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# Stanley depth of squarefree Veronese ideals

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**Abstract**

We compute the Stanley depth for the quotient ring of a square free Veronese ideal and we give some bounds for the Stanley depth of a square free Veronese ideal. In particular, it follows that both satisfy the Stanley's conjecture.

## Introduction

Let $K$ be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over $K$. Let $M$ be a $\mathbb{Z}^n$-graded $S$-module. A *Stanley decomposition* of $M$ is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i]$ as $K$-vector space, where $m_i \in M$, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i]$ is a free $K[Z_i]$-module. We define $\operatorname{sdepth}(\mathcal{D}) = min_{i=1}^{r}|Z_i|$ and $\operatorname{sdepth}(M) = max\{\operatorname{sdepth}(M) |\ \mathcal{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of $M$. Stanley conjecture [1] says that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$.

Herzog, Vladoiu and Zheng show in [5] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. There are two important particular cases, $I$ and $S/I$. The Stanley conjecture for $S/I$ and $I$ was proved for $n \leq 5$ and in other special cases, but it remains open in the general case. See for example, [6]. Also, the explicit computation of the Stanley depth turns out to be a difficult problem, even for simpler monomial ideals, or quotient of monomial ideals. See for instance

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[2], where the authors compute the Stanley depth for the monomial maximal ideal $(x_1,\ldots,x_n)\subset S$.

For any $d\in[n]$, we denote $I_{n,d}:=(u\in S$ square free monomial : $deg(u)=d)$. It is well known that $dim(S/I_{n,d})=\operatorname{depth}(S/I_{n,d})=d-1$. Let $\mathbf{m}=(x_1,\ldots,x_n)\subset S=K[x_1,\ldots,x_n]$ be the maximal monomial ideal of $S$. We showed in [4] that $\operatorname{sdepth}(\mathbf{m}^k)\leq\left\lceil\frac{n}{k+1}\right\rceil$, for any positive integer $k$. In this paper, we use similar techniques to give an upper bound for $\operatorname{sdepth}(I_{n,d})$. More precisely, we show that $\operatorname{sdepth}(S/I_{n,d})=d-1$ and $d\geq\operatorname{sdepth}(I_{n,d})\frac{n-d}{d+1}+d$, see Theorem 1.1. As a consequence, it follows that $I_{n,d}$ and $S/I_{n,d}$ satisfy the Stanley conjecture, see Corollary 1.2. Also, we prove that $\operatorname{sdepth}(I_{n,d})=d+1$, if $2d+1\leq n\leq 3d$, see Corollary 1.5. In order to do so, we use some combinatorics results, see Theorem 1.3 and Corollary 1.4. Finally, we conjecture that $\operatorname{sdepth}(I_{n,d})=\left\lfloor\frac{n-d}{d+1}\right\rfloor+d$.

## 1 Main results

**Theorem 1.1.** *(1)* $\operatorname{sdepth}(S/I_{n,d})=d-1$.

*(2)* $d\leq\operatorname{sdepth}(I_{n,d})\leq\frac{n-d}{d+1}+d$.

*Proof.* (1) Firstly, note that $\operatorname{sdepth}(S/I_{n,d})\leq d-1=dim(S/I_{n,d})$. We use induction on $n$ and $d$. If $n=1$, there is nothing to prove. If $d=1$, it follows that $I_{n,1}=(x_1,\ldots,x_n)$ and thus $\operatorname{sdepth}(S/I_{n,1})=0$, as required. If $d=n$, it follows that $I_{n,n}=(x_1\cdots x_n)$ and therefore $\operatorname{sdepth}(S/I_{n,n})=n-1$, as required. Now, assume $n>1$ and $1<d<n$. Note that

$$S/I_{n,d}=\bigoplus_{|supp(u)|<d}u\cdot K=\sum_{Z\subset\{x_1,\ldots,x_n\},\ |Z|=d-1}K[Z].$$

We denote $S'=K[x_1,\ldots,x_{n-1}]$. By previous equality, we get

$$S/I_{n,d}=\sum_{Z\subset\{x_1,\ldots,x_{n-1}\},\ |Z|=d-1}K[Z]\oplus x_n(\sum_{Z\subset\{x_1,\ldots,x_{n-1}\},\ |Z|=d-1}K[Z])[x_n]=$$

$$=S'/I_{n-1,d}\oplus x_n(S'/I_{n-1,d-1})[x_n].$$

By induction hypothesis, it follows that $\operatorname{sdepth}(S/I_{n,d})=d-1$.

(2) We consider the following simplicial complex, associated to $I_{n,d}$,

$$\Delta_{n,d}:=\{supp(u):\ u\in I_{n,d}\ monomial\}.$$

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Note that, by [5, Theorem 2.4], there exists a partition of $\Delta_{n,d} = \bigcup_{i=1}^{r}[F_i, G_i]$, such that $\min_{i=1}^{r} |G_i| = \operatorname{sdepth}(I_{n,d}) := s$. Note that $\Delta_{n,d} = \{F \subset [n] : |F| \geq d\}$. It follows that $\operatorname{sdepth}(I_{n,d}) \geq d$.

We consider an interval $[F_i, G_i]$ with $|F_i| = d$. Since $|G_i| \geq s$, it follows that there exists at least $(s-d)$ distinct sets in $[F_i, G_i]$ of cardinality $d+1$. Since $\Delta_{n,d} = \bigcup_{i=1}^{r}[F_i, G_i]$ is a partition, it follows that $\binom{n}{d+1} = \frac{n-d}{d+1}\binom{n}{d} \geq (s-d)\binom{n}{d}$. Thus, $s \leq d + \frac{n-d}{d+1}$. $\square$

**Corollary 1.2.** *$I_{n,d}$ and $S/I_{n,d}$ satisfy the Stanley's conjecture. Also,*

$$\operatorname{sdepth}(I_{n,d}) \geq \operatorname{sdepth}(S/I_{n,d}) + 1.$$

Let $k \leq n$ be two positive integers. We denote $A_{n,k} = \{F \subset [n] | \, |F| = k\}$. We present the following well known result from combinatorics. In order of completeness, we give also a sketch of the proof.

**Theorem 1.3.** *For any positive integers $d \leq n$ such that $d \leq n/2$, there exists a bijective map $\Phi_{n,d} : A_{n,d} \to A_{n,d}$, such that $\Phi_{n,d}(F) \cap F = \emptyset$ for any $F \in A_{n,d}$.*

*Proof.* We use induction on $n$ and $d$. If $n \leq 2$ the statement is obvious. If $d = 1$, for any $i \in [n]$, we define $\Phi_{n,1}(\{i\}) = \{j\}$, where $j = \max([n] \setminus \{\Phi_{n,1}(\{1\}), \ldots, \Phi_{n,1}(\{i-1\})\})$. $\Phi_{n,1}$ is well defined and satisfy the required conditions. Now, assume $n \geq 3$ and $d \geq 2$. If $n = 2d$ we define $\Phi_{n,d}(F) = [n] \setminus F$. Obviously, $\Phi_{n,d}$ satisfy the required conditions. Thus, we may also assume $d < n/2$.

On $A_{n,d}$, we consider the lexicographic order, recursively defined by $F < G$ if and only if $max\{F\} < max\{G\}$ or $max\{F\} = max\{G\} = k$ and $F \setminus \{k\} < G \setminus \{k\}$ on $A_{n,d-1}$. For any $F \in A_{n,d}$, we define $G := \Phi_{n,d}(F)$ to be the maximum set, with respect to "$<$", such that $G \cap F = \emptyset$ and $G \neq \Phi_{n,d}(H)$ for all $H < F$. In order to complete the proof, it is enough to show that each collection of sets

$$\mathcal{M}_F^n = \{G \subset [n] \ : \ |G| = d, \ G \cap F = \emptyset, \ G \neq \Phi_{n,d}(H) \ \ (\forall) \ H < F\}$$

is nonempty, for all $F \subset [n]$. Assume there exists some $F \subset [n-1]$ such that $\mathcal{M}_F^n = \emptyset$. It obviously follows that $M_F^{n-1} = \emptyset$ and thus $\Phi_{n-1,d}$ is not well defined, a contradiction. Also, if $M_F = \emptyset$ for some $F \subset [n]$ with $n \in F$, it follows similarly that $\Phi_{n-1,d-1}$ is not well defined, again a contradiction. Therefore, the required conclusion follows. $\square$

**Corollary 1.4.** *For any positive integers $d$ and $n$ such that $d < n/2$, there exists an injective map $\Psi_{n,d} : A_{n,d} \to A_{n,d+1}$, such that $F \subset \Psi_{n,d}(F)$ for any $F \in A_{n,d}$.*

*Proof.* We use induction on $n$. If $n \leq 2$ there is nothing to prove. If $d = 1$, we define $\Psi_{n,1} : A_{n,1} \to A_{n,2}$ by $\Psi_{n,1}(\{1\}) = \{1, 2\}, \ldots, \Psi_{n,1}(\{n-1\}) = \{n-1, n\}$ and $\Psi_{n,1}(\{n\}) = \{1, n\}$. Now, assume $n \geq 3$ and $d \geq 2$. If $n = 2d + 1$, we consider the bijective map $\Phi_{n,d} : A_{n,d} \to A_{n,d}$ such that $\phi(F) \cap F = \emptyset$ for all $F \in A_{n,d}$ and we define $\Psi_{n,d}(F) := [n] \setminus \Phi_{n,d}(F)$. The map $\Psi_{n,d}$ satisfies the required condition.

If $n < 2d+1$, we define $\Psi_{n,d}(F) := \Psi_{n-1,d}(F)$ if $F \subset [n-1]$ and $\Psi_{n,d}(F) := \Psi_{n-1,d-1}(F \setminus \{n\}) \cup \{n\}$ if $n \in F$. Note that both $\Psi_{n-1,d}$ and $\Psi_{n-1,d-1}$ are well defined and injective by induction hypothesis, since $n - 1 \leq 2d + 1$. It follows that $\Psi_{n,d}$ is well defined and injective, as required. □

**Corollary 1.5.** *Let $n, d$ be two positive integers such that $2d + 1 \leq n \leq 3d$. Then* $\operatorname{sdepth}(I_{n,d}) = d + 1$.

*Proof.* As in the proof of 1.1, we denote

$$\Delta_{n,d} := \{supp(u) : \ u \in I_{n,d} \ monomial\} = \{F \subset [n] : \ |F| \geq d\}.$$

We consider the following partition of $\Delta_{n,d}$:

$$\Delta_{n,d} = \bigcup_{|F|=d} [F, \Psi_{n,d}(F)] \cup \bigcup_{|F|>d+1} [F, F],$$

where $\Psi_{n,d}$ is given by the previous corollary. It follows that $\operatorname{sdepth}(I_{n,d}) \geq d+1$. On the other hand, by 1.1, $\operatorname{sdepth}(I_{n,d}) \leq d+1$ and thus $\operatorname{sdepth}(I_{n,d}) = d + 1$, as required. □

We conclude this paper with the following conjecture.

**Conjecture 1.6.** *For any positive integers $d \leq n$ such that $d \leq n/2$, we have*

$$\operatorname{sdepth}(I_{n,d}) = \left\lfloor \frac{n-d}{d+1} \right\rfloor + d.$$

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