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# NONLINEAR NEUMANN BOUNDARY VALUE PROBLEMS WITH $\phi$–LAPLACIAN OPERATORS

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*To Professor Dan Pascali, at his 70's anniversary*

**Abstract**

Using the Leray-Schauder degree theory we obtain existence results for Neumann boundary value problems

$$(\phi(u'))' = f(t,u,u'), \quad u'(0) = 0 = u'(T),$$

where $\phi$ is an homeomorphism between $\mathbb{R}$ and $]-a,a[$ (or between $]-a,a[$ and $\mathbb{R}$), $\phi(0) = 0$ and $f$ is a suitable nonlinearity.

## 1 Introduction

Some nonlinear operators in suitable functions spaces have been introduced in [2] (see also [3]), whose fixed points coincide with the solutions of nonlinear boundary value problems of the type

$$(\phi(u'))' = f(t,u,u'), \quad l(u,u') = 0, \tag{1}$$

where $l(u,u')$ denotes the Dirichlet, Neumann or periodic boundary conditions on $[0,T]$, $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a suitable monotone homeomorphism and $f : [0,T]\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function. Applications are given to existence results when $\phi$ is the vector p-Laplacian ($p > 1$), $f$ is asymptotically homogeneous and $l(u,u')$ is the Dirichlet condition.

Key Words: Nonlinear Boundary value problem; Laplacian Ooperators.

The aim of this paper is to study the existence of solutions for the Neumann boundary value problem

$$(\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T), \tag{2}$$

where $\phi : \mathbb{R} \to ]-a, a[$ is an homeomorphism such that $\phi(0) = 0$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some growth and sign conditions. An analogous result can be obtained for problems of type (2) with $\phi : ]-a, a[\to \mathbb{R}$.

To prove the main results of this article we reformulate problem (2) in an abstract way which allows us to apply the Leray-Schauder degree. When $\phi : ]-a, a[\to \mathbb{R}$, new difficulties occur because the function $\phi^{-1}$ is not defined everywhere. Our existence conditions require $f$ to be everywhere bounded, with a bound depending upon $a$ and $T$, and to satisfy a sign condition (see Theorem 1). When $\phi : \mathbb{R} \to ]-a, a[$, a sign condition is sufficient (see Theorem 2). Examples are given. The method used here is inspired by the continuation theorem of coincidence degree theory [4] and by Theorem 3.1 in [2].

## 2 Notations and preliminaries

We first introduce some notations. Let $C$ denote the Banach space of continuous functions on $[0, T]$ endowed with the norm $||u||_\infty = \max_{t\in[0,T]} |u(t)|$, $C^1$ denote the Banach space of continuously differentiable functions on $[0, T]$ equipped with the norm $||u|| = ||u||_\infty + ||u'||_\infty$ and $C^1_\#$ denote the closed subspace of $C^1$ defined by $C^1_\# = \{u \in C^1 : u'(0) = 0 = u'(T)\}$. We denote by $P, Q$ the projectors

$$P, Q : C \to C, \quad Pu(t) = u(0), \quad Qu(t) = \frac{1}{T}\int_0^T u(s)ds,$$

and we define $H : C \to C$ by

$$Hu(t) = \int_0^t u(s)ds.$$

If $u \in C$, we write

$$[u]_{\mathrm{L}} = \min_{t\in[0,T]} u(t), \quad [u]_{\mathrm{M}} = \max_{t\in[0,T]} u(t).$$

We need the following elementary inequality.

**Lemma 1** *If $w \in C$, then*

$$\|H(I-Q)w\|_\infty \leq \frac{T}{\sqrt{3}} \left( \frac{1}{T} \int_0^T w^2(t)\,dt \right)^{1/2} \leq \frac{T}{\sqrt{3}} \|w\|_\infty. \tag{3}$$

*Proof.* If $v = H(I-Q)w$, then $v \in C^1$ and $v(0) = v(T) = 0$, so that

$$v(t) = \sum_{n=1}^{\infty} A_n \sin n\omega t,$$

where $\omega = \frac{\pi}{T}$, and, as $w \in C \subset L^2(0,T)$, we have

$$w(t) \sim \sum_{n=1}^{\infty} n\omega A_n \cos n\omega t + \frac{1}{T} \int_0^T w(s)ds$$

with $\sum_{n=1}^{\infty} n^2 A_n^2 < +\infty$. Letting $a_n = n\omega A_n$ $(n \geq 1)$, so that $\sum_{n=1}^{\infty} a_n^2 < +\infty$, we get, for each $t \in [0,T]$,

$$\begin{aligned} |H(I-Q)w(t)| &= \left| \sum_{n=1}^{\infty} \frac{a_n}{n\omega} \sin n\omega t \right| \leq \frac{1}{\omega} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \\ &\leq \frac{T}{\sqrt{3}} \left( \frac{1}{T} \int_0^T w^2(t)\,dt \right)^{1/2} \leq \frac{T}{\sqrt{3}} \|w\|_\infty. \end{aligned}$$

■

Finally, to each continuous function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we associate its Nemytskii operator $N_f : C^1 \to C$ defined by

$$N_f(u)(t) = f(t, u(t), u'(t)).$$

All the above defined operators $P, Q, H, N_f$ are continuous.

## 3 Abstract formulation

Let $N : C^1_\# \to C$ be a continuous operator. We consider the operator $\mathcal{G}_N$ given for $u \in C^1_\#$ by

$$\mathcal{G}_N(u) = Pu + QN(u) + H \circ \phi^{-1} \circ H(I-Q)N(u).$$

**Lemma 2** *If $N$ satisfies the condition*

$$\|N(u)\|_\infty \leq K < \sqrt{3}\,\frac{a}{T} \quad \textit{for all} \quad u \in C^1_\# \tag{4}$$

*then the operator $\mathcal{G}_N$ is well defined on $C^1_\#$ and $u$ is a solution of*

$$(\phi(u'))' = N(u), \quad u'(0) = 0 = u'(T) \tag{5}$$

*if and only if $u$ is a fixed point of $\mathcal{G}_N$.*

*Proof.* Let $u \in C^1_\#$. Using (4) and (3) we have

$$\|H(I-Q)N(u)\|_\infty \leq \frac{T}{\sqrt{3}}\|N(u)\|_\infty \leq \frac{TK}{\sqrt{3}} < a. \tag{6}$$

From (6) we deduce that $\mathcal{G}_N$ is well defined on $C^1_\#$. It is clear that $\mathcal{G}_N(u) \in C^1$ if $u \in C^1_\#$. We show that, in fact, $\mathcal{G}_N(u) \in C^1_\#$ for $u \in C^1_\#$. If $u \in C^1_\#$, then $(\mathcal{G}_N(u))' = \phi^{-1} \circ H(I-Q)N(u)$. Using the relations

$$H(I-Q)N(u)(0) = 0 = H(I-Q)N(u)(T), \quad \phi^{-1}(0) = 0,$$

it follows that

$$(\mathcal{G}_N(u))'(0) = 0 = (\mathcal{G}_N(u))'(T).$$

Now suppose that $u$ is a solution of (5). Integrating both members over $[0,T]$ we get

$$QN(u) = 0 \tag{7}$$

and, integrating both members over $[0,t]$ we get $\phi(u') = H \circ N(u)$, from where it follows that $\phi(u') = H(I-Q)N(u)$, so, $u' = \phi^{-1} \circ [H(I-Q)N](u)$ and, integrating, $u = Pu + H \circ \phi^{-1} \circ [H(I-Q)N](u)$, which, because of (7) is equivalent to $u = \mathcal{G}_N(u)$. Conversely, if $u = \mathcal{G}_N(u)$, then

$$u - Pu - H \circ \phi^{-1} \circ [H(I-Q)N](u) = QN(u),$$

which gives

$$u = Pu + H \circ \phi^{-1} \circ [H(I-Q)N](u), \quad QN(u) = 0,$$

so that $u \in C^1_\#$ and $u$ is a solution for (5) by differentiating the first equation, applying $\phi$ to both of its members, differentiating again and using the second equation. ∎

## 4 A compact homotopy

Assume now that $f$ satisfies the condition

$$|f(t,u,v)| \leq K < \sqrt{3}\,\frac{a}{T} \quad \text{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \tag{8}$$

For $\lambda \in [0,1]$ consider the family of abstract Neumann problems

$$(\phi(u'))' = \lambda N_f(u) + (1-\lambda)QN_f(u), \quad u'(0) = 0 = u'(T). \tag{9}$$

As

$$\|\lambda N_f(u) + (1-\lambda)QN_f(u)\|_\infty \leq K < \sqrt{3}\,\frac{a}{T}, \tag{10}$$

for all $u \in C^1_\#$, it follows from Lemma 2 that the operator $\mathcal{M}$ associated to (9), which is, as easily shown, given by

$$\mathcal{M}(\lambda, u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ [\lambda H(I-Q)N_f](u) \tag{11}$$

is well defined and continuous on $[0,1] \times C^1_\#$, and that $u$ is a solution for (9) if and only if $u = \mathcal{M}(\lambda, u)$.

To use Leray-Schauder degree [1, 5] for finding fixed points of $\mathcal{M}$, we prove in the next lemma that the continuous operator $\mathcal{M}$ is completely continuous on $C^1_\#$, i.e. that for any sequence $(\lambda_n, u_n)_n \subset [0,1] \times C^1_\#$ with $(||u_n||)_n$ bounded, the sequence $(\mathcal{M}(\lambda_n, u_n))_n$ has a convergent subsequence.

**Lemma 3** *$\mathcal{M}$ is completely continuous on $C^1_\#$.*

*Proof.* Let $(\lambda_n, u_n)_n \subset [0,1] \times C^1_\#$ with $(\|u_n\|)_n$ bounded. We may assume that $\lambda_n \to \lambda_0$. Let $v_n = \mathcal{M}(\lambda_n, u_n)$, $(n \in \mathbb{N})$. Then

$$v_n = Pu_n + QN_f(u_n) + H \circ \phi^{-1} \circ [\lambda_n H(I-Q)N_f](u_n), \ (n \in \mathbb{N}).$$

Because of (8),

$$\begin{aligned} \|QN_f(u_n)\|_\infty &\leq K, \\ \|\phi^{-1} \circ [\lambda_n H(I-Q)N_f](u_n)\|_\infty &\leq \max\left\{\left|\phi^{-1}\left(-\frac{KT}{\sqrt{3}}\right)\right|, \left|\phi^{-1}\left(\frac{KT}{\sqrt{3}}\right)\right|\right\} := M, \\ (n &\in \mathbb{N}). \end{aligned} \tag{12}$$

From (12) it follows that $(v_n)_n$ is bounded in $C$. Let $t_1, t_2 \in [0,T]$. Then, for all $n \in \mathbb{N}$, using (12) we have

$$|v_n(t_1) - v_n(t_2)| = \left|\int_{t_1}^{t_2} \phi^{-1} \circ [\lambda_n H(I-Q)N_f](u_n)(s)ds\right| \leq M|t_1 - t_2|,$$

which implies that $(v_n)_n$ is equicontinuous. Applying Arzela-Ascoli theorem, passing if necessary to a subsequence, we may assume that $v_n \longrightarrow v$ in $C$. On the other hand

$$v_n' = \phi^{-1} \circ [\lambda_n H(I-Q)N_f](u_n), \quad (n \in \mathbb{N})$$

so, using (12), it follows that $\|v_n'\|_\infty \leq M$ for all $n \in \mathbb{N}$. Furthermore, if $t_1, t_2 \in [0,T]$, then

$$|\phi(v_n'(t_2)) - \phi(v_n'(t_1))| \leq \left| \int_{t_1}^{t_2} (I-Q)N_f(u_n)(s)ds \right| \leq 2K|t_1 - t_2|. \qquad (13)$$

Using (6), (4) and the uniform continuity of $\phi^{-1}$ on compact intervals of $]-a,a[$, it follows that $(v_n')_n$ is equicontinuous. Applying Arzela-Ascoli theorem, we may assume, passing to a subsequence, that $v_n' \to w$ in $C$, with $\|w\|_\infty \leq M$. It follows that $v \in C^1_\#$, $v' = w$, so that $v_n \to v$ in $C^1$. ∎

## 5 A priori estimates

Let $f$ be a function as in Section 3, and $\mathcal{M}$ the corresponding nonlinear operator given by (11).

**Lemma 4** *If there exists $R > 0$ and $\epsilon \in \{-1, 1\}$ such that, with*

$$M = \max\{\left|\phi^{-1}(-\frac{KT}{\sqrt{3}})\right|, \left|\phi^{-1}(\frac{KT}{\sqrt{3}})\right|\},$$

*one has*

$$\epsilon u f(t,u,v) > 0 \quad \text{if} \quad |u| \geq R,\ |v| \leq M,\ t \in [0,T], \qquad (14)$$

*then there is a constant $\rho > R$ such that for each $\lambda \in [0,1]$, each possible fixed point $u$ of $\mathcal{M}(\lambda, \cdot)$ verifies the inequality $\|u\| < \rho$.*

*Proof.* Let $\lambda \in [0,1]$ and $u = \mathcal{M}(\lambda, u)$. Hence $u' = \phi^{-1} \circ [\lambda H(I-Q)N_f(u)]$, and, from (6) and from the choice of $M$ it follows that

$$\|u'\|_\infty \leq M. \qquad (15)$$

Because $u = \mathcal{M}(\lambda, u)$, it follows from Lemma 2 that $u$ is a solution of (9), which implies that

$$\int_0^T f(t, u(t), u'(t))dt = 0. \qquad (16)$$

If $[u]_M \leq -R$ (respectively $[u]_L \geq R$) then, from (15) and (14), it follows that

$$\epsilon \int_0^T f(t, u(t), u'(t))dt < 0 \quad \text{(respectively} \quad \epsilon \int_0^T f(t, u(t), u'(t))dt > 0).$$

Using (16) we have that

$$[u]_M > -R \text{ and } [u]_L < R. \tag{17}$$

It is clear that

$$[u]_M \leq [u]_L + \int_0^T |u'(t)|dt. \tag{18}$$

From relations (17), (18) and (15), we obtain that

$$-(R+M) < [u]_L \leq [u]_M < R+M. \tag{19}$$

It follows that $||u|| < R+2M$ and it suffices to take $\rho = R+2M$. ■

## 6 Main results. Examples

The existence result when $\phi : ]-a, a[ \to \mathbb{R}$ follows from the above a priori estimates and Leray-Schauder theory.

**Theorem 1** *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function verifying conditions (8) and (14). Then (2) has at least one solution.*

*Proof.* Let $\mathcal{M}$ be the operator given by (11). We have that $\mathcal{M}(1,\cdot) = \mathcal{G}_{N_f}$ and $\mathcal{N}(0,\cdot) = P + QN_f$. Using Lemma 3, Lemma 4 and the homotopy invariance of the Leray-Schauder degree [1, 5], we obtain that $d_{\mathrm{LS}}[I - \mathcal{N}(1,\cdot), B_\rho(0), 0]$ and $d_{\mathrm{LS}}[I - \mathcal{N}(0,\cdot), B_\rho(0), 0]$ are well defined and equal. But the range of $\mathcal{N}(0,\cdot)$ is contained in the subset of constant functions, isomorphic to $\mathbb{R}$, so, using a property of the Leray-Schauder degree we have that

$$\begin{aligned} & d_{\mathrm{LS}}[I - \mathcal{N}(0,\cdot), B_\rho(0), 0] = d_{\mathrm{B}}[I - \mathcal{N}(0,\cdot)|_{\mathbb{R}}, (-\rho,\rho), 0] \\ &= d_{\mathrm{B}}[-QN_f, (-\rho,\rho), 0] = \frac{-\mathrm{sign}(QN_f(\rho)) + \mathrm{sign}(QN_f(-\rho))}{2}, \end{aligned}$$

where $d_{\mathrm{B}}$ denotes the Brouwer degree. But, using (14) and the fact that $\rho > R$ we see that $QN_f(\pm\rho) = \frac{1}{T}\int_0^T f(t, \pm\rho, 0)dt$ have opposite signs, which implies that

$$|d_{\mathrm{LS}}[I - \mathcal{N}(1,\cdot), B_\rho(0), 0]| = |d_{\mathrm{LS}}[I - \mathcal{N}(0,\cdot), B_\rho(0), 0]| = 1.$$

Then, from the existence property of the Leray-Schauder degree, there is $u \in B_\rho(0)$ such that $u = \mathcal{N}(1, u) = \mathcal{G}_{N_f}(u)$, and $u$ is a solution for (2) by Lemma 2. ■

The case where $\phi : ]-a, a[ \to \mathbb{R}$ is simpler to treat because $\phi^{-1}$ is now defined over $\mathbb{R}$, so that the fixed point operator $\mathcal{G}_N$ is well defined without growth restriction upon $N$. Notice now that a solution of (2) or of (5) must satisfy the estimate $-a < u'(t) < a$ for all $t \in [0, T]$ in order to be defined. This estimate is satisfied for any possible fixed point of $\mathcal{G}_N$ or $\mathcal{M}$. The complete continuity of $\mathcal{M}$ is proved like in Lemma 3. We have the following result

**Theorem 2** *Let $\phi : ]-a, a[\to \mathbb{R}$ be a homeomorphism such that $\phi(0) = 0$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for some $R > 0$ and some $\epsilon \in \{-1, 1\}$,*

$$\epsilon u f(t, u, v) > 0 \quad \text{if} \quad |u| \geq R,\ |v| < a,\ t \in [0, T]. \tag{20}$$

*Then (2) has at least one solution.*

*Proof.* If $\lambda \in [0, 1]$ and $u$ is a possible fixed point of $\mathcal{M}(\lambda, \cdot)$, then

$$u' = \phi^{-1} \circ [\lambda H(I - Q)N](u), \tag{21}$$

and

$$\int_0^T f(t, u(t), u'(t))\, dt = 0. \tag{22}$$

If follows from (21) that

$$|u'(t)| < a \quad (t \in [0, T]). \tag{23}$$

Now, if $[u]_M \leq -R$, we have, using (21) and (20),

$$\epsilon f(t, u(t), u'(t)) < 0 \quad (t \in [0, T]),$$

which gives a contradiction to (22). Similarly if $[u]_L \geq R$. Hence,

$$[u]_M > -R, \quad [u]_L < R. \tag{24}$$

Now, using (23),

$$[u]_M - [u]_L \leq \int_0^T |u'(t)|\, dt < aT,$$

which implies, together with (24) that

$$\|u\|_\infty < R + aT,$$

and hence

$$\|u\| < R + a(T + 1) \tag{25}$$

for all possible fixed points of $\mathcal{M}(\lambda, \cdot)$. The end of the proof is then entirely similar to that of Theorem 1. ■

**Example 1** Using Theorem 1 we obtain that *the Neumann boundary value problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \alpha(\arctan u + \sin t), \quad u'(0) = u'(1) = 0$$

*has at least one solution if* $|\alpha| \leq 0.835$.

**Example 2** Using Theorem 1 we obtain that *the Neumann boundary value problem*

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \frac{\sqrt{3}}{4}\arctan(u+t) + \frac{\sqrt{3}}{3}\sin(u'+t^2), \quad u'(0) = u'(1) = 0$$

*has at least one non constant solution.*

**Example 3** Using Theorem 2 we obtain that *the Neumann boundary value problem*

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = (u+t)^3 + \sin^2(u'), \quad u'(0) = 0 = u'(T)$$

*has at least one non constant solution.*

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