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GENERALIZATION OF A THEOREM OF GAUSS-KUZMIN

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Abstract

A Gauss-Kuzmin theorem for the natural extension of the regular continued fraction expansion is given.

Let Ω denote the set of irrational numbers in $I = [0,1]$. Given $\omega \in \Omega$, let $a_1(\omega), a_2(\omega), ...$ be the sequence of partial quotients of the continued fraction expansion of ω constructed as follows.

Define $\tau : \Omega \to \Omega$ by

$$\tau(\omega) = \frac{1}{\omega} - \left[\frac{1}{\omega}\right], \ \omega \neq 0; \ \tau(0) = 0. \tag{1}$$

Then $a_{n+1}(\omega) = a_1(\tau^n(\omega)), \ n \in N^* = \{1, 2, ..., n\}$,with $a_1(\omega) =$ the integer part of $1/\omega$.

Let λ be an arbitrary non-atomic probability measure on the σ-algebra $\mathcal{B}$ of Borel subsets of I and let γ be the Gauss probability measure on $\mathcal{B}_I$ defined as

$$\gamma(A) = \frac{1}{\log 2} \int\limits_A \frac{dx}{1+x}, \ A \in \mathcal{B}_I.$$

Put $F_n(x) = \lambda\left(\tau^{-n}\left((0,x)\right)\right), \ x \in I$ for all $n \in N^* = \{0, 1, ...\}$, with $\tau^0 =$ the identity map on I. Clearly $F_0(x) = \lambda((0,x)), \ x \in I$. For any fixed $n \in N$ and $x \in I$, the set $\tau^{-n}((0,x))$ consists of all $\omega \in \Omega$ for which $\tau^n(\omega) < x$, i.e. the continued fractions

$$\cfrac{1}{a_{n+1}(\omega) + \cfrac{1}{a_{n+2}(\omega)} + \ddots} \text{ is less than } x.$$

Key Words: regular continued fraction expansion

Then, noting that we have $\tau^{n+1}(\omega) < x$ if and only if $\frac{1}{x+i} < \tau^n(\omega) < \frac{1}{i}$ for some $i \in N^*$, we obtain Gauss'equation

$$F_{n+1}(x) = \sum_{i \in N^*} \left(F_n(\frac{1}{i}) - F_n\left(\frac{1}{x+i}\right)\right),\ n \in N, x \in I.$$

Assuming that for some $m \in N$ the derivative F'_m exists everywhere in I and is bounded, it is easy to see by induction that F'_{m+n} exists and it is bounded for all $n \in N^*$, and we have

$$F'_{n+1}(x) = \sum_{i \in N^*} \frac{1}{(x+i)^2} \cdot F'_n\left(\frac{1}{x+i}\right),\ n \geq m,\ x \in I. \tag{2}$$

Now, write $f_n(x) = (x+1)F'_n(x),\ x \in I, n \geq m$ to get $f_{n+1} = Uf_n,\ n \geq m$, with U is the linear operator defined as

$$Uf(x) = \sum_{i \in N^*} \frac{x+1}{(x+i)(x+i+1)} f\left(\frac{1}{x+i}\right),\ f \in \mathcal{B}(I), x \in I \tag{3}$$

$\mathcal{B}(I)$ being the Banach space of bounded measurable complex-valued functions f on I under the supremum norm $|f| = \sup\{|f(x)|\, |x \in I\}$.

Hence

$$F_{m+n}(x) = \int_0^x \frac{U^n f_m(u)}{u+1} du,\ n \in N,\ x \in I \tag{4}$$

The asymptotic behaviour of F_n as $n \to \infty$ including the rate of convergence for $\mu = \lambda =$ the Lebesgue measure is a problem stated by Gauss in a letter to Laplace exactly 180 years ago.

On October 25, 1800, Gauss wrote in his diary that (in modern notation)

$$\pm \lim_{n \to \infty} \lambda \left(\{\omega \in [0,1) \backslash Q;\ \tau^n_\omega \leq z\} = \frac{\log(1+z)}{\log 2}\right),\ 0 \leq z \leq 1. \tag{5}$$

Later, in a letter dated January 30, 1812, Gauss asked Laplace to give an estimate of the error term $r_n(z)$, defined by $r_n(z)$, defined by

$$r_N(z) = \lambda\left(\tau^{-n}[0, Z]\right) - \frac{\log(1+z)}{\log 2},\ n \geq 1.$$

The first one who proves and in the same time answering Gauss'question was Kuzmin. In 1928 Kuzmin showed that $r_n(z) = \mathcal{O}(q^{\sqrt{n}})$ with $q \in (0,1)$, uniformly for z.

Independently, Lévy showed one year later that $r_n(z) = \mathcal{O}(q^n)$ with $q = 0,7...$, uniformly for z.

Theorem 1.1. *For every Borel set $E \subset [0,1)$, one has* $|\lambda(\tau^{-n}E) - \mu(E)| < b\lambda(E)\sigma(n)$,*where μ is the so-called Gauss measure on* $([0,1], \mathcal{B})$, *$\mathcal{B}$ being the collection of Borel sets of $[0,1)$, defined by*

$$\mu(E) = \frac{1}{\log 2} \int\limits_E \frac{dx}{1+x},\ E \in \mathcal{B} \tag{6}$$

b is a constant and $\sigma : N \to R_+$ satisfies

$$\sigma(n) < 3q^n,\ n \geq 1 \text{ where } q = \frac{3-\sqrt{5}}{2}.$$

Proof. An essential ingredient in any proof of any proof of the Gauss-Kuzmin theorem is the following observation.

Let $\omega \in [0,1)\backslash Q$ and put $\tau_k = \tau^k \omega,\ k \geq 0$, where $\tau : [0,1) \to [0,1)$ is the operator defined in (1). From (1) it follows at once that

$$0 \leq \tau_{n+1} \leq x \Leftrightarrow \tau_n \in \bigcup_{k=1}^{\infty} \left[\frac{1}{r+x}, \frac{1}{k}\right].$$

Thus if we put $m_n(x) = \lambda\left(\{\omega \in [0,1); \tau^n \omega \leq x\}\right),\ n \geq 0$, then

$$m_{n+1}(x) = \sum_{k=1}^{\infty} \left(m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+x}\right)\right),\ n \geq 0 \tag{7}$$

To be more precise, a Gauss-Kuzmin theorem is related to the natural extension

$$\left(\overline{\Omega}, \overline{\mathcal{B}}, \bar{\mu}, \mathcal{T}\right),\ \overline{\Omega} = [0,1) \times [0,1],$$

where $\bar{\mu}$ is a probability measure on $\left(\overline{\Omega}, \overline{\mathcal{B}}\right)$writh density $\frac{1}{\log 2} \cdot \frac{1}{(1+xy)^2}$, and $\mathcal{T} : \overline{\Omega} \to \overline{\Omega}$ is defined by

$$\mathcal{T}(\xi, \mu) = \left(\tau\xi, \frac{1}{\left[\frac{1}{\xi}\right] + \eta}\right), (\xi, \eta) \in \overline{\Omega}. \tag{8}$$

Let $\omega \in [0,1)\backslash Q$, the regular continued fraction expansion

$$\cfrac{1}{a_1 + \ddots \cfrac{}{+ \cfrac{1}{a_n + \ddots}}} = [0; a_1, ..., a_n, ...]. \tag{9}$$

Finite truncation (9) yields the sequence of regular convergents of ω

$$\frac{p_n(\omega)}{q_n(\omega)} = [0; a_1, ..., a_n]\,,\ n \geq 1.$$

One easily shows that

$$q_{-1}(\omega) = 0,\ q_0(\omega) = 1,\ q_n(\omega) = a_n q_{n-1}(\omega) + q_{n-2}(\omega),\ n \geq 1$$

and

$$\frac{1}{2q_n(\omega)q_{n+1}(\omega)} < \left|\omega - \frac{p_n(\omega)}{q_n(\omega)}\right| < \frac{1}{q_n(\omega)\cdot q_{n+1}(\omega)};\ n \geq 1. \tag{10}$$

Put

$$(T_m, V_m) = \mathcal{T}^m(\xi, \eta),\ \text{for } (\xi, \eta) \in \overline{\Omega},\ m \geq 1$$

and $(T_0, V_0) = (\xi, \eta)$. Then

$$T_m = [0; a_{m+1}, ..., a_{m+n}, ...]\,,\ V_m = [0; a_m, ..., a_2, a_1 + \eta]\,,\ m \geq 1.$$

Finally, we define for $m \geq 1$ the function $m_n(x, y)$ by

$$m_n(x, y) = \bar{\lambda}\left(\left\{(\xi, \eta) \in \overline{\Omega};\ (T_n, V_n) \in \mathcal{T}_{x,y}\right\}\right),$$

where $\bar{\lambda}$ is the Lebesgue measure on $\overline{\Omega}$ and

$$\mathcal{T}_{x,y} = [0, x] \times [0, y].$$

Theorem 1.2. *For all $N \geq 2$ and all $(x, y) \in \overline{\Omega}$, one has*

$$m_N(x, y) = \frac{1}{\log 2} \cdot \log(1 + xy) + \mathcal{O}(g^N)$$

and the constant of the $\mathcal{O}$ symbol is universal.

Proof. The definition of $\mathcal{T}$ yields

$$0 \leq V_{n+1} \leq y \Leftrightarrow 0 \leq \frac{1}{a_{n+1} + V_n} \leq y \Leftrightarrow \frac{1}{y} - a_{n+1} \leq V_n \leq 1.$$

Thus, putting $l_1 = \left[\frac{1}{y}\right]$, one has

$$(T_{n+1}, V_{n+1}) \in \mathcal{T}_{x,y} \Leftrightarrow (T_n, V_n) \in \left(\bigcup_{k=l_1+1}^{\infty}\left[\frac{1}{k+x}, \frac{1}{k}\right] \times [0, 1]\right) \cup$$

$$\cup \left(\left[\frac{1}{l_1 + x}, \frac{1}{l_1} \right] \times \left[\frac{1}{y} - l_1, 1 \right] \right).$$

Since $\bar{\lambda} \left(\left\{ (\xi, \eta) \in \overline{\Omega}; (T_n, V_n) \in \left[\frac{1}{l_1 + x}, \frac{1}{l_1} \right] \times \left[\frac{1}{y} - l_1, 1 \right] \right\} \right) =$

$$= m_n \left(\frac{1}{l_1}, 1 \right) - m_n \left(\frac{1}{l_1 + x}, 1 \right) + m_n \left(\frac{1}{l_1 + x}, \frac{1}{y} - l_1 \right) - m_n \left(\frac{1}{l_1}, \frac{1}{y} - l_1 \right),$$

one finds $m_{n+1}(x, y) = \sum_{k=l_1}^{\infty} \left(m_n \left(\frac{1}{k}, 1 \right) - m_n \left(\frac{1}{k+x}, 1 \right) \right) -$

$$- \left(m_n \left(\frac{1}{l_1}, \frac{1}{y}, l_1 \right) - m_n \left(\frac{1}{l_1 + x}, \frac{1}{y} - l_1 \right) \right). \qquad (*)$$

Let $f_0(x, y)$ be a continuous function on $\overline{\Omega}$, and define the sequence of functions $f_n(x, y)$ on $\overline{\Omega}$ recursively by

$$f_{n+1}(x, y) = \sum_{k=l_1}^{\infty} \left(f_n \left(\frac{1}{k}, 1 \right) - f_n \left(\frac{1}{k + x}, 1 \right) \right) -$$

$$- \left(f_n \left(\frac{1}{l_1}, \frac{1}{y} - l_1 \right) - f_n \left(\frac{1}{l_1 + x}, \frac{1}{y} - l_1 \right) \right),$$

where $l_1 = \left[\frac{1}{y} \right]$. Then one easily shows that $\bar{\mu}$ is an eingenfunction of the above equation.

Lemma 1.3. *Let $N \in \mathbb{N}$, $N \geq 2$, and let $y \in (0, 1) \cap Q$, with regular continued fraction expansion*

$$y = [0; l_1, ..., l_d], \; l_1, ..., l_d \in \mathbb{N}, 2 \leq d \leq [N/2].$$

Then one has for each x,x* $\in [0, 1]$ *with* $x^* < x$,

$$\left| (m_N(x, y) - m_N(x^*, y)) - \frac{1}{\log 2} \cdot \log \left(\frac{1 + xy}{1 + x^* y} \right) \right| <$$

$$< 4\bar{\lambda} \left(\mathcal{T}_{x,y} \backslash \mathcal{T}_{x^*,y} \right) b\sigma(N - d),$$

where $q = g^2$ and b, $\sigma(N - d)$ as given in Theorem 1.1.

Proof. (A). Put $y_i = \tau_y^i = [0; l_{i+1}, ..., l_d]$ $i = 0, ..., d$. Note that $y_0 = y$ and $y_d = 0$).

From $(*)$ we at once have that $m_N(x,y) - m_N(x^*,y) =$

$$=\sum_{k=l_1}^{\infty}\left(m_{N-1}\left(\frac{1}{k},1\right)\text{-}m_{N-1}\left(\frac{1}{k+x},1\right)\right)$$

$$\text{-}m_{N-1}\left(\frac{1}{l_1},y_1\right)+m_{N-1}\left(\frac{1}{l_1+x},y_1\right)+$$

$$+\sum_{k=l_1}^{\infty}\left(m_{N-1}\left(\frac{1}{k},1\right)\text{-}m_{N-1}\left(\frac{1}{k+x^*},1\right)\right)$$

$$\text{-}m_{N-1}\left(\frac{1}{l_1},y_1\right)+m_{N-1}\left(\frac{1}{l_1+x^*},y_1\right).$$

Now for each $D \in \mathcal{B}$ one has

$$\frac{1}{2\log 2}\bar{\lambda}(D) \leq \bar{\mu}(D) \leq \frac{1}{\log 2}\bar{\lambda}(D). \tag{11}$$

For each $n \in N$ and $\bar{a} = (a_1, ..., a_n) \in N^n$, we consider the fundamental intervals

$$\Delta_n(\bar{a}) = \{\omega \in [0,1); p_n(\omega)/q_n(\omega) = [0; a_1, ..., a_n]\}.$$

From (11) and the fact that $\mathcal{T}$ is measure-preserving with respect to $\bar{\mu}$, it follows that

$$\sum_{k=l_1}^{\infty}\left(\frac{1}{k+x^*} - \frac{1}{k+x}\right) = \sum_{k=l_1}^{\infty}\bar{\lambda}\left(([0,k+x],[0,k+x^*])\times[0,1]\right) \leq$$

$$\leq 2\log 2\sum_{k=l_1}^{\infty}\bar{\mu}\left((x^*,x)\times\Delta_1(k)\right) \leq 2(x-x^*)\lambda\left(0,\frac{1}{l_1}\right) \leq 4\,(x-x^*)\,y.$$

From this and Theorem 1.1, it follows

$$\sum_{k=l_1}^{\infty}\left(m_{N-1}\left(\frac{1}{k+x^*},1\right) - m_{N-1}\left(\frac{1}{k+x},1\right)\right) =$$

$$= \sum_{k=l_1}^{\infty}\left(\mu\left(\left[\frac{1}{k+x},\frac{1}{k+x^*}\right]\right) + \left(\frac{1}{k+x^*} - \frac{1}{k+x}\right)\mathcal{O}\left(q^{N-1}\right)\right) =$$

$$= \frac{1}{\log 2}\log\left(\frac{l_1+x}{l_1+x^*}\right) + \bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)\mathcal{O}\left(q^{N-1}\right).$$

For each $2 \leq i \leq d$,

$$\sum_{k=l_1}^{\infty} |[0;k,l_{i-1},...,l_1+x^*]-[0;k,l_{i-1},...,l_{1+x}]| \leq$$

$$\leq 2\log 2 \sum_{k=l_1}^{\infty} \bar{\mu}\left(\tau^i\left([0;k,l_{i-1},...,l_1+x^*],[0;k,l_{i-1},...,l_1+x]\right)\right) \leq$$

$$\leq \sum_{k=l_1}^{\infty} \bar{\lambda}\left((x^*,x)\times\Delta_i\left(l_1,...,l_{i-1},k\right)\right) \leq 2(x\text{-}x^*)\lambda\left(\Delta_{i-1}\left(l_1,...,l_{i-1}\right)\right) \leq 4\left(x\text{-}x^*\right)\cdot y.$$

Now applying $(*)$ to

$$m_{N-1}\left(\frac{1}{l_1+x},y_1\right)-m_{N-1}\left(\frac{1}{l_1+x^*},y_1\right)$$

yields

$$m_N(x,y)-m_N(x^*,y)=\frac{1}{\log 2}\cdot\log\left(\frac{l_1+x}{l_1+x^*}\right)+\frac{1}{\log 2}\log\left(\frac{l_2+\dfrac{1}{l_1+x}}{l_2+\dfrac{1}{l_1+x^*}}\right)+$$

$$+\bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)\mathcal{O}\left(q^{N-1}\right)+\bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)\mathcal{O}\left(q^{N-2}\right)+$$

$$+m_{N-2}\left(\frac{1}{l_2+\dfrac{1}{l_1+x}},y_2\right)-m_{N-2}\left(\frac{1}{l_2+\dfrac{1}{l_1+x^*}},y_2\right).$$

After the step d, we get

$$\begin{aligned} m_N(x,y)-m_N(x^*,y)&=\frac{1}{\log 2}\cdot\log\left(\frac{l_1+x}{l_1+x^*}...\frac{[l_d;l_{d-1},...,l_2,l_1+x]}{[l_d;l_{d-1},...,l_2,l_1+x^*]}\right)+\\ &+\bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)\mathcal{O}\left(q^{N-1}\right)+...+\bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)\mathcal{O}\left(q^{N-d}\right)+\\ &+m_{N-d}\left([0;l_d,...,l_2,l_1+x],y_d\right)-m_{N-d}\left([0;l_d,...,l_2,l_1+x^*],y_d\right). \end{aligned} \tag{12}$$

(B). Now define

$$P_{-1}=1,\ P_0=0;\ P_i=\alpha_i P_{i-1}+P_{i-2},\ \ i=1,...,d$$

$$Q_{-1}=0,\ Q_0=1;\ Q_i=\alpha_i Q_{i-1}+Q_{i-2},\ i=1,...,d$$

where $\alpha_1=l_1+x,\ \alpha_2=l_2,...,\alpha_d=l_d$.

Then one has $\cfrac{1}{l_1+\ddots+\cfrac{1}{l_1+x}} = [0; l_i, ..., l_1+x] = \dfrac{Q_{i-1}}{Q_i},\ i = 1, ..., d$ and therefore

$$(l_1+x)\left([l_2; l_1+x]\right)\left([l_3; l_2, l_1+x]\right)... = \frac{Q_1}{Q_0}\cdot\frac{Q_2}{Q_1}...\frac{Q_d}{Q_{d-1}} = Q_d.$$

Furthermore $\dfrac{P_d}{Q_d} = [0; \alpha_1, \alpha_2, ..., \alpha_d] = [0; l_1+x, l_2, ..., l_d]$.
Similarly, one has

$$\frac{P_d^*}{Q_d^*} = [0; l_1+x^*, l_2, ..., l_d].$$

Note that $P_d = P_d^*$, so that

$$\frac{(l_1+x)\left([l_2, l_1+x]\right)...\left([l_d; l_{d-1}, ..., l_2, l_1+x]\right)}{(l_1+x^*)\left([l_2, l_1+x^*]\right)...\left([l_d; l_{d-1}, ..., l_2, l_1+x^*]\right)} = \frac{Q_d}{Q_d^*} =$$

$$= \frac{P_d^*}{Q_d^*}\cdot\frac{Q_d}{P_d} = \frac{1}{x^*+[l_1; l_2, ..., l_d]}\cdot\left(x+[l_1; l_2, ..., l_d]\right) = \frac{x+\dfrac{1}{y}}{x^*+\dfrac{1}{y}} = \frac{1+xy}{1+x^*y}.$$

(C). Since $q^{N-d}+q^{N-d+1}+...+q^{N-1} = q^{N-d}\left(1+q+...+q^{d-1}\right) \leq$

$$\leq q^{N-d}\cdot\left(\sum_{i=0}^{\infty} q^i\right) = q^{N-d}\cdot\frac{1}{1-q} = g\cdot q^{N-d}$$

and $y_d = 0$.

$$\left|\left(m_N(x,y) - m_N(x^*,y)\right) - \frac{1}{\log 2}\log\left(\frac{1+xy}{1+x^*y}\right)\right| \leq 12gb\bar{\lambda}\left(\mathcal{T}_{x,y}\backslash\mathcal{T}_{x^*,y}\right)q^{N-d}.$$

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