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INSTABILITY CRITICAL LOADS OF THE FIBER REINFORCED ELASTIC COMPOSITES

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To Professor Dan Pascali, at his 70's anniversary

Abstract

The internal and superficial instability of a prestressed fiber reinforced orthotropic elastic composite is considered in the paper. Using Guz's formalism, boundary and far field conditions we get the critical values of compressive stresses which are producing the internal and superficial instability in the body. The superficial instability appears before that the internal instability. To avoid such a situation, due to the structured character of the composites, the admissible compressive load must be drastically limited. Numerical results for the particular case of a graphite/epoxy orthotropic composite materials are obtained in the paper.

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1 Introduction

The instability of the fiber-reinforced elastic composites is studied in the paper. We suppose that the body is a linear orthotropic elastic body supposed to small, infinitesimal initial deformations.

The phenomenon of internal instability was analised for the first time by Biot [1] and Guz [2] and it concerns the loss of stability of the structure. It depends on the geometrical and mechanical characteristics of the internal structure and it is independent of geometrical characteristics of the body.

Key Words: Instability, orthotropic material, graphite/epoxy composite.

The material is considered as a structureless continuum and its behavior is described by constitutive equations containing material constants. The structural properties of the body are implicitely reflected by the value of the elasticities occuring in the stress-strain relation of the material.

The homogeneous equilibrium state is studied. Using Guz's representation of incremental fields, boundary and far field conditions we get the values of the critical stresses which produce internal and superficial instabilities.

The other result is that the critical load producing superficial instability is greater than those producing internal instability, *i.e.* the superficial instability appears the first.

For the graphite/epoxy orthotropic composite material our results are numerically verified.

2 Equilibrium state of a prestressed infinite body

We study the behavior of an infinite body, submited to well-defined given loads, acting at large distances. We assume in above conditions that the initial deformed and prestressed infinite body is in a homogeneous equilibrium state. The incremental behavior is governed by the differential system (see [2])

$$P_{lm} u_m = 0 \tag{1}$$

where P_{lm} are the differential operators defined by

$$P_{lm} = \omega_{klmn} \frac{\partial^2}{\partial x_k \partial x_n} \ , \tag{2}$$

and ω_{klmn} represent the instantaneous elasticities depending on the elasticities c_{klmn} of the material, as well as on the initial applied stress $\overset{\circ}{\sigma}$.

We assume that the *stress free* reference configuration of the body is *locally stable.* Hence, the differential system (1) corresponding to $\overset{\circ}{\sigma} = \mathbf{0}$ is *elliptic.* If, on a given loading paths, the instantaneous elasticity ω is positive definite; *i.e.* the system (1) conserves its ellipticity, the solutions of various incremental boundary value problems are unique, have a *local* character, and internal instability does not occur. If for some critical values of the loading parameters, the system (1) *ceases* to be elliptic, and becomes hyperbolic, the behavior of the perturbation changes radically and their local character is lost. A perturbation, appearing in a small domain, can propagate along the characteristics, producing considerable damages in the material. By internal instability we mean just the occurrence of such essential change in the behavior of the perturbation. Hence, internal instability occurs when on a given loading path the differential system (1) ceases to be elliptic. The corresponding critical values of the loading parameters are determined using the above criterion.

According to Guz's representation theorem (see [2]), we can replace the system (1) by a *simple* equation,

$$(detP)\,\varphi^{(j)} = 0,\ P = [P_{lm}]\,,\ j = 1,2,3, \tag{3}$$

satisfied by the displacement potentials $\varphi^{(j)}$. Consequently, the critical values of the loading parameters, producing internal instability, are those values for which on a given loading path the ellipticity of the equation (2) is for the first time lost.

We can say that the occurrence of the internal instability is guaranteed if the instantaneous elasticities satisfy the condition

$$\zeta_{lk}\omega_{klmn}\zeta_{mn} = 0 \text{ for any } \zeta_{mn} \text{ such that } \zeta_{kn}\zeta_{mn} \neq 0. \tag{4}$$

In the following we shall illustrate these general ideas by some special cases. We observe that the implications of the general criterion (4) can be more easily determined using the factorized forms of the equations satisfied by the displacement potentials.
Following [3] the equation (3) satisfied by the displacements potentiales $\varphi^{(\alpha)}$ can be factorized becoming

$$\left(\frac{\partial^2}{\partial x_2^2} + \eta_1^2\,\frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2} + \eta_2^2\,\frac{\partial^2}{\partial x_1^2}\right)\varphi^{(\alpha)} = 0, \tag{5}$$

where the parameters η_1^2 and η_2^2 satisfy equations

$$f(\eta) \equiv \eta^4 - 2A\eta^2 + B = 0, \tag{6}$$

with

$$A = \frac{\omega_{1111}\omega_{2222} + \omega_{1221}\ \omega_{2112} - (\omega_{1122} + \omega_{1212})^2}{2\ \omega_{2222}\omega_{2112}},\ B = \frac{\omega_{1111}\omega_{1221}}{\omega_{2222}\omega_{2112}}. \tag{7}$$

We start the analysis considering *incremental plane states*. Examining the equation (5), it can be seen that internal instability occurs when

$$\eta_1 = 0 \text{ or } \eta_2 = 0 \text{ for } \omega_{2222} \neq 0 \text{ and } \omega_{2112} \neq 0. \tag{8}$$

According to equations (6) and (7), the above condition will be satisfied if

$$B = 0,$$

or, more exactly, if

$$\omega_{1221} = 0 \text{ or } \omega_{1111} = 0 \text{ for } \omega_{2222} \neq 0 \text{ and } \omega_{2112} \neq 0. \tag{9}$$

Taking into account the values of the involved instantaneous elasticities, the above conditions become

$$C_{66} + \overset{\circ}{\sigma}_{11} = 0 \text{ or } C_{11} + \overset{\circ}{\sigma}_{11} = 0 \text{ for } C_{22} + \overset{\circ}{\sigma}_{22} \neq 0 \text{ and } C_{66} + \overset{\circ}{\sigma}_{22} \neq 0. \tag{10}$$

The reference configuration of the body being assumed locally stable, the elasticities of the material satisfy the inequalities

$$C_{11}, C_{22}, C_{66}, C_{11}C_{22} - C_{12}^2 > 0. \tag{11}$$

Thus, the relations (10) show that internal instability can occur only if $\overset{\circ}{\sigma}_{11}$ is a compressive stress; *i.e.*

$$\overset{\circ}{\sigma}_{11} < 0.$$

We assume now that, on the considered loading path,

$$\overset{\circ}{\sigma}_{22} = 0. \tag{12}$$

In this case, the restrictions $(10)_{3,4}$ are satisfied. Internal instability occurs only if the applied compressive stress $\overset{\circ}{\sigma}_{11}$ satisfies the condition

$$\overset{\circ}{\sigma}_{11} = -C_{66} \text{ or } \overset{\circ}{\sigma}_{11} = -C_{11}. \tag{13}$$

We suppose that the material is a *fiber-reinforced composite*, and the fibers have the direction of the applied compressive force. For such composite materials, the transverse shear rigidity C_{66} is much smaller as the longitudinal axial rigidity; *i.e.*

$$C_{66} \ll C_{11}. \tag{14}$$

Hence, internal instability occurs if the compressive stress, applied in the fibers direction, reaches the critical value

$$\overset{\circ}{\sigma}{}_{11}^{ci} = -C_{66}. \tag{15}$$

The internal deformation produced by $\overset{\circ}{\sigma}{}_{11}^{ci}$ are *infinitesimal*, since the compressive stress acts in the fibers direction and (12) is true. Hence, the conditions in which the used incremental theory is applicable are fulfilled. Consequently, the loss of internal stability actually can occur in a fiber-reinforced composite, if the applied compressive force acts in the fibers direction.

If the material is isotropic, C_{66} and C_{11} have the same order of magnitude and internal instability can not occur for compressive force, for which the linear theory of elasticity is applicable.

The loss of internal stability for fiber-reinforced composite materials, for relatively small compressive stresses, is a direct consequence of their structured character, reflected by the *strong* anisotropy of the composite.

We assume that the material is a fiber-reinforced elastic composite with the fibers in the direction of Ox_1 axis.

From Eqs. (5) using Baggio's theorem, and from far field conditions we get for the displacements potentials the following forms

$$\begin{aligned} \varphi^{(1)} &= (A_1 e^{a\eta_1 x_2} + A_2 e^{a\eta_2 x_2}) \sin ax_1, \\ \varphi^{(2)} &= (B_1 e^{a\eta_1 x_2} + B_2 e^{a\eta_2 x_2}) \cos ax_1, \end{aligned} \tag{16}$$

where A_1, A_2, B_1, B_2 are arbitrary constants, and

$$a, \eta_1, \eta_2 > 0. \tag{17}$$

First we suppose that $\varphi^{(2)} \equiv 0$ and look for a possible eigenmode described by $\varphi^{(1)}$.

Using Guz's representation formula given the incremental displacement and stress field we obtain that homogeneous boundary conditions are satisfied if and only if there exists non vanishing constants A_1 and A_2 satisfying the following homogeneous system

$$\begin{aligned} &\left(\omega_{1212}\omega_{1111} + \omega_{2112}\omega_{1122}\eta_1^2\right) A_1 + \left(\omega_{1212}\omega_{1111} + \omega_{2112}\omega_{1122}\eta_2^2\right) A_2 = 0, \\ &\eta_1 \left\{\omega_{1122}\left(\omega_{1122} + \omega_{1212}\right) - \omega_{1111}\omega_{2222} + \omega_{2222}\omega_{2112}\eta_1^2\right\} A_1 + \\ &+ \eta_2 \left\{\omega_{1122}\left(\omega_{1122} + \omega_{1212}\right) - \omega_{1111}\omega_{2222} + \omega_{2222}\omega_{2112}\eta_2^2\right\} A_2 = 0, \end{aligned} \tag{18}$$

i.e. the determinant of the system is vanishing.

After long computations we obtain that superficial instability are occur if and only if the following equation containing the unknown $\overset{\circ}{\sigma}_{11}$ is fulfilled:

$$\sqrt{\left(C_{11} + \overset{\circ}{\sigma}_{11}\right) C_{22}} C_{66} \overset{\circ}{\sigma}_{11} + \sqrt{\left(C_{66} + \overset{\circ}{\sigma}_{11}\right) C_{66}} \left(C_{11}C_{22} - C_{12}^2 + C_{22} \overset{\circ}{\sigma}_{11}\right) = 0. \tag{19}$$

If the stress-free reference configuration is locally stable, the inequalities (11) are satisfied. Then we can conclude that the left-hand side of the above equation is positive if $\overset{\circ}{\sigma}_{11} = 0$. Hence, a static analogue of Rayleigh's surface waves can not exist, if the stress-free reference configuration of the body is locally stable.

3 Critical load producing superficial instability

The critical value of $\overset{\circ}{\sigma}_{11}$, for which superficial instability can occur in a *pre-stressed* equilibrium configuration, must satisfy equation (19).

We shall analyze now if such critical value may exist. After some obvious transformations, (19) becomes

$$\left(C_{11}+\overset{\circ}{\sigma}_{11}\right)C_{22}C_{66}\,\overset{\circ}{\sigma}{}_{11}^{2}-\left(C_{66}+\overset{\circ}{\sigma}_{11}\right)\left(C_{11}C_{22}-C_{12}^{2}+C_{22}\,\overset{\circ}{\sigma}_{11}\right)^{2}=0. \quad (1)$$

We introduce the dimensionless ratios

$$x=\overset{\circ}{\sigma}_{11}/C_{11},\ \varepsilon=C_{66}/C_{11}>0. \quad (2)$$

In this way, (1) takes the form

$$\frac{C_{22}}{C_{11}}\left(\frac{C_{22}}{C_{11}}-\varepsilon\right)x^{3}+\frac{C_{22}}{C_{11}}\left[\varepsilon\left(\frac{C_{22}}{C_{11}}-1\right)+2\left(\frac{C_{22}}{C_{11}}-\frac{C_{12}^{2}}{C_{11}^{2}}\right)\right]x^{2}+$$
$$+\left(\frac{C_{22}}{C_{11}}-\frac{C_{12}^{2}}{C_{11}^{2}}\right)\left(\frac{C_{22}}{C_{11}}-\frac{C_{12}^{2}}{C_{11}^{2}}+2\varepsilon\frac{C_{22}}{C_{11}}\right)x+\varepsilon\left(\frac{C_{22}}{C_{11}}-\frac{C_{12}^{2}}{C_{11}^{2}}\right)^{2}=0. \quad (3)$$

We recall that for a fiber-reinforced composite,

$$C_{66}\ll C_{11};$$

hence,

$$\varepsilon\ll 1. \quad (4)$$

Consequently, using an iterative method, we look for a root having the following form:

$$x=x_{0}+\varepsilon x_{1}+\varepsilon^{2}x_{2}+\varepsilon^{3}x_{3}. \quad (5)$$

Introducing (5) in (3), and neglecting terms of order ε^{4} and higher, we determine successively the unknowns x_0, x_1, x_2 and x_3.

Elementary, but a long computation, gives

$$x_{0}=0,\ x_{1}=-1,\ x_{2}=0,\ x_{3}=\frac{C_{11}}{C_{22}}\frac{C_{11}^{2}C_{22}^{2}}{\left(C_{11}C_{22}-C_{12}^{2}\right)^{2}}. \quad (6)$$

Introducing (6) in (5), we get

$$x=-\varepsilon\left(1-\varepsilon^{2}\frac{C_{11}}{C_{22}}\frac{C_{11}^{2}C_{22}^{2}}{\left(C_{11}C_{22}-C_{12}^{2}\right)^{2}}\right). \quad (7)$$

Using the notation (2) for the *critical value* $\overset{\circ}{\sigma}{}^{cs}_{11}$ *for which superficial instability occurs*, we obtain the following expression:

$$\overset{\circ}{\sigma}{}^{cs}_{11} = -C_{66}\left(1 - \frac{C_{66}^2}{C_{11}C_{22}} \frac{C_{11}^2 C_{22}^2}{\left(C_{11}C_{22} - C_{12}\right)^2}\right). \tag{8}$$

We recall that the critical value $\overset{\circ}{\sigma}{}^{ci}_{11}$, for which internal (structural) instability occurs, is given by the equation (15). Hence, it results

$$\overset{\circ}{\sigma}{}^{cs}_{11} = \overset{\circ}{\sigma}{}^{ci}_{11}\left(1 - \frac{C_{66}^2}{C_{11}C_{22}} \frac{C_{11}^2 C_{22}^2}{\left(C_{11}C_{22} - C_{12}^2\right)^2}\right), \tag{9}$$

or using the engineering constants of the material, we get

$$\overset{\circ}{\sigma}{}^{ci}_{11} = -G_{12},\ \overset{\circ}{\sigma}{}^{cs}_{11} = -G_{12}\left(1 - \frac{G_{12}^2}{E_1E_2}\left(1 - \nu_{13}\nu_{31}\right)\left(1 - \nu_{23}\nu_{32}\right)\right). \tag{10}$$

In a fiber-reinforced composite

$$G_{12}^2 \ll E_1E_2 \text{ and } 0 < \frac{G_{12}^2}{E_1E_2}\left(1 - \nu_{13}\nu_{31}\right)\left(1 - \nu_{23}\nu_{32}\right) < 1.$$

Consequently, according to (10),

$$\overset{\circ}{\sigma}{}^{ci}_{11} < \overset{\circ}{\sigma}{}^{cs}_{11} < 0. \tag{11}$$

Hence, the critical load-producing superficial instability is a compressive one, as well as the critical load-producing structural (internal) instability. Moreover, the superficial instability appears before the structural one.

4 Numerical results

In this Section we consider the particular case of a graphite/epoxy fiber reinforced orthotropic composite material.

We compute the critical values of the compresive stresses $\overset{\circ}{\sigma}{}^{ci}$ and $\overset{\circ}{\sigma}{}^{cs}$ producing internal, and respectivelly superficial instability and the non-vanishing components of produced strain $\overset{\circ}{\varepsilon}$ by critical compressive stresses $\overset{\circ}{\sigma}{}^{cs}$.

A graphite/epoxy fiber reinforced orthotropic composite material is characterized by the following technical constants:

$$E_1 = 190GPa,\ E_2 = E_3 = 10GPa,\ G_{12} = 7GPa,\ G_{13} = G_{23} = 6GPa,$$
$$\nu_{12} = 0.3,\ \nu_{13} = \nu_{23} = 0.2.$$

Using the reciprocity relations

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$$

we find

$$\nu_{21} = 0.16, \nu_{31} = \nu_{32} = 0.1.$$

The critical compressive stress $\overset{\circ}{\sigma}{}^{ci}_{11}$ acting in the fibers direction and producing internal(structural) instability of the material is given by the equation (15). Since $C_{66} = G_{12}$ we find

$$\overset{\circ}{\sigma}{}^{ci}_{11} = -7GPa.$$

The critical compressive stress $\overset{\circ}{\sigma}{}^{cs}_{11}$ producing superficial instability of the material is given by the equation $(10)_2$. We have

$$\overset{\circ}{\sigma}{}^{cs}_{11} = -G_{12}(1 - \frac{G_{12}^2}{E_1 E_2}(1 - \nu_{13}\nu_{31})(1 - \nu_{23}\nu_{32})).$$

Hence

$$\overset{\circ}{\sigma}{}^{cs}_{11} = -0.9883.$$

Consequently, superficial instability occurs before internal instability and the compressive stress leading to superficial instability produces only infinitesimal deformations.

In order to find the deformation produced by the above determined critical compressive stress $\overset{\circ}{\sigma}{}^{ci}_{11}$we must use the constitutive equations (see [3]) describing the behavior of an orthotopic material. Thus, for the non-vanishing components of the produced strain $\overset{\circ}{\varepsilon}$, we find

$$\overset{\circ}{\varepsilon}_{11} = \frac{1}{E_1}\,\overset{\circ}{\sigma}{}^{ci}_{11},\ \overset{\circ}{\varepsilon}_{22} = \overset{\circ}{\varepsilon}_{33} = -\frac{\nu_{12}}{E_1}\,\overset{\circ}{\sigma}{}^{ci}_{11}\,.$$

Hence

$$\overset{\circ}{\varepsilon}_{11} = 0.036, \overset{\circ}{\varepsilon}_{22} = \overset{\circ}{\varepsilon}_{33} = 0.0108.$$

These results show that in a fiber reinforced, hence structured composite material, internal instability can occur for relatively small compressive stresses, producing infinitesimal deformations. Hence the danger represented by the occurrence of the internal (structural) instability actually exists and can be detected using the linear elastic model of the material.

5 Final remarks

We considered a prestressed fiber reinforced elastic composite. The incremental behavior of the body is studied using Guz's representation with the displacement potentials $\varphi^{(\alpha)}, \ \alpha = 1, 2$. We obtained the critical values of the compresive stresses which produce the loss of internal and respectivelly of superficial instability and we observed that the superficial instability appears before the structural one.

Since the critical load leading to superficial instability has the order at magnitude of the transverse shear modulus, the losing of superficial stability, for relatively small compressive stresses, can really occur in a fiber-reinforced composite loaded by compressive forces acting in the fibers direction. To avoid a dangerous situation, due to the structured character of the composites, the admissible compressive forces must be drastically limited.

On results were numerically verified for the particular case of a graphite/epoxy fiber reinforced elastic composite.

References

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