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EIGENVALUES AND EIGENVECTORS FOR THE QUATERNION MATRICES OF DEGREE TWO

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Abstract

In this paper we give a computation method, in a particular case, for eigenvalues and eigenvectors of the quaternion matrices of degree two with elements in the generalized quaternion division algebra $\mathbb{H}(\alpha,\beta)$. It is known (see[1]) that every quaternion matrix has at least one characteristic root , but there is not yet giving a computing method. By using [4] we give such a computing method for eigenvalues and eigenvectors of the quaternion matrices of degree two with elements in the generalized quaternion division algebra $\mathbb{H}(\alpha,\beta)$.

Let $\mathbb{H}(\alpha,\beta)$ be the generalized quaternion division algebra over the comutative field K with *char* $K \neq 2$.

Definition 1 *Let $A \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ and $\lambda \in \mathbb{H}(\alpha,\beta)$. The quaternion λ is called an **eigenvalue** of the matrix A (or a **characteristic root**), if there exists a matrix $x \in \mathcal{M}_{n\times 1}(\mathbb{H}(\alpha,\beta))$, $x \neq 0$, such that $Ax = x\lambda$. The matrix x is called the **eigenvector** of the matrix A.*

Proposition 1 *Two similar matrices have the same characteristic roots.*

Proof. Let $A \sim B$, i.e. there exists an invertible matrix $T \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ such that $B = TAT^{-1}$. Let $\lambda \in \mathbb{H}(\alpha,\beta)$ be an eigenvalue for the matrix A, then we find the matrix $x \in \mathcal{M}_{n\times 1}(\mathbb{H}(\alpha,\beta))$ such that $Ax = x\lambda, x \neq 0$. Let $y = Tx$. Then $By = TAT^{-1}y = TAx = Tx\lambda = y\lambda$.□

Proposition 2 *Let $A \in \mathcal{M}_n(\mathbb{H}(\alpha,\beta))$ and let $\lambda \in \mathbb{H}(\alpha,\beta)$ be an eigenvalue of the matrix A. If $\rho \in \mathbb{H}(\alpha,\beta)$, $\rho \neq 0$, then $\rho^{-1}\lambda\rho$ is also an eigenvalue of the matrix A.*

Key Words: quaternion matrices; generalized quaternion algebra.

Proof. From $Ax = x\lambda$, we get $A(x\rho) = x\lambda\rho = (x\rho)\rho^{-1}\lambda\rho$.$\square$

Remark 1 From the Proposition 2, we see that, if the vector corresponding to the eigenvalue λ is x, then $x\rho$ is the eigenvector corresponding to the characteristic root $\rho^{-1}\lambda\rho$.

Proposition 3 ([1])*Let K be an arbitrary field, not necessarily commutative, with char $K \neq 2$. If $A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(K)$, then we have a triangular invertible matrix T such that $C = T^{-1}AT$, $C = (c_{ij})_{i,j=\overline{1,n}}$, where $c_{ij} = 0$, for all $i > j + 1$, $i, j \in \{1, 2, ..., n\}$.*$\square$

Let $\mathbb{H}$ be the real quaternion algebra and let f be the polynomial of degree n:

$$f(X) = a_0 X a_1 X ... X a_n + g(X),$$

where $a_0, a_1, ..., a_n \in \mathbb{H}$, $a_i \neq 0$ for every $i = \overline{1,n}$ and $g(X)$ is a finite sum of monomials of the form $b_0 X b_1 X ... X b_m$, where $m \lneqq n$.

In [2], it is shown that, if the polynomial f has a single term of degree n, then the equation $f(x) = 0$ has exactly n solutions in $\mathbb{H}$.

Proposition 4 ([1])*Let $A \in \mathcal{M}_n(\mathbb{H})$, then the matrix A has an eigenvalue.*$\square$

In the next, let $\mathbb{H}(\alpha, \beta)$ be the generalized quaternion division algebra over the commutative field K with *char* $K \neq 2$. It is known that $\mathbb{H}(\alpha, \beta)$ is an algebra of degree two, then every element $x \in \mathbb{H}(\alpha, \beta)$ satisfies a relation of the form:

$$x^2 + t(x)x + n(x) = 0,$$

where $t(x), n(x) \in K$ are the **trace** and the **norm** of the element x.

If $\{1, e_1, e_2, e_3\}$ is a basis in $\mathbb{H}(\alpha, \beta)$ and $x \in \mathbb{H}(\alpha, \beta)$, then, for $x = a + be_1 + ce_2 + de_3$, the element $\bar{x} = a - be_1 - ce_2 - de_3$ is called the **conjugate** of the element x and we have the relations:

$$x + \bar{x} = t(x) \text{ and } x\bar{x} = n(x)$$

Proposition 5 ([4]) *Let $a, b \in \mathbb{H}(\alpha, \beta), a \neq 0, b \neq 0$. Then the linear equation*

$$ax = xb \tag{5.1.}$$

has nonzero solutions, $x \in \mathbb{H}(\alpha, \beta)$, if and only if :

$$t(a) = t(b) \text{ and } n(a - a_0) = n(b - b_0), \tag{5.2.}$$

where $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3, b = b_0 + b_1e_1 + b_2e_2 + b_3e_3$.$\Box$

Proposition 6 ([4]) *i) If $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}(\alpha, \beta)$ with $b \neq \bar{a}$, $a, b \notin K$, then the solutions of the equation (5.1.), with $t(a) = t(b)$ and $n(a - a_0) = n(b - b_0)$, are found in $\mathcal{A}(a, b)$ (the algebra generated by the elements a and b) and have the form :*

$$x = \lambda_1(a - a_0 + b - b_0) + \lambda_2 (n(a - a_0) - (a - a_0)(b - b_0)), \tag{6.1.}$$

where $\lambda_1, \lambda_2 \in K$ are arbitrary.
ii) If $b = \bar{a}$, then the general solution of the equation (5.1.) is $x = x_1e_1 + x_2e_2 + x_3e_3$, where $x_1, x_2, x_3 \in K$ and they satisfy the identity :

$$\alpha a_1x_1 + \beta a_2x_2 + \alpha\beta a_3x_3 = 0.\Box \tag{6.2.}$$

Proposition 7 ([4]) *Let $a \in \mathbb{H}(\alpha, \beta)$, $a \notin K$. If there exists $r \in K$ such that $n(a) = r^2$, then $a = \bar{q}rq^{-1}$, where $q = r + \bar{a}$, $q^{-1} = \frac{\bar{q}}{n(q)}$.*

Proof. By hypothesis we have $a(r + \bar{a}) = ar + a\bar{a} = ar + n(a) = = ar + r^2 = (a + r)r$. From $\bar{q} = r + a$ it results $\bar{q}r = aq$.$\Box$

Proposition 8 ([4]) *Let $a \in \mathbb{H}(\alpha, \beta)$ with $a \notin K$, if there exist $r, s \in K$ with the properties $n(a) = r^4$, $n\left(r^2 + \bar{a}\right) = s^2$, then the quadratic equation $x^2 = a$ has two solutions of the form: $x = \pm\frac{r\left(r^2+a\right)}{s}$.*

Proof. By *Proposition 7,* it results that a has the form
$a = \bar{q}r^2q^{-1}$, where $q = r^2 + \bar{a}$. Because $q^{-1} = \frac{\bar{q}}{n(q)}$, we obtain
$a = r^2\bar{q}q^{-1} = r^2\bar{q}\frac{\bar{q}}{n(q)} = r^2\frac{\bar{q}^2}{s^2} = \left(\frac{r}{s}\bar{q}\right)^2$, therefore

$$x_1 = \frac{r}{s}\bar{q},\ x_2 = -\frac{r}{s}\bar{q}$$

are the claimed solutions.$\Box$

Proposition 9 ([4]) *Let $a, b, c \in \mathbb{H}(\alpha, \beta)$ such that ab and $b^2 - c$ do not belong to K. If ab and $b^2 - c$ satisfy the conditions in Proposition 8, then the equations $xax = b$ and $x^2 + bx + xb + c = 0$ have solutions.*

Proof. $xax = b \Longleftrightarrow (ax)^2 = ab$ and $x^2 + bx + xb + c = 0 \Longleftrightarrow (x+b)^2 = b^2 - c$.$\Box$

Proposition 10 ([4]) *If $b, c \in \mathbb{H}(\alpha, \beta)\backslash\{K\}$ satisfy the conditions $bc{=}cb$, $\frac{b^2}{4} - c \neq 0$ and there exists $r \in K$ such that $n\left(\frac{b^2}{4} - c\right) = r^4$ and $n\left(r^2 + \frac{\bar{b}^2}{4} - \bar{c}\right) = s^2,\ s \neq 0$, then the equation*

$$x^2 + bx + c = 0 \tag{10.1}$$

has solutions in $\mathbb{H}(\alpha, \beta)$.

Proof. Let $x_0 \in \mathbb{H}(\alpha, \beta)$ be a solution of the equation (10.1.). Because $x_0^2{=}t(x_0)\,x_0{-}n(x_0)$ şi $x_0^2{+}bx_0{+}c{=}0$, it results that $t(x_0)\,x_0{-}n(x_0){+}bx_0{+}c{=}0$, therefore $(t(x_0){+}b)\,x_0{=}c{+}n(x_0)$.

Because $t(x_0){+}b \neq 0,\ t(x_0),\ n(x_0) \in K, 1 \in \mathcal{A}(b, c)$, we have

$$t(x_0) + b\,c \text{ şi } c + n(x_0) \in \mathcal{A}(b, c).$$

Therefore $x_0 \in \mathcal{A}(b, c)$. Because $bc = cb$, we obtain that $\mathcal{A}(b, c)$ is commutative, therefore x_0 commutes with every element of $\mathcal{A}(b, c)$. Then the equation (10.1.) can be written:

$$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$$

and we use *Proposition 8*.$\Box$

We consider now the case $n = 2$, hence we take $A = (a_{ij})_{i,j=\overline{1,2}} \in \mathbb{H}(\alpha, \beta)$.

Case I. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{H}(\alpha, \beta)$ with $a_{21} \neq 0$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0$ be the eigenvector corresponding to the eigenvalue λ of the matrix A. We suppose that $x_2 \neq 0$. Then the vector $xx_2^{-1} = \begin{pmatrix} x_1x_2^{-1} \\ 1 \end{pmatrix}$ is the eigenvector corresponding to the eigenvalue $x_2\lambda x_2^{-1}$ for the matrix A. Therefore we have got an eigenvector of the form $x = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$. Then the relation $Ax = x\lambda$ is equivalent to the next system:

$$\begin{cases} a_{11}x_1 + a_{12} = x_1\lambda \\ a_{21}x_1 + a_{22} = \lambda \end{cases} \tag{*}$$

We replace λ from the second equation in the first one and we get: $a_{11}x_1+a_{12}=x_1(a_{21}x_1+a_{22})$, hence $x_1a_{21}x_1+x_1a_{22}-a_{11}x_1-a_{12}=0$. We multiply this last relation to the left side with a_{21}. It results $a_{21}x_1a_{21}x_1+a_{21}x_1a_{22}-a_{21}a_{11}x_1-a_{21}a_{12}=0$. We denote $a_{21}x_1=t$and we obtain

$$t^2+ta_{22}-a_{21}a_{11}a_{21}^{-1}t-a_{21}a_{12}=0. \tag{**}$$

If $a_{22}=-a_{21}a_{11}a_{21}^{-1}=b$, we denote $c=-a_{21}a_{12}$, and if, $b^2-c\notin K$ and there exist $r,s\in K$ with the properties $n\left(b^2-c\right)=r^4$ and $n\left(r^2+\overline{b^2-c}\right)=s^2$, then we may use the *Proposition 8* getting $(t+b)^2=b^2+a_{21}a_{12}$, therefore: $t=\pm\frac{r}{s}\left(r^2+b^2-c\right)-b$.

It results that $a_{21}x_1==\pm\frac{r}{s}\left(r^2+b^2-c\right)-b$ hence $a_{21}x_1=\pm\frac{r}{s}\left(r^2+a_{21}a_{11}^2a_{21}^{-1}+a_{21}a_{12}\right)+a_{21}a_{11}a_{21}^{-1}$. Therefore

$$x_1=\pm\frac{r}{s}\left(r^2a_{21}^{-1}+a_{11}^2a_{21}^{-1}+a_{12}\right)+a_{11}a_{21}^{-1},$$

and, for the eigenvalue λ, we have the expression:

$$\lambda=\pm\frac{r}{s}\left(r^2+a_{22}^2+a_{21}a_{12}\right),$$

because $a_{22}=-a_{21}a_{11}a_{21}^{-1}$ and $a_{21}a_{11}^2a_{21}^{-1}=a_{21}a_{11}a_{11}a_{21}^{-1}=-a_{22}a_{21}a_{11}a_{21}^{-1}= =a_{22}^2$.

Case II. If $a_{22}\neq-a_{21}a_{11}a_{21}^{-1}, a_{21}\neq 0$, then the equation $(**)$ is written $(t+a_{22})^2-a_{22}^2-a_{22}t-a_{21}a_{11}a_{21}^{-1}t-a_{21}a_{12}=0$. Equivalently, we get: $(t+a_{22})^2-\left(a_{22}+a_{21}a_{11}a_{21}^{-1}\right)(t+a_{22})+a_{21}a_{11}a_{21}^{-1}a_{22}-a_{21}a_{12}=0$. Denoting $-\left(a_{22}+a_{21}a_{11}a_{21}^{-1}\right)=b$, $a_{21}a_{11}a_{21}^{-1}a_{22}-a_{21}a_{12}=c$ and $t+a_{22}=v$, we obtain the equation:

$$v^2+bv+c=0. \tag{***}$$

If $b,c\in\mathbb{H}(\alpha,\beta)\backslash\{K\}$, $bc{=}c\,b$, $\frac{b^2}{4}-c\neq 0$ and there exists $r\in K$ such that $n\left(\frac{b^2}{4}-c\right){=}r^4$ and $n\left(r^2+\frac{\bar{b}^2}{4}-\bar{c}\right)=s^2$, $s\neq 0$, we may use *Proposition 10* and we obtain the solutions. If these conditions are not satisfied, we can say only that the solutions of the equation $(***)$ are in the algebra generated by b and c.

Case III. If $a_{21}=0,$ and $a_{12}\neq 0$, then the vector $\begin{pmatrix}1\\0\end{pmatrix}$ is the eigenvector for the eigenvalue $\lambda=a_{11}$. If $a_{21}=0$ and $a_{12}=0$, we have $a_{22}=\lambda$ and then the system $(*)$ is equivalent to the equation $a_{11}x_1 = a_{22}x_1$ and its nonzero solutions are given by *Proposition 6.* If we have $t(a_{11}) = t(a_{22})$ and $n(a'_{11}) = n(a'_{22})$, where $a'_{11} = a_{11} - t(a_{11})$ and $a'_{22} = a_{22} - t(a_{22})$, then the solutions have the form (6.1.) for $a_{11} \neq \bar{a}_{22}$ or have the form (6.2.) for $a_{11} = \bar{a}_{22}$.

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