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$\mathcal{F}$–multipliers and the localization of $LM_{n\times m}$–algebras

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Abstract

In this note, we introduce the notion of $n \times m$-ideal on $n \times m$–valued Łukasiewicz-Moisil algebras (or $LM_{n\times m}$–algebras) which allows us to consider a topology on them. Besides, we define the concept of $\mathcal{F}$–multiplier, where $\mathcal{F}$ is a topology on an $LM_{n\times m}$–algebra L, which is used to construct the localization $LM_{n\times m}$–algebra $L_{\mathcal{F}}$. Furthermore, we prove that the $LM_{n\times m}$–algebra of fractions L_S associated with an $\wedge$–closed subset S of L is an $LM_{n\times m}$–algebra of localization. Finally, in the finite case we prove that L_S is isomorphic to a special subalgebra of L. Since n-valued Łukasiewicz-Moisil algebras are a particular case of $LM_{n\times m}$–algebras, all these results generalize those obtained in [4] (see also [3]).

1 Introduction

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A ([10, 15]). Using the notion of $\mathcal{F}$–multiplier, G. Georgescu ([8]) associated with every topology $\mathcal{F}$ on a lattice L, a lattice $L_{\mathcal{F}}$ having the same role for lattices as the localization ring in ring theory. Furthermore, if $\mathcal{F}_S$ is the topology corresponding to an $\wedge$–closed subset S of L, $L_{\mathcal{F}_S}$ is the lattice of fractions of L_S ([2]). In 2005, D. Buşneag

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and F. Chirteş ([4, 3]) obtained, among others, similar results for n–valued Łukasiewicz-Moisil algebras.

On the other hand, in 1975 W. Suchoń ([16]) defined matrix Łukasiewicz algebras so generalizing n–valued Łukasiewicz algebras without negation ([9]). In 2000, A. V. Figallo and C. Sanza ([6]) introduced $n \times m$–valued Łukasiewicz algebras with negation which are both a particular case of matrix Łukasiewicz algebras and a generalization of n–valued Łukasiewicz–Moisil algebras ([1]). It is worth noting that unlike what happens in n–valued Łukasiewicz-Moisil algebras, generally the De Morgan reducts of $n \times m$–valued Łukasiewicz algebras with negation are not Kleene algebras. Furthermore, in [13] an important example which legitimated the study of this new class of algebras is provided. Following the terminology established in [1], these algebras were called $n \times m$–valued Łukasiewicz-Moisil algebras (or $LM_{n\times m}$–algebras for short).

The aim of this paper is to generalize some of the results established in [4], using the model of bounded distributive lattices from [8] to $LM_{n\times m}$–algebras. To this end, we introduce the notion of $n \times m$–ideal on $LM_{n\times m}$–algebras, dual to that of Stone filter (see [13]), which allows us to consider a topology on them. Besides, we define the concept of $\mathcal{F}$–multiplier, where $\mathcal{F}$ is a topology on an $LM_{n\times m}$–algebra L, which is used to construct the localization $LM_{n\times m}$–algebra $L_{\mathcal{F}}$. Furthermore, we prove that the $LM_{n\times m}$–algebra of fractions L_S associated with an $\wedge$–closed subset S of L is an $LM_{n\times m}$–algebra of localization. In the last part of this paper we give an explicit description of the $LM_{n\times m}$–algebras $L_{\mathcal{F}}$ and L_S in the finite case.

2 Preliminaries

In [13], *$n \times m$–valued Łukasiewicz-Moisil algebras* (or $LM_{n\times m}$–algebras), in which n and m are integers, $n \geq 2$, $m \geq 2$, were defined as algebras $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L verifying these conditions for all $(i, j) \in (n \times m)$ and $x, y \in L$:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,

(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,

(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,

(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i, j) \in (n \times m)$ imply $x = y$, (C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$,

(C7) $\sigma_{ij} \sim x = \sim \sigma_{(n-i)(m-j)}x$.

These algebras were extensively investigated in [12, 14, 7, 13]. Let us observe that by identifying the set $\{1, \ldots, n-1\} \times \{1\}$ with $\{1, \ldots, n-1\}$ we infer that every $LM_{n\times 2}$–algebra is isomorphic to an n–valued Łukasiewicz-Moisil algebra. In what follows we will indicate with $\mathbf{LM_{n\times m}}$ the variety of $LM_{n\times m}$–algebras ([13]) and we will denote them by L.

In Lemma 2.1 we summarize some properties of these algebras necessary in what follows. It is worth mentioning that (C16) will play an important role in the development of this paper.

Lemma 2.1. ([12]) *Let $L \in \mathbf{LM_{n\times m}}$. Then the following properties are satisfied:*

(C8) $\sigma_{ij}(x \wedge y) = \sigma_{ij}x \wedge \sigma_{ij}y$,

(C9) $\sigma_{ij}x \wedge \sim \sigma_{ij}x = 0$,

(C10) $x \leq y$ *iff* $\sigma_{ij}x \leq \sigma_{ij}y$ *for all* $(i,j) \in (n \times m)$,

(C11) $x \leq \sigma_{(n-1)(m-1)}x$,

(C12) $\sigma_{ij}0 = 0$, $\sigma_{ij}1 = 1$,

(C13) $\sigma_{11}x \leq x$,

(C14) $x \wedge \sim \sigma_{(n-1)(m-1)}x = 0$,

(C15) $x \vee \sim \sigma_{11}x = 1$,

(C16) $x \wedge [x,y] = y \wedge [y,x]$ *where* $[a,b] = \bigwedge\limits_{(i,j)\in(n\times m)} ((\sim \sigma_{ij}a \vee \sigma_{ij}b) \wedge (\sim \sigma_{ij}b \vee \sigma_{ij}a))$.

Remark 2.1. *Let $L \in \mathbf{LM_{n\times m}}$. We will denote by $B(L)$ the set of all Boolean or complemented elements of L. In [13], it was proved that $B(L) = \{x \in L : \sigma_{ij}x = x$ for each $(i,j) \in (n \times m)\}$. These elements will play an important role in what follows.*

Definition 2.1. *Let L, $L' \in \mathbf{LM_{n\times m}}$. A function $f : L \to L'$ is an $LM_{n\times m}$–homomorphism if it verifies the following conditions, for all $x, y \in L$:*

(i) $f(x \wedge y) = f(x) \wedge f(y)$,

(ii) $f(x \vee y) = f(x) \vee f(y)$,

(iii) $f(0) = 0, \ f(1) = 1$,

(iv) $f(\sigma_{ij} x) = \sigma_{ij} f(x)$, *for every* $(i,j) \in (n \times m)$,

(v) $f(\sim x) = \sim f(x)$.

Remark 2.2. *Let us observe that condition* (v) *in Definition 2.1 is a direct consequence of* (C5), (C7) *and the conditions* (i) *to* (iv).

Definition 2.2. *Let* $L \in \mathbf{LM_{n\times m}}$. *A non-empty subset* I *of* L *is an* $n \times m$*–ideal of* L*, if* I *is an ideal of the lattice* L *which verifies this condition:* $x \in I$ *implies* $\sigma_{(n-1)(m-1)} x \in I$.

It is worth noting that $\{0\}$ and L are $n \times m$–ideals of L. We will denote by $\mathcal{I}_{n\times m}(L)$ the set of all $n \times m$–ideals of L.

Remark 2.3. *If* $I \in \mathcal{I}_{n\times m}(L)$ *and* $x \in I$*, then from* (C2) *and* (C3) *we infer that* $\sigma_{ij} x \in I$*, for every* $(i,j) \in (n \times m)$.

If X is a non-empty subset of L, we will denote by $\langle X \rangle$ the $n \times m$–ideal generated by X. In particular, if $X = \{a\}$ we will write $\langle a \rangle$ instead of $\langle \{a\} \rangle$. We have that

$\langle X \rangle = \{ y \in L : \text{there are } x_1, \ldots, x_k \in X \text{ such that } y \leq \sigma_{(n-1)(m-1)}(\bigvee_{i=1}^{k} x_i) \}$.

Moreover, if $a \in L$ then $\langle a \rangle = \{ x \in L : x \leq \sigma_{(n-1)(m-1)} a \}$ and so, if $a \in B(L)$ we infer that $\langle a \rangle = \{ x \in L : x \leq a \}$.

Let $I \in \mathcal{I}_{n\times m}(L)$ and $x \in L$. We will denote by $(I : x) = \{ y \in L : x \wedge y \in I \}$.

Lemma 2.2. *Let* $I \in \mathcal{I}_{n\times m}(L)$ *and* $x \in L$*. The set* $(I : x)$ *is an* $n \times m$*–ideal of* L.

Proof. It is a direct consequence of Definition 2.2, (C8) and (C11). □

The notion of congruence in $LM_{n\times m}$–algebras is defined as usual. However, compatibility with $\sim$ follows from the other conditions as it is shown in Lemma 2.3.

Lemma 2.3. *Let* $L \in \mathbf{LM_{n\times m}}$ *and let* R *be an equivalence relation on* L*. Then the following conditions are equivalent:*

(i) R *is a congruence on* L,

(ii) R *is compatible with* $\wedge$, $\vee$ *and* σ_{ij} *for all* $(i,j) \in (n \times m)$.

Proof. We only prove (ii)$\Rightarrow$ (i). Suppose that xRy. Then $\sigma_{ij}xR\sigma_{ij}y$ for all $(i,j)\in(n\times m)$ and so, from Remark 2.1, $1R\sim\sigma_{ij}x\vee\sigma_{ij}y$ and $1R\sim\sigma_{ij}y\vee\sigma_{ij}x$. Therefore, $1R(\sim\sigma_{ij}x\vee\sigma_{ij}y)\wedge(\sim\sigma_{ij}y\vee\sigma_{ij}x)$ which allows us to infer that $\sim\sigma_{ij}xR\sim\sigma_{ij}x\wedge\sim\sigma_{ij}y$ and $\sim\sigma_{ij}yR\sim\sigma_{ij}x\wedge\sim\sigma_{ij}y$ for all $(i,j)\in(n\times m)$. From these statements we have that $\sigma_{ij}\sim xR\sigma_{ij}\sim y$ for all $(i,j)\in(n\times m)$. Hence, $1R\sim\sigma_{ij}\sim x\vee\sigma_{ij}\sim y$ and $1R\sim\sigma_{ij}\sim y\vee\sigma_{ij}\sim x$ for all $(i,j)\in(n\times m)$ and so, $1R[\sim x,\sim y]$. Then $\sim xR\sim x\wedge[\sim x,\sim y]$ and $\sim yR\sim y\wedge[\sim y,\sim x]$. By (C16) we conclude that $\sim xR\sim y$. This completes the proof. $\square$

3 $LM_{n\times m}$–algebra of fractions relative to an $\wedge$–closed subset

Definition 3.1. *A non-empty subset S of an $LM_{n\times m}$–algebra L is an $\wedge$–closed subset of L, if it satisfies the following conditions:*

(S1) $1\in S$,

(S2) $x,y\in S$ *implies* $x\wedge y\in S$.

We will denote by $S(L)$ the set of all $\wedge$–closed subsets of L.

Lemma 3.1. *Let S be an $\wedge$–closed subset of an $LM_{n\times m}$–algebra L. Then, the binary relation θ_S defined by*

$$(x,y)\in\theta_S\Leftrightarrow \text{there is } s\in S\cap B(L) \text{ such that } x\wedge s=y\wedge s$$

is a congruence on L.

Proof. We will only prove that θ_S is compatible with $\wedge$, $\vee$ and σ_{ij} for all $(i,j)\in(n\times m)$. Let $(x,y)\in\theta_S$. Then, there exists $s\in S\cap B(L)$ such that $x\wedge s=y\wedge s$. Therefore, $(x\wedge z)\wedge s=(y\wedge z)\wedge s$ and $(x\vee z)\wedge s=(y\vee z)\wedge s$ for all $z\in L$. Hence, $(x\wedge z,y\wedge z)\in\theta_S$ and $(x\vee z,y\vee z)\in\theta_S$ for all $z\in L$. Besides, from (C8), $\sigma_{ij}x\wedge\sigma_{ij}s=\sigma_{ij}y\wedge\sigma_{ij}s$ for all $(i,j)\in(n\times m)$ and so, by Remark 2.1 we infer that $(\sigma_{ij}x,\sigma_{ij}y)\in\theta_S$ for all $(i,j)\in(n\times m)$. $\square$

Remark 3.1. *Let S be an $\wedge$–closed subset of an $LM_{n\times m}$–algebra L. Then, from the definition of θ_S it is easy to see that $\theta_S=\theta_{S\cap B(L)}$.*

Let $L\in\mathbf{LM_{n\times m}}$ and $x\in L$. Then $[x]_S$ and $L[S]$ denote the congruence class of x relative to θ_S and the quotient algebra L/θ_S, respectively. Besides, $q_S: L\to L[S]$ is the canonical homomorphism.

Remark 3.2. *Since for every $s\in S\cap B(L)$, $s\wedge s=s\wedge 1$, we deduce that $[s]_S=[1]_S$, hence $q_S(S\cap B(L))=\{[1]_S\}$.*

Theorem 3.1. *If $L \in \mathbf{LM_{n\times m}}$ and $f : L \to L'$ is an $LM_{n\times m}$–homomorphism such that $f(S \cap B(L)) = \{1\}$, then there is a unique $LM_{n\times m}$–homomorphism $f' : L[S] \to L'$ such that $f' \circ q_S = f$.*

Proof. It follows from [11, Theorem 4.1] and Remark 3.1. □

Theorem 3.1 allows us to call $L[S]$ the $LM_{n\times m}$*–algebra of fractions relative to the $\wedge$–closed subset S of L.*

Remark 3.3. *From Theorem 3.1 we have that*

(i) *If $S \cap B(L) = \{1\}$, then θ_S coincides with the identity congruence on L and so, $L[S] \simeq L$.*

(ii) *If S is an $\wedge$–closed subset of L such that $0 \in S$ (for example $S = L$ or $S = B(L)$), then $\theta_S = L \times L$. Hence, $L[S] = \{[0]_S\}$.*

4 $\mathcal{F}$–multipliers and the localization of $LM_{n\times m}$–algebras

Taking into account the notion of topology for bounded distributive lattices introduced in [8], we will consider this concept in the particular case of $LM_{n\times m}$–algebras.

Definition 4.1. *Let $L \in \mathbf{LM_{n\times m}}$ and $\mathcal{F}$ a non–empty set of $n \times m$–ideals of L. $\mathcal{F}$ will be called a topology on L, if the following conditions hold:*

(T1) *If $I \in \mathcal{F}$ and $x \in L$, then $(I : x) \in \mathcal{F}$,*

(T2) *If $I_1, I_2 \in \mathcal{I}_{n\times m}(L)$, $I_2 \in \mathcal{F}$ and $(I_1 : x) \in \mathcal{F}$ for all $x \in I_2$, then $I_1 \in \mathcal{F}$.*

Lemma 4.1. ([8]) *Let $\mathcal{F}$ be a topology on an $LM_{n\times m}$–algebra L.*

(i) *If $I_1 \in \mathcal{F}$ and $I_2 \in \mathcal{I}_{n\times m}(L)$ is such that $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$,*

(ii) *If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.*

Any intersection of topologies on L is a topology. Hence, the set of the topologies on L is a complete lattice with respect to inclusion. Next, we will show that each $\wedge$–closed subset of L determines a topology on L.

Proposition 4.1. *Let $L \in \mathbf{LM_{n\times m}}$ and $S \in S(L)$. Then $\mathcal{F}_S = \{I \in \mathcal{I}_{n\times m}(L) : I \cap S \cap B(L) \neq \emptyset\}$ is a topology on L.*

Proof. If $I \in \mathcal{F}_S$ and $x \in L$, then from Lemma 2.2 and the fact that $I \subseteq (I : x)$, it follows that $(I : x) \in \mathcal{F}_S$. So, (T1) holds. In order to prove (T2), let $I_1, I_2 \in \mathfrak{I}_{n\times m}(L)$ be such that $I_2 \in \mathcal{F}_S$ and $(I_1 : x) \in \mathcal{F}_S$ for every $x \in I_2$. Let $x_0 \in I_2 \cap S \cap B(L)$. Hence, from (T1), $(I_1 : x_0) \in \mathcal{F}_S$ and so, there is $y_0 \in (I_1 : x_0) \cap S \cap B(L)$. Therefore, $x_0 \wedge y_0 \in I_1 \cap S \cap B(L)$ which allows us to conclude that $I_1 \in \mathcal{F}_S$. □

The topology $\mathcal{F}_S$ will be called *the topology associated with the $\wedge$–closed subset S of L.*

The notion of multiplier was introduced by W. Cornish in [5]. Using the concept of $\mathcal{F}$–multiplier we will associate with every topology $\mathcal{F}$ on an $LM_{n\times m}$–algebra L an algebra $L_{\mathcal{F}}$ which plays the same role for these algebras as the localization ring in ring theory.

Let $\mathcal{F}$ be a topology on an $LM_{n\times m}$–algebra L. Let us consider the binary relation $\theta_{\mathcal{F}}$ on L as follows:

$(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there is $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for all $e \in I$.

Lemma 4.2. *$\theta_{\mathcal{F}}$ is a congruence on L.*

Proof. It is simple to verify that reflexive and symmetric laws hold. The transitive law follows from (ii) in Lemma 4.1. On the other hand, let $(x, y) \in \theta_{\mathcal{F}}$. Then, there exist $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for all $e \in I$. Then, for all $z \in L$ we have that $e \wedge (x \wedge z) = e \wedge (y \wedge z)$ and $e \wedge (x \vee z) = e \wedge (y \vee z)$ for all $e \in I$. Therefore, $\theta_{\mathcal{F}}$ is compatible with $\wedge$ and $\vee$. Besides, from Remark 2.3 we infer that $\sigma_{ij}e \wedge x = \sigma_{ij}e \wedge y$, for all $(i, j) \in (n \times m)$. Hence, from (C4) and (C8) we infer that for all (i, j), $(r, s) \in (n \times m)$, $\sigma_{ij}e \wedge \sigma_{rs}x = \sigma_{ij}e \wedge \sigma_{rs}y$ and so, $\sigma_{ij}(e \wedge \sigma_{rs}x) = \sigma_{ij}(e \wedge \sigma_{rs}y)$ for all $(i, j) \in (n \times m)$. From this last assertion and (C5) we obtain $e \wedge \sigma_{rs}x = e \wedge \sigma_{rs}y$ for all $e \in I$, which allows us to conclude that $(\sigma_{rs}x, \sigma_{rs}y) \in \theta_{\mathcal{F}}$, for all $(r, s) \in (n \times m)$. □

Proposition 4.2. *Let $L \in \mathbf{LM_{n\times m}}$ and $a \in L$. Then $[a]_{\theta_{\mathcal{F}}} \in B(L/\theta_{\mathcal{F}})$ iff $\sigma_{ij}a \in [a]_{\theta_{\mathcal{F}}}$ for all $(i, j) \in (n \times m)$.*

Proof. $[a]_{\theta_{\mathcal{F}}} \in B(L/\theta_{\mathcal{F}}) \Leftrightarrow \sigma_{ij}[a]_{\theta_{\mathcal{F}}} = [a]_{\theta_{\mathcal{F}}}$, for all $(i, j) \in (n \times m) \Leftrightarrow [\sigma_{ij}a]_{\theta_{\mathcal{F}}} = [a]_{\theta_{\mathcal{F}}}$, for all $(i, j) \in (n \times m)$. □

Definition 4.2. *Let $\mathcal{F}$ be a topology on L and $I \in \mathcal{F}$. An $\mathcal{F}$–multiplier on L is a map $f : I \to L/\theta_{\mathcal{F}}$, which verifies the following condition:*

$$f(e \wedge x) = [e]_{\theta_{\mathcal{F}}} \wedge f(x), \text{ for each } e \in L \text{ and } x \in I.$$

Lemma 4.3. *For each $\mathcal{F}$–multiplier $f : I \to L/\theta_{\mathcal{F}}$ the following properties hold:*

(i) $f(x) \leq [x]_{\theta_{\mathcal{F}}}$, *for all* $x \in I$,

(ii) $f(x \wedge y) = f(x) \wedge f(y)$,

(iii) $[x]_{\theta_{\mathcal{F}}} \wedge f(y) = [y]_{\theta_{\mathcal{F}}} \wedge f(x)$.

The maps $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = [0]_{\theta_{\mathcal{F}}}$ and $\mathbf{1}(x) = [x]_{\theta_{\mathcal{F}}}$ for all $x \in L$ are $\mathcal{F}$–multipliers. Furthermore, for $a \in L$ and $I \in \mathcal{F}$, the map $f_a : I \to L/\theta_{\mathcal{F}}$ defined by $f_a(x) = [a]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}}$ for every $x \in I$, is an $\mathcal{F}$–multiplier on L called principal.

We will denote by $M(I, L/\theta_{\mathcal{F}})$ the set of the $\mathcal{F}$–multipliers having the domain $I \in \mathcal{F}$ and by

$$M(L/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}}).$$

If $I, J \in \mathcal{F}$ and $I \subseteq J$, we have a canonical map $\delta_{I,J} : M(J, L/\theta_{\mathcal{F}}) \to M(I, L/\theta_{\mathcal{F}})$ defined by $\delta_{I,J}(f) = f \mid_I$ for every $f \in M(J, L/\theta_{\mathcal{F}})$.

Let us consider the direct system of sets

$$\langle \{M(I, L/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}}, \{\delta_{J,I}\} \rangle$$

and denote by $L_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$L_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, L/\theta_{\mathcal{F}}).$$

For each $\mathcal{F}$–multiplier $f : I \to L/\theta_{\mathcal{F}}$ we will denote by $\widehat{(I, f)}$ the congruence class of f in $L_{\mathcal{F}}$.

Remark 4.1. *If $f_i : I_i \to L/\theta_{\mathcal{F}}$, $i = 1, 2$ are $\mathcal{F}$–multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ iff there exists $K \in \mathcal{F}$, $K \subseteq I_1 \cap I_2$ such that $f_1 \mid_K = f_2 \mid_K$.*

Let $f_i \in M(I_i, L/\theta_{\mathcal{F}})$, $i = 1, 2$. Let us consider the maps

$$f_1 \wedge f_2,\ f_1 \vee f_2 : I_1 \cap I_2 \to L/\theta_{\mathcal{F}}$$

defined by

$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$

$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$

for all $x \in I_1 \cap I_2$.

Lemma 4.4. *$f_1 \wedge f_2,\ f_1 \vee f_2 \in M(I_1 \cap I_2, L/\theta_{\mathcal{F}})$.*

Proof. It is straightforward. □

We define on $L_{\mathcal{F}}$ the following operations:

$\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2),$

$\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2).$

We denote $\widehat{(L, \mathbf{0})}$ and $\widehat{(L, \mathbf{1})}$ by $\widehat{\mathbf{0}}$ and $\widehat{\mathbf{1}}$, respectively.

For each $f \in M(I, L/\theta_{\mathcal{F}})$, let us consider the map

$$f^* : I \to L/\theta_{\mathcal{F}}$$

defined by

$$f^*(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\sigma_{(n-1)(m-1)}x)$$

for all $x \in I$.

Lemma 4.5. *$f^* \in M(I, L/\theta_{\mathcal{F}})$.*

Proof. It is a direct consequence of (C8) and (C14). □

We define on $L_{\mathcal{F}}$ the following operation:

$\widehat{(I, f)}^* = \widehat{(I, f^*)}.$

Remark 4.2. *For all $x \in L$, we have that $\mathbf{0}^*(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sim [0]_{\theta_{\mathcal{F}}} = [x]_{\theta_{\mathcal{F}}} \wedge [1]_{\theta_{\mathcal{F}}} = [x]_{\theta_{\mathcal{F}}}$, that is to say, $\mathbf{0}^* = \mathbf{1}$. Similarly, $\mathbf{1}^* = \mathbf{0}$.*

Let $f \in M(I, L/\theta_{\mathcal{F}})$. For each $(i, j) \in (n \times m)$ let us consider the map

$$\widetilde{\sigma}_{ij} : M(I, L/\theta_{\mathcal{F}}) \to M(I, L/\theta_{\mathcal{F}})$$

defined by

$$\widetilde{\sigma}_{ij} f(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}x))$$

for all $x \in I$.

Lemma 4.6. *$\widetilde{\sigma}_{ij} f \in M(I, L/\theta_{\mathcal{F}})$ for all $(i,j) \in (n \times m)$.*

Proof. It follows from (C8), (C4) and (C11). □

For each $(i,j) \in (n \times m)$, we define on $L_{\mathcal{F}}$ the following operation:

$$\sigma_{ij}^{\mathcal{F}} \widehat{(I,f)} = (\widehat{I, \widetilde{\sigma}_{ij} f}).$$

Lemma 4.7. *For each $I \in \mathcal{F}$, $\langle M(I, L/\theta_{\mathcal{F}}), \wedge, \vee, *, \{\sigma_{ij}^{\mathcal{F}}\}_{(i,j)\in(n\times m)}, \mathbf{0}, \mathbf{1}\rangle$ is an $LM_{n\times m}$–algebra.*

Proof. It is easy to verify that $\langle M(I, L/\theta_{\mathcal{F}}), \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is a bounded distributive lattice. To prove that it is a De Morgan algebra, we have for all $f_1, f_2, f \in M(I, L/\theta_{\mathcal{F}})$ and $x \in I$,

$$\begin{aligned}
(f_1 \vee f_2)^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sim (f_1 \vee f_2)(\sigma_{(n-1)(m-1)} x) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim (f_1(\sigma_{(n-1)(m-1)} x) \vee f_2(\sigma_{(n-1)(m-1)} x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge (\sim f_1(\sigma_{(n-1)(m-1)} x) \wedge \sim f_2(\sigma_{(n-1)(m-1)} x)) \\
&= ([x]_{\theta_{\mathcal{F}}} \wedge \sim f_1(\sigma_{(n-1)(m-1)} x)) \wedge ([x]_{\theta_{\mathcal{F}}} \wedge \sim f_2(\sigma_{(n-1)(m-1)} x)) \\
&= f_1^*(x) \wedge f_2^*(x).
\end{aligned}$$

Hence, $(f_1 \vee f_2)^* = f_1^* \wedge f_2^*$.

Furthermore, bearing in mind (C4), (C11), (C14) and Lemma 4.3 we have

$$\begin{aligned}
(f^*)^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sim f^*(\sigma_{(n-1)(m-1)} x) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim ([\sigma_{(n-1)(m-1)} x]_{\theta_{\mathcal{F}}} \wedge \sim f(\sigma_{(n-1)(m-1)}(\sigma_{(n-1)(m-1)} x))) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge (\sim [\sigma_{(n-1)(m-1)} x]_{\theta_{\mathcal{F}}} \vee f(\sigma_{(n-1)(m-1)} x)) \\
&= ([x]_{\theta_{\mathcal{F}}} \wedge \sim [\sigma_{(n-1)(m-1)} x]_{\theta_{\mathcal{F}}}) \vee ([x]_{\theta_{\mathcal{F}}} \wedge f(\sigma_{(n-1)(m-1)} x)) \\
&= [0]_{\theta_{\mathcal{F}}} \vee ([x]_{\theta_{\mathcal{F}}} \wedge f(\sigma_{(n-1)(m-1)} x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge f(\sigma_{(n-1)(m-1)} x) \\
&= [\sigma_{(n-1)(m-1)} x]_{\theta_{\mathcal{F}}} \wedge f(x) \\
&= \sigma_{(n-1)(m-1)} [x]_{\theta_{\mathcal{F}}} \wedge f(x) = f(x).
\end{aligned}$$

Therefore, $(f^*)^* = f$.

To complete the proof it remains to verify

(C1): For all $x \in I$ and $(i,j) \in (n \times m)$,

$$\begin{aligned}
\tilde{\sigma}_{ij}(f_1 \vee f_2)(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}x) \vee f_2(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge (\sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}x)) \vee \sigma_{ij}(f_2(\sigma_{(n-1)(m-1)}x))) \\
&= ([x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}x))) \vee ([x]_{\theta_{\mathcal{F}}} \wedge \\
&\quad\ \sigma_{ij}(f_2(\sigma_{(n-1)(m-1)}x))) \\
&= \tilde{\sigma}_{ij}(f_1(x)) \vee \tilde{\sigma}_{ij}(f_2(x)) \\
&= (\tilde{\sigma}_{ij}f_1 \vee \tilde{\sigma}_{ij}f_2)(x).
\end{aligned}$$

Hence, $\tilde{\sigma}_{ij}(f_1 \vee f_2) = \tilde{\sigma}_{ij}f_1 \vee \tilde{\sigma}_{ij}f_2$.

(C2): For all $x \in I$ and $(i,j) \in (n \times m)$,

$$\begin{aligned}
\tilde{\sigma}_{ij}f(x) \wedge \tilde{\sigma}_{(i+1)j}f(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}x)) \wedge \sigma_{(i+1)j} \\
&\quad\ (f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}x)) = \tilde{\sigma}_{ij}f(x).
\end{aligned}$$

Hence, $\tilde{\sigma}_{ij}f(x) \wedge \tilde{\sigma}_{(i+1)j}f(x) = \tilde{\sigma}_{ij}f(x)$ for all $(i,j) \in (n \times m)$.

(C3): It is analogous to (C2).

(C4): For all $x \in I$ and $(i,j),(r,s) \in (n \times m)$,

$$\begin{aligned}
\tilde{\sigma}_{ij}\tilde{\sigma}_{rs}(f)(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(\tilde{\sigma}_{rs}(f)(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}[\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(\sigma_{rs}(f(\sigma_{(n-1)(m-1)} \\
&\quad\ (\sigma_{(n-1)(m-1)}x)))) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge [\sigma_{ij}\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(\sigma_{rs}f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge [\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sigma_{rs}(f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{rs}(f(\sigma_{(n-1)(m-1)}x)) = \tilde{\sigma}_{rs}(f)(x).
\end{aligned}$$

Therefore, $\tilde{\sigma}_{ij}(\tilde{\sigma}_{rs}f) = \tilde{\sigma}_{rs}f$.

(C5): Let $\tilde{\sigma}_{ij}f_1 = \tilde{\sigma}_{ij}f_2$ for all $(i,j) \in (n \times m)$. Then, for all $x \in I$ we have that the following statements hold:

1. $\tilde{\sigma}_{ij}(f_1)(x) = \tilde{\sigma}_{ij}(f_2)(x)$,
2. $[x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}x) = [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f_2(\sigma_{(n-1)(m-1)}x)$,

3. $[\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}(\sigma_{(n-1)(m-1)}x))) = [\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}}$ $\wedge\, \sigma_{ij}(f_2(\sigma_{(n-1)(m-1)}(\sigma_{(n-1)(m-1)}x)))$,

4. $\sigma_{ij}([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge f_1(\sigma_{(n-1)(m-1)}x)) = \sigma_{ij}([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}}$ $\wedge\, f_2(\sigma_{(n-1)(m-1)}x)$,

5. $\sigma_{ij}(f_1(\sigma_{(n-1)(m-1)}x)) = \sigma_{ij}(f_2(\sigma_{(n-1)(m-1)}x))$ for all $(i,j) \in (n \times m)$,

6. $f_1(\sigma_{(n-1)(m-1)}x) = f_2(\sigma_{(n-1)(m-1)}x)$.

From this last equality we conclude that for all $x \in I$

$$\begin{aligned} f_1(x) &= f_1(x \wedge \sigma_{(n-1)(m-1)}x) \\ &= [x]_{\theta_{\mathcal{F}}} \wedge f_1(\sigma_{(n-1)(m-1)}x) \\ &= [x]_{\theta_{\mathcal{F}}} \wedge f_2(\sigma_{(n-1)(m-1)}x) \\ &= f_2(x \wedge \sigma_{(n-1)(m-1)}x) \\ &= f_2(x). \end{aligned}$$

Therefore, $f_1 = f_2$.

(C6): For all $x \in I$ and $(i,j) \in (n \times m)$,

$$\begin{aligned} (\tilde{\sigma}_{ij}(f) \vee (\tilde{\sigma}_{ij}(f))^*)(x) &= \tilde{\sigma}_{ij}(f)(x) \vee (\tilde{\sigma}_{ij}(f))^*(x) \\ &= ([x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}x))) \vee ([x]_{\theta_{\mathcal{F}}} \wedge \\ &\quad \sim \tilde{\sigma}_{ij}(f)(\sigma_{(n-1)(m-1)}x)) \\ &= [x]_{\theta_{\mathcal{F}}} \wedge (\sigma_{ij}(f(\sigma_{(n-1)(m-1)}x)) \vee \sim ([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \\ &\quad \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}(\sigma_{(n-1)(m-1)}x)))) \\ &= [x]_{\theta_{\mathcal{F}}} \wedge (\sigma_{ij}(f(\sigma_{(n-1)(m-1)}x)) \vee [\sim \sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \\ &\quad \vee \sim \sigma_{ij}(f\sigma_{(n-1)(m-1)}x)))) \\ &= [x]_{\theta_{\mathcal{F}}} \wedge [1]_{\theta_{\mathcal{F}}} = [x]_{\theta_{\mathcal{F}}}. \end{aligned}$$

Therefore, $\tilde{\sigma}_{ij}(f) \vee (\tilde{\sigma}_{ij}(f))^* = \mathbf{1}$ for all $(i,j) \in (n \times m)$.

(C7): For all $x \in I$ and $(i,j) \in (n\times m)$,

$$\begin{aligned}
(\tilde{\sigma}_{(n-i)(m-j)}f)^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sim \tilde{\sigma}_{(n-i)(m-j)} f(\sigma_{(n-1)(m-1)}x) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim ([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sigma_{(n-i)(m-j)} \\
&\quad (f(\sigma_{(n-1)(m-1)}\sigma_{(n-1)(m-1)}x))) \\
&= ([x]_{\theta_{\mathcal{F}}} \wedge \sim [\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}}) \vee ([x]_{\theta_{\mathcal{F}}} \wedge \sim \sigma_{(n-i)(m-j)} \\
&\quad (f(\sigma_{(n-1)(m-1)}x))) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim \sigma_{(n-i)(m-j)}(f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(\sim f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}}) \wedge \sigma_{ij}(\sim f(\sigma_{(n-1)(m-1)}x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}([\sigma_{(n-1)(m-1)}x]_{\theta_{\mathcal{F}}} \wedge \sim f(\sigma_{(n-1)(m-1)} \\
&\quad (\sigma_{(n-1)(m-1)}x))) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f^*(\sigma_{(n-1)(m-1)}x)) = \tilde{\sigma}_{ij}(f^*)(x).
\end{aligned}$$

Hence, $(\tilde{\sigma}_{(n-i)(m-j)}f)^* = \tilde{\sigma}_{ij}f^*$. □

Proposition 4.3. *$\langle L_{\mathcal{F}}, \wedge, \vee, *, \{\sigma_{ij}^{\mathcal{F}}\}_{(i,j)\in(n\times m)}, \widehat{\mathbf{0}}, \widehat{\mathbf{1}}\rangle$ is an $LM_{n\times m}$–algebra.*

Proof. It follows as a special case of Corollary 2.1 in [11]. Indeed, condition (ii) in Lemma 4.1 is stronger than the property of being down directed, the operations $\vee$, $\wedge$, $*$, σ_{ij}, $\mathbf{0}$ and $\mathbf{1}$ of $M(I, L/\theta_{\mathcal{F}})$ obviously satisfy conditions (2.1) and (2.2) in [11, Section 2.1] and $M(I, L/\theta_{\mathcal{F}})$ is an $LM_{n\times m}$–algebra by Lemma 4.7. □

The $LM_{n\times m}$–algebra $L_{\mathcal{F}}$ will be called *the localization $LM_{n\times m}$–algebra of L with respect to the topology $\mathcal{F}$.*

Lemma 4.8. *Let $\mathcal{F}_S$ be the topology associated with the $\wedge$–closed subset S. Then $\theta_{\mathcal{F}_S} = \theta_S$.*

Proof. Let $(x,y) \in \theta_{\mathcal{F}_S}$. Then there is $I \in \mathcal{F}_S$ such that $s\wedge x = s \wedge y$, for all $s \in I$. Since there exists $s_0 \in I \cap S \cap B(L)$ verifying $s_0 \wedge x = s_0 \wedge y$, we infer that $(x,y) \in \theta_S$. Conversely, let $(x,y) \in \theta_S$. Then there is $s_0 \in S \cap B(L)$ such that $x \wedge s_0 = y \wedge s_0$. By considering $I = \langle s_0\rangle$ we conclude that $(x,y) \in \theta_{\mathcal{F}_S}$. □

It is worth mentioning that Lemma 4.8 is a particular case of Lemma 4.3 in [11] considering $S \cap B(L)$ instead of S and the fact that $\mathcal{F}_S = \mathcal{F}_{S\cap B(L)}$.

Remark 4.3. *From Lemma 4.8, we have that $L/\theta_{\mathcal{F}_S} = L[S]$. Then an $\mathcal{F}_S$–multiplier can be consider as a map $f : I \to L[S]$ where $I \in \mathcal{F}_S$ and $f(e \wedge x) = [e]_S \wedge f(x)$ for all $x \in I$ and $e \in L$.*

Lemma 4.9. *Let $\widehat{(I_1, f_1)}$, $\widehat{(I_2, f_2)} \in L_{\mathcal{F}_S}$ be such that $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$. Then there exists $I \subseteq I_1 \cap I_2$ such that $f_1(s_0) = f_2(s_0)$ for all $s_0 \in I \cap S \cap B(L)$.*

Proof. From the hypothesis and Remark 4.1, we have that there exists $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ such that $f_1 \mid_I = f_2 \mid_I$ and so, $f_1(s_0) = f_2(s_0)$ for each $s_0 \in I \cap S \cap B(L)$. □

Theorem 4.1. *Let $L \in \mathbf{LM_{n\times m}}$. If $\mathcal{F}_S$ is the topology associated with the $\wedge$–closed subset S, then $L_{\mathcal{F}_S}$ is isomorphic to $L[S]$.*

Proof. Let $\alpha : L_{\mathcal{F}_S} \to L[S]$ be defined by $\alpha\widehat{(I, f)} = f(s)$ for all $s \in I \cap S \cap B(L)$. From Lemma 4.9, we have that α is well-defined. Besides, α is one-to-one. Indeed, suppose that $\alpha\widehat{(I_1, f_1)} = \alpha\widehat{(I_2, f_2)}$. Then there exist $s_1 \in I_1 \cap S \cap B(L)$ and $s_2 \in I_2 \cap S \cap B(L)$ such that $f_1(s_1) = f_2(s_2)$. Hence, by considering $f_1(s_1) = [x]_S$ and $f_2(s_2) = [y]_S$, we have that there is $s \in S \cap B(L)$ verifying $x \wedge s = y \wedge s$. If $s' = s \wedge s_1 \wedge s_2$, then we infer that $f_1(s') = f_1(s' \wedge s_1) = [s']_S \wedge f_1(s_1) = [s']_S \wedge f_2(s_2) = f_2(s')$. Let $I = \langle s' \rangle$. So, $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_1 \mid_I = f_2 \mid_I$. Remark 4.1 allows us to infer that $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$. In order to prove that α is surjective, let $[a]_S \in L[S]$ and $f_a : L \to L[S]$ defined by $f_a(x) = [a \wedge x]_S$ for all $x \in L$. It is simple to verify that f_a is an $\mathcal{F}_S$-multiplier. Moreover, from Remark 3.2, $\alpha\widehat{(L, f_a)} = f_a(s) = [a \wedge s]_S = [a]_S$, being $s \in S \cap B(L)$. It is simple to verify that this map is an homomorphism of bounded distributive lattices. Furthermore, from Lemma 2.2 it only remains to prove that $\alpha(\sigma_{ij}^{\mathcal{F}_S}\widehat{(I, f)}) = \sigma_{ij}(\alpha\widehat{(I, f)})$. Indeed, $\alpha(\sigma_{ij}^{\mathcal{F}_S}\widehat{(I, f)}) = \alpha(\widehat{(I, \widetilde{\sigma}_{ij} f)}) = \widetilde{\sigma}_{ij} f(s) = [s]_{\theta_{\mathcal{F}_S}} \wedge \sigma_{ij} f(s) = \sigma_{ij} f(s) = \sigma_{ij}(\alpha\widehat{(I, f)})$. □

Remark 4.4. *Theorem 4.1 is valid under the more general hypothesis of Theorem 4.2 in [11] namely: the algebra L has a meet–semilattice reduct, S is a subsemilattice of L, θ_S is a congruence, the multipliers form a subset of our multipliers, including the present multiplier with domain L and the isomorphism is the same in both theorems.*

Finally, in this section in order to establish a relationship between the localization of a $LM_{n\times m}$–algebra L and the Boolean elements of $L[S]$ we have to consider another theory of multipliers (meaning we add a new axiom for $\mathcal{F}$–multipliers). More precisely,

Definition 4.3. *Let $\mathcal{F}$ be a topology on L and $I \in \mathcal{F}$. An strong $\mathcal{F}$–multiplier is an $\mathcal{F}$–multiplier $f : I \to L/\theta_{\mathcal{F}}$ which verifies the following condition:*

(f) *if $e \in B(L) \cap I$, then $f(e) \in B(L/\theta_{\mathcal{F}})$.*

Remark 4.5. *If $L \in \mathbf{LM_{n\times m}}$, the $\mathcal{F}$–multipliers $\mathbf{0}, \mathbf{1} : L \to L/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = [0]_{\theta_{\mathcal{F}}}$ and $\mathbf{1}(x) = [x]_{\theta_{\mathcal{F}}}$ for all $x \in L$ are strong $\mathcal{F}$–multipliers. Furthermore, if $f_i : I_i \to L/\theta_{\mathcal{F}}$, $i = 1, 2$ are strong $\mathcal{F}$–multipliers, then $f_1 \wedge f_2$, $f_1 \vee f_2$ defined as above are strong $\mathcal{F}$–multipliers. Moreover if $f : I \to L/\theta_{\mathcal{F}}$ is a strong multiplier, then taking into account Remark 2.1 and Proposition 4.2 we have that $f^* : I \to L/\theta_{\mathcal{F}}$ and $\widetilde{\sigma}_{ij} f : I \to L/\theta_{\mathcal{F}}$ defined by $f^*(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\sigma_{(n-1)(m-1)}x)$ and $\widetilde{\sigma}_{ij} f(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sigma_{ij}(f(\sigma_{(n-1)(m-1)}x))$, respectively for all $x \in I$ are strong $\mathcal{F}$–multipliers.*

Remark 4.6. *Analogous as in the case of $\mathcal{F}$–multipliers if we work with strong $\mathcal{F}$–multipliers we obtain an $LM_{n\times m}$–subalgebra of $L_{\mathcal{F}}$ denoted by s-$L_{\mathcal{F}}$ which will be called the strong–localization $LM_{n\times m}$–algebra of L with respect to the topology $\mathcal{F}$.*

Theorem 4.2. *Let $L \in \mathbf{LM_{n\times m}}$. If $\mathcal{F}_S$ is the topology associated with a $\wedge$–closed subset S of L, then the $LM_{n\times m}$–algebra s-$L_{\mathcal{F}_S}$ is isomorphic to $B(L[S])$ (see Example 5.1).*

Proof. From Theorem 4.1 there is an isomorphism $\alpha : L_{\mathcal{F}_S} \to L[S]$ defined by $\alpha\widehat{(I, f)} = f(s)$ for all $s \in I \cap S \cap B(L)$. Let us consider the restriction of α to s-$L_{\mathcal{F}_S}$ which we will denote by α_s. Since f is a strong $\mathcal{F}$–multiplier it follows immediately that $\alpha_s\widehat{(I, f)} \in B(L[S])$ for all $\widehat{(I, f)} \in s - L_{\mathcal{F}_S}$. Furthermore, since s-$L_{\mathcal{F}_S}$ is an $LM_{n\times m}$–subalgebra of $L_{\mathcal{F}_S}$ we have that α_s is an injective homomorphism. To prove the surjectivity of α_s, let $[a]_S \in B(L[S])$. Hence, there is $e_0 \in S \cap B(L)$ such that $a \wedge e_0 \in B(L)$. We consider $I_0 = \langle e_0 \rangle$ and since $e_0 \in I_0 \cap S \cap B(L)$ we infer that $I_0 \in \mathcal{F}_S$. Let $f_a : I_0 \to L[S]$ be the function defined by $f_a = [a \wedge e_0]_S$ for all $x \in I_0$. It is simple to verify that f_a is an $\mathcal{F}$–multiplier. Furthermore, f_a is strong. Indeed, $f_a(e) = [a \wedge e]_S = [a]_S \wedge [e]_S \in B(L[S])$ for all $e \in B(L) \cap I$. Moreover, from Remark 3.2 and the fact that $e \in S$ we have that $\alpha_s\widehat{(I_0, f_a)} = f_a(e_0) = [e_0]_S \wedge [a]_S = 1 \wedge [a]_S = [a]_S$. □

5 Localization and fractions in finite $LM_{n\times m}$–algebras

In this section, our attention is focus on considering the above results in the particular case of finite $LM_{n\times m}$–algebras. More precisely, we will prove that for each finite $LM_{n\times m}$–algebra L and $S \in S(L)$ the algebra $L[S]$ is isomorphic to a special subalgebra of L. In order to do this, the following propositions will be fundamental.

Proposition 5.1. *Let L be a finite $LM_{n\times m}$–algebra and $I \subseteq L$. Then, the following conditions are equivalent:*

(i) $I \in I_{n\times m}(L)$,

(ii) $I = \langle a\rangle$ *for some* $a \in B(L)$.

Proof. (i) $\Rightarrow$ (ii): Since L is finite, from a well-known result of finite lattices we have that $I = \langle a\rangle$ for some $a \in L$. Furthermore, from the hypothesis we have that $\sigma_{(n-1)(m-1)}a \in \langle a\rangle$ and so, $\sigma_{(n-1)(m-1)}a \leq a$. Hence, by (C11) we infer that $a = \sigma_{(n-1)(m-1)}a$ which implies that $a \in B(L)$.

(ii) $\Rightarrow$ (i): Let $x \in I$. Then $x \leq a$ and so, $\sigma_{(n-1)(m-1)}x \leq \sigma_{(n-1)(m-1)}a = a$. Therefore, $\sigma_{(n-1)(m-1)}x \in I$. $\square$

Proposition 5.2. *Let L be a finite $LM_{n\times m}$–algebra and $S \in S(L)$. Then* $\mathcal{F}_S = \{\langle a\rangle : a \in B(L), \bigwedge\limits_{x\in S\cap B(L)} x \leq a\}$.

Proof. Let us consider $\mathcal{T} = \{\langle a\rangle : a \in B(L), \bigwedge\limits_{x\in S\cap B(L)} x \leq a\}$. Assume that $I \in \mathcal{F}_S$. Then, by Proposition 5.1 we have that $I = \langle a\rangle$ for some $a \in B(L)$. On the other hand, from Proposition 4.1 there is $c \in S \cap \langle a\rangle \cap B(L)$ which implies that $\bigwedge\limits_{x\in S\cap B(L)} x \leq c \leq a$. Therefore, $I \in \mathcal{T}$. Conversely, suppose that $I \in \mathcal{T}$. Hence, $\bigwedge\limits_{x\in S\cap B(L)} x \in I \cap S \cap B(L)$. Furthermore, by Proposition 5.1 we have that $I \in \mathcal{I}_{n\times m}(L)$. From these last assertions and Proposition 4.1 we conclude that $I \in \mathcal{F}_S$. $\square$

Proposition 5.3. *Let L be a finite $LM_{n\times m}$–algebra and $S \in S(L)$. Then, the following conditions are equivalent:*

(i) $(x, y) \in \theta_{\mathcal{F}_S}$,

(ii) $x \wedge b = y \wedge b$ *where* $b = \bigwedge\limits_{x\in S\cap B(L)} x$.

Proof. It is routine. $\square$

Proposition 5.4. *Let L be a finite $LM_{n\times m}$–algebra and $\langle a\rangle \in \mathcal{I}_{n\times m}(L)$. Then, $L_a = \langle\langle a\rangle, \wedge, \vee, \sim_a, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, a\rangle$ is an $LM_{n\times m}$–algebra, where $\sim_a x = \sim x \wedge a$.*

Proof. It is easy to cheek that $\langle\langle a\rangle, \wedge, \vee, 0, a\rangle$ is a bounded distributive lattice. Furthermore, if $x, y \in \langle a\rangle$, then we have that $\sim_a\sim_a x = \sim_a (a\wedge \sim x) = \sim (a\wedge \sim x) \wedge a = (\sim a \vee x) \wedge a = 0 \vee (x \wedge a) = x$ and $\sim_a (x\wedge y) = \sim (x\wedge y) \wedge a = (\sim x \vee \sim y) \wedge a = \sim_a x \vee \sim_a y$. Moreover, $\sigma_{ij}x \in \langle a\rangle$ for all $x \in \langle a\rangle$ and $(i,j) \in (n\times m)$. Indeed, since $x \leq a$ we have by (C10) that $\sigma_{ij}x \leq \sigma_{ij}a = a$ for all $(i,j) \in (n\times m)$. $\square$

Finally, we obtain our desired goal.

Proposition 5.5. *Let L be a finite $LM_{n\times m}$–algebra and $S \in S(L)$. Then $L[S]$ is isomorphic to L_b where $b = \bigwedge_{x\in S\cap B(L)} x$.*

Proof. Let $\beta : L \to L_b$ be the function defined by the prescription $\beta(x) = x\wedge b$. It is easy to check that β is a $0,1$–lattice epimorphism. Furthermore, for all $x \in L$, $\beta(\sim x) = \sim x \wedge b = (\sim x \wedge b) \vee (\sim b \wedge b) = \sim (x \wedge b) \wedge b = \sim \beta(x) \wedge b = \sim_a \beta(x)$ and $\beta(\sigma_{ij}x) = \sigma_{ij}x \wedge b = \sigma_{ij}x \wedge \sigma_{ij}b = \sigma_{ij}(x\wedge b) = \sigma_{ij}\beta(x)$ for all $(i,j) \in (n\times m)$. Therefore, β is an $LM_{n\times m}$–epimorphism. Moreover, $x \in [1]_{\theta_S} \Leftrightarrow (x,1) \in \theta_S \Leftrightarrow$ there is $s \in S\cap B(L)$ such that $x \wedge s = s \Leftrightarrow x\wedge b = b \Leftrightarrow \beta(x) = b \Leftrightarrow x \in Ker(\beta)$. Therefore, taking into account a well-known result of universal algebra ([1, p. 59]) we conclude that $L[S]$ is isomorphic to L_b $\square$

Corollary 5.1. *Let L be a finite $LM_{n\times m}$–algebra and $S \in S(L)$. Then, $L_{\mathcal{F}_S}$ is isomorphic to L_b where $b = \bigwedge_{x\in S\cap B(L)} x$. More precisely, $L_{\mathcal{F}_S} = \{\widehat{(\langle b\rangle, f_x)} : x \in \langle b\rangle\}$.*

Proof. It follows as a consequence of Theorem 4.1 and Proposition 5.5 $\square$

Example 5.1. *Let us consider the $LM_{3\times 3}$–algebra L shows in Figure 1 where the operations $\sim$ and σ_{ij} for all $(i,j) \in (3\times 3)$ are defined as follows:*

x	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
$\sim x$	1	a_{25}	a_{24}	a_{21}	a_{23}	a_{22}	a_{20}	a_{19}	a_{18}	a_{17}	a_{16}	a_{15}
$\sigma_{11}x$	0	0	0	0	a_4	a_5	a_4	a_5	a_8	0	0	0
$\sigma_{12}x$	0	a_4	a_5	a_8	a_4	a_5	a_8	a_8	a_8	0	a_4	a_5
$\sigma_{21}x$	0	0	0	0	a_4	a_5	a_4	a_5	a_8	a_{18}	a_{18}	a_{18}
$\sigma_{22}x$	0	a_4	a_5	a_8	a_4	a_5	a_8	a_8	a_8	a_{18}	a_{22}	a_{23}

x	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}
$\sim x$	a_{12}	a_{14}	a_{13}	a_{11}	a_{10}	a_9	a_8	a_7	a_6	a_3	a_5
$\sigma_{11}x$	0	a_4	a_5	a_4	a_5	a_8	a_{18}	a_{18}	a_{18}	a_{18}	a_{22}
$\sigma_{12}x$	a_8	a_4	a_5	a_8	a_8	a_8	a_{18}	a_{22}	a_{23}	1	a_{22}
$\sigma_{21}x$	a_{18}	a_{22}	a_{23}	a_{22}	a_{23}	1	a_{18}	a_{18}	a_{18}	a_{18}	a_{22}
$\sigma_{22}x$	1	a_{22}	a_{23}	1	1	1	a_{18}	a_{22}	a_{23}	1	a_{22}

x	a_{23}	a_{24}	a_{25}	1
$\sim x$	a_4	a_2	a_1	0
$\sigma_{11}x$	a_{23}	a_{22}	a_{23}	1
$\sigma_{12}x$	a_{23}	1	1	1
$\sigma_{21}x$	a_{23}	a_{22}	a_{23}	1
$\sigma_{22}x$	a_{23}	1	1	1

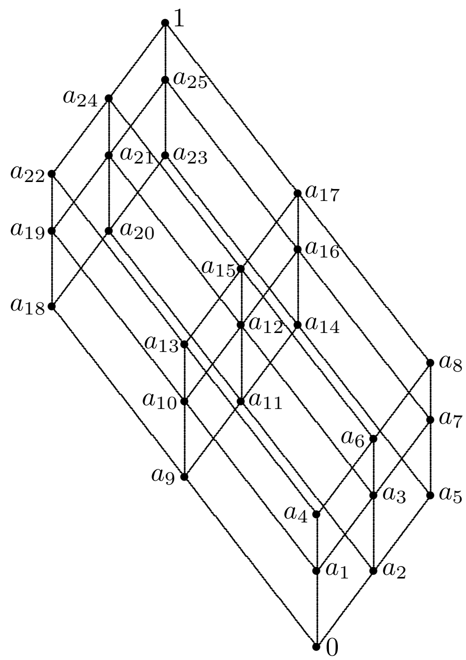

Figure 1

Then, $B(L) = \{0, a_4, a_5, a_8, a_{18}, a_{22}, a_{23}, 1\}$*. If we consider* $S = \{a_{18}, a_{20}, a_{21}, a_{23}, a_{24}, a_{25}, 1\}$*, then according to Proposition 5.5 we have that* $L[S]$ *is iso-*

morphic to $L_{a_{18}} = \{0, a_9, a_{18}\}$. *Furthermore,* $\mathcal{F}_S = \{\langle a_{18}\rangle, \langle a_{22}\rangle, \langle a_{23}\rangle, \langle 1\rangle = L\}$ *and taking into account Corollary 5.1 we have that* $L_{\mathcal{F}_S} = \{(\widehat{\langle a_{18}\rangle, f_0}), (\widehat{\langle a_{18}\rangle, f_{a_9}}), (\widehat{\langle a_{18}\rangle, f_{a_{18}}})\}$.

On the other hand, since $B(L[S]) = \{[0]_S, [a_{18}]_S\}$ *from Theorem 4.2 we conclude that* s-$L_{\mathcal{F}_S} = \{(\widehat{\langle a_{18}\rangle, f_0}), (\widehat{\langle a_{18}\rangle, f_{a_{18}}})\}$. *This example also shows that there are* $\mathcal{F}$*-multipliers which are not strong* $\mathcal{F}$*-multipliers. Indeed, that is the case of* $f_{a_9} : \langle a_{18}\rangle \to L[S]$ *because* $a_{23} \in B(L) \cap \langle a_{18}\rangle$ *and* $f_{a_9}(a_{23}) = [a_9]_S \notin B(L[S])$.

Concluding remark. Since $n-$valued Łukasiewicz-Moisil algebras are particular case of $LM_{n\times m}$–algebras, all the results obtained generalize those established in [4] and in [3].

Bearing in mind that $LM_{n\times m}$–algebras are a particular case of matrix Łukasiewicz algebras introduced by W. Suchoń, a subject for future research would be to develop the theory of localization for these algebras.

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