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VANISHING RESULTS FOR SEMISIMPLE LIE ALGEBRAS

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Abstract

In these papers we have proved the vanishing of H^i (g,U(g)) for *i=1,2* which is called the i-th cohomology group of the Lie algebra g with coefficients in $\mathcal{U}$ (g), called the enveloping algebra of g.

We start with the following result.

Proposition 1.1. *If g is a finite-dimensional complex semisimple Lie algebra and M is a finite-dimensional non-trivial simple left g-module, then $H^n(g, M) = 0$ for all $n \geq 0$. Here non-trivial means that M is not one-dimensional g-module on which the Lie algebra acts by zero.*

Proof. We need the Casimir element $C = \sum\limits_k x_k x^k$. We know that C acts on any finite-dimensional non-trivial simple g-module by a non-zero scalar. In order to prove the proposition, we construct for all n a map $h : C^n(g, M) \to C^{n-1}(g, M)$ such that

$$Cf = \delta h f + h \delta f \tag{1.1}$$

for all $f \in C^n(g, M)$. By Cf we mean the n-liniar map defined by

$$(Cf)(y_1, ..., y_n) = C(f(y_1, ..., y_n))$$

where $y_1, ..., y_n$ belong to g. Let f be an n-cocycle with values in M, i.e. such that $\delta f = 0$. By (1.1) we get $Cf = \delta(hf)$, which means that Cf is a n-coboundary.

Since C acts by a non-zero scalar in M we see that f is a coboundary. This proves the vanishing of $H^n(g, M)$.

We are left with building a map h satisfying (1.1). Given $f \in C^n(g, M)$ with $n > 0$ and the Casimir element, we define an antisymmetric $(n-1)$-linear map hf with values in M by

$$(hf)(y_1, ..., y_{n-1}) = \sum_k x_k f(x^k, y_1, ..., y_{n-1}) \tag{1.2}$$

for all $y_1, ..., y_{n-1} \in g$. If $f \in C^0(g, M)$, let $hf = 0$.. Using (1.2) we get

$$(\delta hf + h\delta f)(y_1, ..., y_n) = Cf(y_1, ..., y_n) + \sum_{1 \le i \le n} (-1)^i Z_i$$

where

$$Z_i = \sum_k \left([x_k, y_i] f \left(x^k, y_1, ..., \widehat{y_i}, ..., y_n \right) + x_k f \left(\left[x^k, y_i \right], y_1, ..., \widehat{y_i}, ..., y_n \right) \right)$$

Relation (1.1) will be proved if we show that all Z_i vanish. Using the linear forms α_{kl} and β_{kl}, where $[x_k, x] = \sum\limits_l \alpha_{kl}(x) x_l$ and $\left[x^k, x \right] = \sum\limits_l \beta_{kl}(x) x^l$, we get

$$Z_i = \sum_{k,l} \left(\alpha_{kl}(y_i) x_l f \left(x^k, y_1, ..., \widehat{y_i}, ..., y_n \right) + \beta_{kl}(y_i) x_k f \left(x^l, y_1, ..., \widehat{y_i}, ..., y_n \right) \right).$$

Exchanging k and l in the second summand, we obtain

$$Z_i = \sum_{k,l} \left(\alpha_{kl}(y_i) + \beta_{lk}(y_i) \right) x_l f \left(x^k, y_1, ..., \widehat{y_i}, ..., y_n \right),$$

which vanishes in view of the fact $\beta_{ij} = -\alpha_{ji}$ for all i, j.

As a consequence, we get the so-called ,,*Whitehead lemmas*".

Corollary 1.2. *If g is a semisimple Lie algebra and M is any finite-dimensional left g-module, then $H^1(g, M) = H^2(g, M) = 0$.*

Proof. We know that any finite-dimensional module M over a semisimple Lie algebra is a direct sum $M = \oplus_i M_i$ of simple module M_i. Since the complex $C^*(g, M)$ is the direct sum of the subcomplex $C^*(g, M_i)$ we have $H^n(g, M) = \oplus H^n(g, M_i)$. In view of Proposition (1.1), it is enough to prove Corollary 1.2 when M is the trivial one-dimensional g-module C.

a) We first prove the vanishing of $H^1(g, C)$ which will imply the vanishing of $H^1(g, M)$ for all finite-dimensional modules M. Let f be a 1-cocycle with values in the trivial module C. We know that $f([x, y]) = 0$ for all $x, y \in g$ and Serre's relation show that the elements $[x, y]$ span the vector space g. Therefore $f = 0$ on the whole space g.

b) The argument for the vanishing of $H^2(g, C)$ is slightly more insolved. We first claim that if f is a 2-cocycle with values in C, then the linear map $\tilde{f}$ given by $\tilde{f}(x)(y) = f(x, y)$ for all $x, y \in g$, is a 1-cocycle of g with values in the dual vector space g^*. Such a statement presupposes that we have defined a left action of g on g^*. This is done by taking the coadjoint representation given by

$$(x\alpha)(y) = \alpha([y, x]) \tag{1.3}$$

where $x, y \in g$ and $\alpha \in g^*$. Indeed, if f is a 2-cocycle with values in C, we have

$$\widetilde{f}([x,y])(z) = \widetilde{f}(y)([z,x]) - \widetilde{f}(x)([z,y])$$

for all $x, y, z \in g$. Reformulating this, with (1.3), we get

$$\widetilde{f}([x,y]) = x\widetilde{f}(y) - y\widetilde{f}(x),$$

which shows that $\widetilde{f}$ is a 1-cocycle with values in the finite-dimensional g-module g^*.

By the first part of Corollary (1.2), the cocycle $\widetilde{f}$ is a coboundary, i.e., there exists a linear form $\alpha \in g^*$ such that $\widetilde{f}(x) = x\alpha$.

We thus get

$$f(x,y) = \widetilde{f}(x)(y) = (x\alpha)(y) = \alpha([x,y]) = -\alpha([x,y]).$$

In other words, the 2-cocycle f is the coboundary of α. This completes the proof of the vanishing of H^2. Observe that, incidentally, we proved that $H^2(g, C) \cong H^1(g, g^*)$. □

Let us equip $\mathcal{U}(g)$ with the adjoint representation of g for which the Lie algebra acts on $\mathcal{U}(g)$ on the left by $x \cdot u = xu - ux = [x, u]$ where $x \in g$ and $u \in \mathcal{U}(g)$. If $u = x_1....x_n$ with $x_1, ..., x_n$ belonging to g, an easy induction shows that

$$x \cdot u = \sum_{i=1}^{n} x_1 ... x_{i-1} [x, x_i] x_{i+1} ... x_n \tag{1.4}$$

We record the following corollary.

Corollary 1.3. *Let g be a finite-dimensional complex semisimple Lie algebra acting on $\mathcal{U}(g)$ as above. Then $H^1(g,\mathcal{U}(g)) = H^2(g,\mathcal{U}(g)) = 0$.*

Proof. We use the symmetrization map

$$\eta : S(g) \to \mathcal{U}(g)$$

defined by

$$\eta(x_1 ... x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} ... x_{\sigma(n)}$$

where $x_1, ..., x_n \in g$. We know that η is a linear isomorphism. Moreover, if we equip $S(g)$ with left g-module structure given by

$$x \cdot (x_1 ... x_n) = \sum_{i=1}^{n} x_1 ... x_{i-1} [x, x_i] x_{i+1} ... x_n \tag{1.5}$$

the map η becomes an isomorphism of g-module.

Now, as can easily be sum from (1.5), the action of g respects the decomposition of $S(g)$ into its homogeneous components $S^n(g)$. We thus obtain an isomorphism

$$\mathcal{U}(g) \cong \oplus_{n\geq 0} S^n(g)$$

of g-modules. Consequently, for $i = 1, 2$, we have

$$H^1(g,\mathcal{U}(g)) = \oplus_{n\geq 0} H^i(g, S^n(g)) = 0$$

by application of Corollary 1.2 to the finite dimensional modules $S^n(g),\ n \geq 0$.

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