homomorphism $R \to K$ endows K with the structure of an R-algebra and thus it ensures the K-isomorphism of algebras $KG \cong RG \otimes_R K$.

According to this isomorphism property and to the preceding May's statement, Ullery establishes the following assertion.

Theorem. ([U]). *Suppose G is an abelian group and R is an ND-ring or an indecomposable commutative ring with* 1. *If G is R-favorable and $RH \cong RG$*

as R-algebras for any group H, then H is R-favorable as well. Even more, $H \cong G$ provided $char(R) = 0$.

We shall use and extend in the sequel this affirmation by showing that for rings of positive characteristic with non-empty set of invertible primes the ND-rings are the only ones that preserve the property of the torsion group basis to be ring-favorable. Thus we discover that the complete inheritance by RG of this property for another group basis H consists entirely of the specific decomposable ring structure of R.

The next proposition will be helpful for proving once again that the group G modulo its torsion part G_t, that is G/G_t, can be invariantly retrieved by RG over any commutative ring R with identity (see [M], [Ma] and [D] for example).

Proposition. (May, 1976). *Suppose that R is an indecomposable ring and suppose that $supp(G) \cap inv(R) = \emptyset$. Then $V(RG) = GW_{RG}$ and $G \cap W_{RG} = G_t$, where W_{RG} is the multiplicative group (= the group of units) of the maximal integral subalgebra of RG with augmentation 1 and G_t is the maximal torsion subgroup of G.*

This enables us to give our first statement.

Theorem. *Suppose R is a commutative ring with identity of $char(R) \neq 0$ and G is a torsion R-favorable abelian group. Then $RG \cong RH$ as R-algebras over R for any group H will imply that H is a torsion R-favorable abelian group if and only if either 1) $inv(R) = \emptyset$ or 2) $inv(R) \neq \emptyset$ and R is an ND-ring.*

Proof. We foremost note that G being torsion and $RG \cong RH$ being $R-$isomorphic force that H is torsion as well (see [M] or [D] for instance).

After this, we consider two cases about $inv(R)$.

1) $inv(R) = \emptyset$.

Hence every torsion group is R-favorable, so G and H being torsion groups are both R-favorable.

2) $inv(R) \neq \emptyset$.

In this aspect, two subcases are valid:

2.1) If R is indecomposable, we mention that it can be formally interpreted as an ND-ring and everything is done by the foregoing listed result of Ullery from [U].

2.2) Suppose for a moment that R is decomposable, say $R = R_1 \times \cdots \times R_n$ for some rings R_i, $1 \leq i \leq n$, n is natural. Notice that $RG \cong RH \iff R_iG \cong R_iH$, $\forall\ i : 1 \leq i \leq n$.

First of all, if R is an ND-ring, we are done (e.g. [U]).

That is why, we shall presume that R is not an ND-ring, whence $inv(R) \subset inv(R_i)\ \forall\ i : 1 \leq i \leq n$. Thus, we distinguish two possibilities for $char(R)$.

a) Firstly, we note that $char(R) \neq p^k$, for all primes p and all positive integers k. Otherwise, $char(R) = p^k$ for some prime number p and natural number k assures that $p \notin inv(R)$ and thus there is $l : 1 \leq l \leq n$ with $p \notin inv(R_l)$; $p \in inv(R_i) \forall i$ if and only if $p \in inv(R)$. But $(q, p) = 1$ $\forall$ prime numbers $q \neq p$, so $q \in inv(R) \subset inv(R_i)$ $\forall i$. Therefore $inv(R) = \pi \setminus \{p\} = inv(R_l)$, where π is the set of all primes. Henceforth, R is an ND-ring, a contradiction.

b) Secondly, $char(R) = m \neq p^k$, for all nonegative integers k and over any prime number p. We show below that there exist three objects R, G and H, such that R is not an ND-ring, $char(R) = 6$ and G is R-favorable while H is not R-favorable but $RG \cong RH$ are R-isomorphic. This will substantiate our claim that for the group H to be deduced as R-favorable, R must be an ND-ring.

For a counterexample, let $R = F_2 \times F_3$, where F_p is an algebraically closed field of characteristic p, let T_p be an abelian p-group of cardinality $\aleph_0$, and put $G = T_2 \times T_3$ and $H = T_2 \times T_3 \times T_5$. Furthermore, it follows from well-known results due to May [M] on group algebras over algebraically closed fields that $F_2 T_3 \cong F_2(T_3 \times T_5)$ and $F_3 T_2 \cong F_3(T_2 \times T_5)$. Thus $RG \cong F_2 G \times F_3 G \cong (F_2 T_3) T_2 \times (F_3 T_2) T_3 \cong (F_2(T_3 \times T_5)) T_2 \times (F_3(T_2 \times T_5)) T_3 \cong F_2(T_2 \times T_3 \times T_5) \times F_3(T_2 \times T_3 \times T_5) \cong (F_2 \times F_3)(T_2 \times T_3 \times T_5) = RH$. So, the example is shown and the theorem is proved in full generality.

We now examine the extreme case when $char(R) = 0$ with $inv(R) \neq \emptyset$. Consider the ring $R = P \times L$ where $char(P) = char(L) = 0$. Since $inv(R) \neq \emptyset$ it follows that $inv(P) \cap inv(L) \neq \emptyset$, and even more that $inv(R) \subseteq inv(P) \cap inv(L)$. Note that we can take $inv(P) \neq inv(L)$ when R is not an ND-ring.

Next, the following problem is actual.

Problem. Given $P = Z[1/p, 1/q]$ and $L = Z[1/p, 1/s]$ as well as $G \cong Z(q^\infty) \times Z(s^\infty)$ and $H \cong Z(p^\infty) \times Z(q^\infty) \times Z(s^\infty)$. (We indicate that, because only $p \in inv(R)$, so R is not an ND-ring, G is R-favorable while H is not.) Does it follow that $PG \cong PH$ as P-algebras and $LG \cong LH$ as L-algebras, respectively?

Notice that, because of the symmetry, only the first isomorphism of algebras have to be verified. If this question has a positive answer, we conclude that $RG \cong RH$ as R-algebras. Thereby, when $char(R) = 0$, the condition on R to be an ND-ring cannot perhaps be omitted in general as well.

However, that possibility is probably fulfilled in all generality for the mixed case as the following example shows.

Example. ([U], Theorem 2). When G is mixed, R need not be however an ND-ring. Specifically, there exist three objects, namely R, G and H, such that R is not an ND-ring, $char(R) = 0$ and $inv(R) \neq \emptyset$ whereas both G and H are R-favorable mixed groups with $RG \cong RH$. Nevertheless, $G \ncong H$.

It may be given two another independent simple verifications of the periodicity of H.

In fact, if G is torsion and $G_p = 1$ for every prime p, then $G = 1$ hence $H = 1$ and there is nothing to prove. If now $G_p \neq 1$ for some prime number p, then $p \notin inv(R)$ since G is R-favorable. Furthermore, there exists a maximal ideal J of R with $p \in J$. So, $F = R/J$ is a field of $char(F) = p > 0$ and by the tensor multiplication over F, we infer that $RG \cong RH$ as R-algebras guarantees that $FG \cong FH$ as F-algebras. Thus [M], [Ma] or [D] can be employed to derive that H is torsion, as wanted.

For the second confirmation in a special case for G, given that P is a minimal prime ideal of R, whence R/P is an integral domain and so indecomposable. Besides, we obviously observe that $RG \cong RH$ as R-algebras yields $(R/P)G \cong (R/P)H$ as R/P-algebras. We shall assume extraordinary that G is chosen a priory as R/P-favorable, hence it is R-favorable since $p \in inv(R)$ therefore $p \in inv(R/P)$ whence $inv(R) \subseteq inv(R/P)$. Without harm of generality, we shall assume also that $(R/P)G = (R/P)H$. Since, by what we have argued above, both G and H are R/P-favorable groups, whence $supp(H) \cap inv(R/P) = \emptyset$, consulting with the second proposition of May we can write $V((R/P)G) = GW_{(R/P)G} = HW_{(R/P)H} = V((R/P)H)$. Because of the fact that the maximal integral subalgebra is an invariant for the group algebra, we establish that $GW_{(R/P)G}/W_{(R/P)G} \cong G/G_t \cong H/H_t \cong HW_{(R/P)H}/W_{(R/P)H}$. Thus $G/G_t \cong H/H_t$ and G being torsion trivially leads us to H is torsion, as desired.

The proofs are finished.

Remarks. W. May ([Ma], p. 489, line 13(-)) had claimed that $R_1 \leq R$, where R is indecomposable, implies that $inv(R_1) \subseteq inv(R)$, but this is not immediate if it is true or otherwise it holds valid when $1_R \in R_1 \iff 1_{R_1} = 1_R$.

We emphasize that $char(R) = p$ gives $p.1_R = 0$ whence $p \notin inv(R)$ and $p \in zd(R)$, but $char(R) = 0$ when $zd(R) = \emptyset$.

Moreover, an appeal to the foregoing Theorem riches us with the implication that if $char(R) = p \neq 0$ and G is p-mixed, then $RH \cong RG$ insures that H is p-mixed, too. This is so since in that situation we have $G_R = \coprod_{q \neq p} G_q = 1$. In the special case when G is also torsion, each R-favorable torsion group is a p-group.

In the variant when $inv(R)$ coincides with the set of all primes, G being R-favorable torsion gives that $G = G_R = 1$, and so by combining $G = 1$ and $RH = RG$ we elementarily extract that $H = 1$.

Finally, we comment some aspects from the papers [Ulle] and [Ullery].

Inspired by the above argued Theorem, we detect that it is not necessarily in all of the hypotheses in [Ulle] the group H to be a priory given as R-favorable torsion. In this direction, in [Ullery], H need not be a p-group a priory, because as we have just seen $RG \cong RH$ along with $p \notin inv(R)$ imply $FG \cong FH$ for some field F of $char(F) = p$, whence G being a p-group yields that the same holds true for H, and besides $G_p = 1$ ensures $H_p = 1$. Thereby, it is rather natural to consider the another possibility for p. So, we state the following.

Problem. If $G_p = 1$ such that $p \in inv(R)$ and $RH \cong RG$, does it follow that $H_p = 1$?

The Theorem answers when $supp(G) \cap inv(R) = \emptyset$. This query considers the reverse cases when for all primes $q \neq p$ we have $G_q \neq 1$ but $q \in inv(R)$ that is $q \in supp(G) \cap inv(R)$, or when for almost all primes $q \neq p$ we have $G_q = 1$ but $q \notin inv(R)$ such that $supp(G) \cap inv(R) \neq \emptyset$.

However, such an investigation will be a work of some other appropriate research study.

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