An. Şt. Univ. Ovidius Constanţa **Vol. 20(2), 2012, 115–130**

Reidemeister type moves for knots and links in lens spaces

Enrico Manfredi and Michele Mulazzani

Abstract

We introduce the concept of regular diagrams for knots and links in lens spaces, proving that two diagrams represent equivalent links if and only if they are related by a finite sequence of seven Reidemester type moves. As a particular case, we obtain diagrams and moves for links in $\mathbb{RP}^3$ previously introduced by Y.V. Drobothukina.

1 Preliminaries

In this paper we work in the *Diff* category (of smooth manifolds and maps). Every result also holds in the *PL* category.

Let X and Y be two smooth manifolds. A smooth map $f : X \to Y$ is an *embedding* if the differential $d_x f$ is injective for all $x \in X$ and if X and $f(X)$ are homeomorphic. As a consequence, X and $f(X)$ are diffeomorphic and $f(X)$ is a submanifold of Y. An *ambient isotopy* between two embeddings l_0 and l_1 from X to Y is a smooth map $H : Y \times [0,1] \to Y$ such that, if we define $h_t(y) = H(y,t)$ for each $t \in [0,1]$, then $h_t : Y \to Y$ is a diffeomorphism, $h_0 = \mathrm{Id}_Y$ and $l_1 = h_1 \circ l_0$.

A *link* in a closed 3-manifold M^3 is an embedding of ν copies of $\mathbf{S}^1$ into M^3, namely it is $l : \mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1 \to M^3$. A link can also be denoted by L, where $L = l(\mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1) \subset M^3$. A *knot* is a link with $\nu = 1$. Two links L_0 and L_1 are *equivalent* if there exists an ambient isotopy between the two related embeddings l_0 and l_1 (i.e., $h_1(L_0) = L_1$).

Key Words: knots and links, Reidemeister moves, lens spaces, three manifolds.
2010 Mathematics Subject Classification: Primary 57M25, 57M27; Secondary 57N10.
Received: August, 2011.
Accepted: February, 2012.

Let p and q be two integer numbers such that $\gcd(p,q)=1$ and $0 \leqslant q < p$. Consider $B^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2+x_2^2+x_3^2 \leqslant 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3. Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3=0$ with B^3. Label with N and S the "north pole" $(0,0,1)$ and the "south pole" $(0,0,-1)$ of B^3, respectively. Let $g_{p,q}: E_+ \to E_+$ be the rotation of $2\pi q/p$ around the x_3 axis as in Figure 1, and let $f_3: E_+ \to E_-$ be the reflection with respect to the plane $x_3=0$. The *lens space* $L(p,q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identify $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by $F: B^3 \to B^3/\sim$ the quotient map. Notice that on the equator $\partial B_0^2 = E_+ \cap E_-$ there are p points in each class of equivalence.

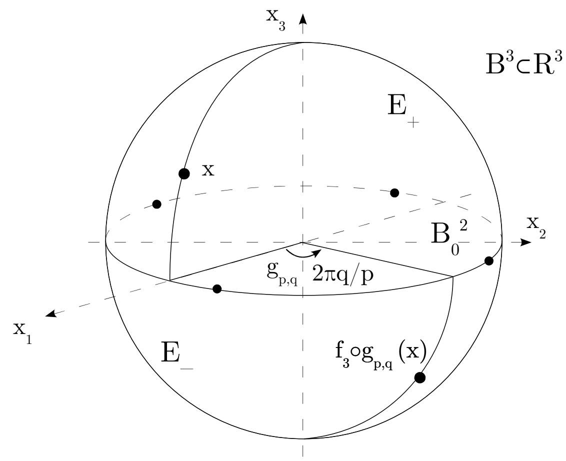

Figure 1: Model for $L(p,q)$.

It is easy to see that $L(1,0) \cong \mathbf{S}^3$ since $g_{1,0} = \mathrm{Id}_{E_+}$. Furthermore, $L(2,1)$ is $\mathbb{RP}^3$, since we obtain the usual model of the projective space where opposite points of the boundary of the ball are identified.

2 Links in $\mathbf{S}^3$

2.1 Diagrams

One of the first tools used to study links in $\mathbf{S}^3$ are diagrams, which are suitable projections of the links on a plane.

If L is a link in $\mathbf{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, we can always suppose, up to equivalence, that L belongs to $\mathrm{int}B^3$. Let $\mathbf{p}: B^3 \smallsetminus \{N,S\} \to B_0^2$ be the projection defined by $\mathbf{p}(x) = c(x) \cap B_0^2$, where $c(x)$ is the circle (possibly a line) through N, x and S.

Now project $L \subset \mathrm{int}B^3$ using $\mathbf{p}_{|L} : L \to B_0^2$. For any $P \in \mathbf{p}(L)$, $\mathbf{p}_{|L}^{-1}(P)$ may contains more than one point; in this case, we say that P is a multiple point. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, up to a "small" isotopy, that the projection $\mathbf{p}_{|L} : L \to B_0^2$ of L is *regular*, namely:

1. the arcs of the projection contain no cusps;

2. the arcs of the projection are not tangent to each other;

3. the set of multiple points is finite, and all of them are actually double points.

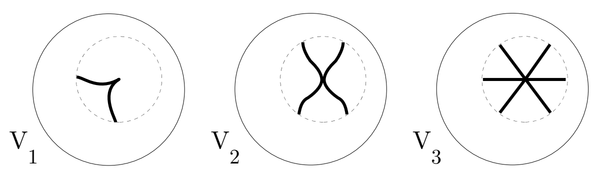

Figure 2:

These conditions correspond to forbidden configurations depicted in Figure 2.

Now let Q be a double point and consider $\mathbf{p}_{|L}^{-1}(Q) = \{P_1, P_2\}$.

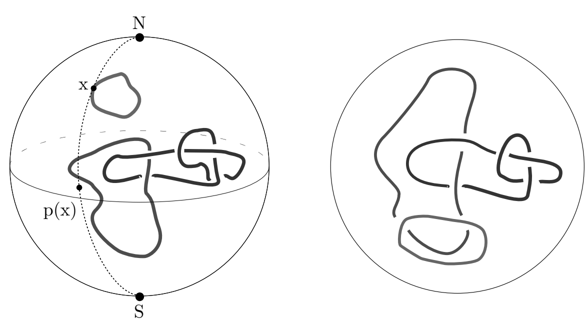

Figure 3: A link in S^3 and the corresponding diagram.

We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L such that $\mathbf{p}(\overline{U})$ does not contain other double points. We call U *underpass*. Take the complementary set in L of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called *overpass*. The overpass/underpass rule in the double points is visualized by removing U from L' before projecting the link (see Figure 3). Observe that some components of the link might have no underpasses.

A *diagram* of a link L in $\mathbf{S}^3$ is a regular projection of L on the equatorial disk B_0^2, with specified overpasses and underpasses.

2.2 Reidemeister moves

There are three (local) moves that allow us to determine when two links in $\mathbf{S}^3$ are equivalent, directly from their diagrams. Reidemeister proved this theorem for *PL* links. For the *Diff* case a possible reference is [3], where the result concerns links in arbitrary dimensions, so the proof is rather complicated.

The *Reidemeister moves* for a diagram of a link $L \subset \mathbf{S}^3$ are the moves R_1, R_2, R_3 of Figure 4.

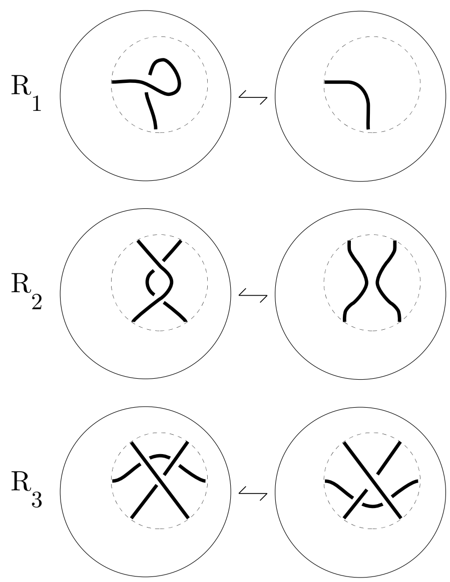

Figure 4: Reidemeister moves.

Theorem 2.1. [3] *Two links L_0 and L_1 in $\mathbf{S}^3$ are equivalent if and only if their diagrams can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : \mathbf{S}^3 \times [0,1] \to \mathbf{S}^3$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. From general position theory (see [3] for details), we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times only one condition is violated. From each type of violation a transformation of the diagram appears, that is to say, a Reidemeister move, as it follows (see Figure 5):

– from V_1 we obtain the move R_1;
– from V_2 we obtain the move R_2;
– from V_3 we obtain the move R_3.

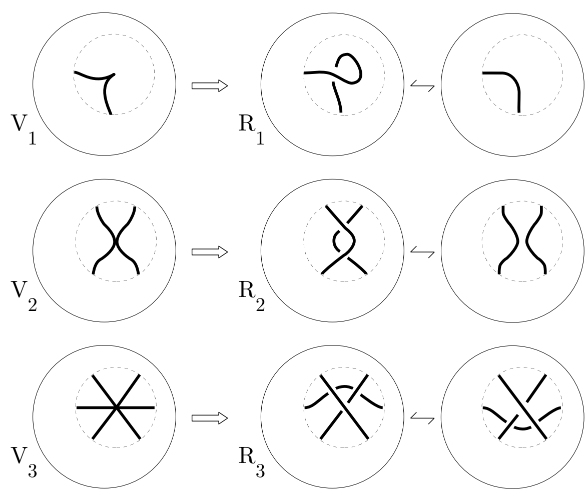

Figure 5:

So diagrams of two equivalent links can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies. $\square$

3 Links in $\mathbb{RP}^3$

3.1 Diagrams

The definition of diagrams for links in the projective space, given by Drobothukina in [1], makes use of the model of the projective space $\mathbb{RP}^3$ explained in Section 1, as a particular case of $L(p,q)$ with $p = 2$ and $q = 1$. Namely, consider B^3 and let $\sim$ the equivalence relation which identify diametrically opposite points on ∂B^3, then $\mathbb{RP}^3 \cong B^3/\sim$.

Let L be a link in $\mathbb{RP}^3$ and consider $L' = F^{-1}(L)$, where F is the quotient map. By moving L with a small isotopy in $\mathbb{RP}^3$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\mathrm{int} B^3$ and arcs properly embedded* in B^3.

Let $\mathbf{p} : B^3 \setminus \{N, S\} \to B_0^2$ be the projection defined in the previous section. Take L' and project it using $\mathbf{p}_{|L'} : L' \to B_0^2$. A point $P \in \mathbf{p}(L')$ such that $\mathbf{p}_{|L'}^{-1}(P)$ contains more than one point is called a multiple point. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, by moving L via small isotopies, that the projection $\mathbf{p}(L')$ is *regular*,

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B_0^2;

5) no double point is on ∂B_0^2.

namely:

These conditions correspond to forbidden configurations $V_1, \ldots, V_5$ depicted in Figures 2 and 6.

Now let Q be a double point and consider $\mathbf{p}_{|L'}^{-1}(Q) = \{P_1, P_2\}$. We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L' such that $\mathbf{p}(\overline{U})$ does not contain other double points and does not meet ∂B_0^2. We call U *underpass*. Take the complementary set in L' of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called

*An arc is properly embedded in a compact 3-manifold M^3 if only the endpoints belong to ∂M^3.

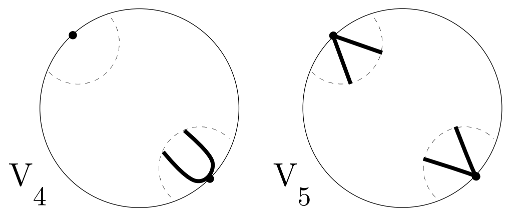

Figure 6:

overpass. Again the overpass/underpass rule in the double points is visualized by removing U from L' before projecting the link (see Figure 7).

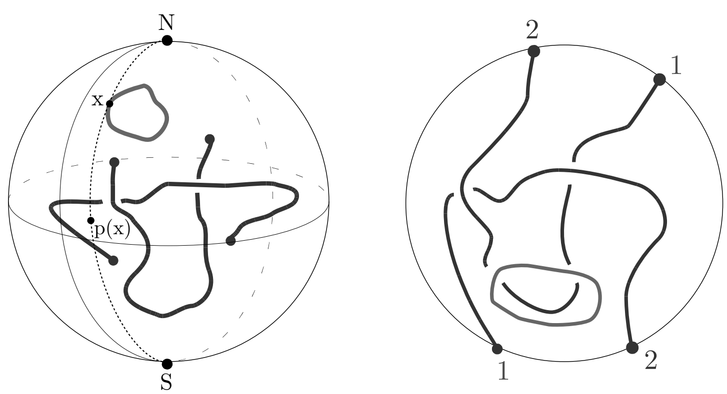

Figure 7: A link in $\mathbb{RP}^3$ and the corresponding diagram.

A *diagram* of a link L in $\mathbb{RP}^3$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B^2_0, with specified overpasses and underpasses.

The boundary points of the link projection are labelled in order to make clear the identifications. Assume that the number of boundary points is $2t$ and orient the equator counterclockwise (looking at it from N). Choose a point of $\mathbf{p}(L')$ on the equator and label it by 1, as well as its antipodal point; then following the orientation of ∂B^2_0, label the points of $\mathbf{p}(L')$ on the equatorial circle, as well as the antipodal ones, by $2, \ldots, t$ respectively (see Figure 7).

3.2 Generalized Reidemeister moves

The *generalized Reidemeister moves* on a diagram of a link $L \subset \mathbb{RP}^3$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5 of Figure 8.

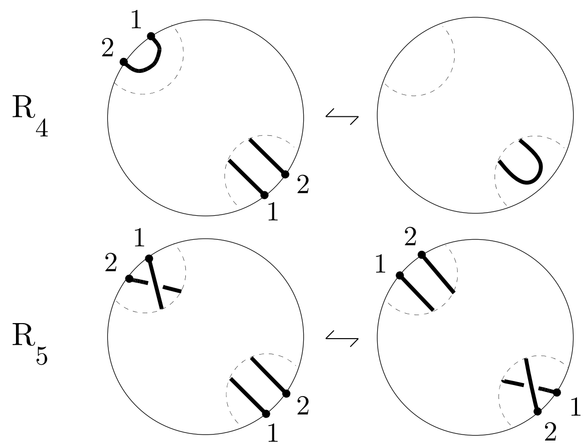

Figure 8: Generalized Reidemeister moves for links in $\mathbb{RP}^3$.

An analogue of the Reidemeister theorem can be proved in this contest:

Theorem 3.1. [1] *Two links L_0 and L_1 in $\mathbb{RP}^3$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_5$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : \mathbb{RP}^3 \times [0,1] \to \mathbb{RP}^3$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. As for links in $\mathbf{S}^3$, using general position theory we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times only one condition is violated. From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as it follows (see Figures 5 and 9):

- from V_1, V_2 and V_3 we obtain the classic Reidemeister moves R_1, R_2 and R_3;

– from V_4 we obtain the move R_4;
– from V_5 we obtain the move R_5.

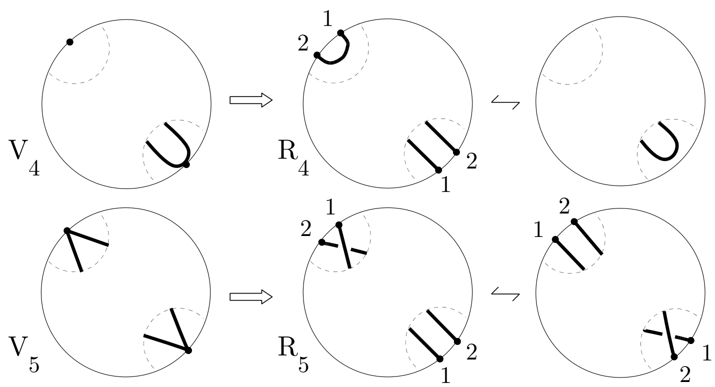

Figure 9:

So diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_5$ and diagram isotopies. $\square$

4 Links in $L(p, q)$

4.1 Diagrams

In this section we improve the definition of diagram for links in lens spaces given in [2]. We can assume $p > 2$, since the particular cases $L(1, 0) \cong \mathbf{S}^3$ and $L(2, 1) \cong \mathbb{RP}^3$ have been already considered in previous sections.

The model for the lens space $L(p, q) = B^3/\sim$ is the one given in Section 1. Let L be a link in $L(p, q)$ and let $L' = F^{-1}(L)$, where F is the quotient map. By moving L via a small isotopy in $L(p, q)$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\text{int} B^3$ and arcs properly embedded in B^3.

Let $\mathbf{p} : B^3 \smallsetminus \{N, S\} \to B^2_0$ be the projection defined in Section 2. Take L' and project it using $\mathbf{p}_{|L'} : L' \to B^2_0$. As before, a point $P \in \mathbf{p}(L')$ such that $\mathbf{p}^{-1}_{|L'}(P)$ contains more than one point is called a multiple point. In particular, if it contains exactly two points, P is called a *double point*. We can assume, by moving L via a small isotopy, that the projection $\mathbf{p}_{|L'} : L' \to B^2_0$ of L is *regular*, namely:

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B^2_0;

5) no double point is on ∂B^2_0;

6) $L' \cap \partial B^2_0 = \emptyset$.

Of course, overpasses and underpasses are defined as in the previous section. A *diagram* of a link L in $L(p,q)$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B^2_0, with specified overpasses and underpasses.

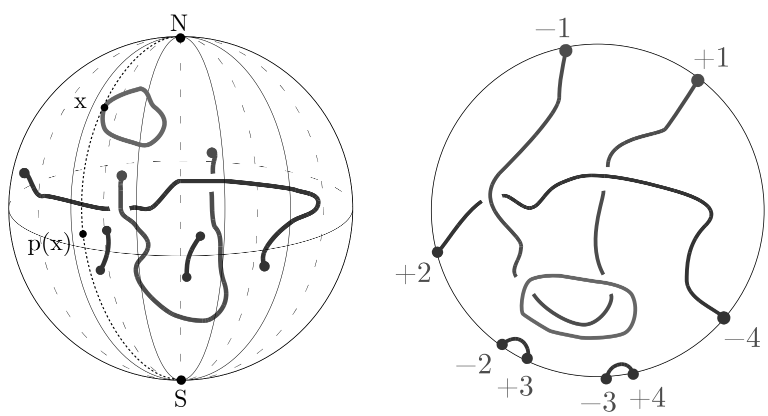

Figure 10: A link in $L(9,1)$ and the corresponding diagram.

We label the boundary points of the link projection, in order to make clear the identification. Assume the equator is oriented counterclockwise, and consider the t endpoints of the overpasses belonging to the upper hemisphere. Label their projecton by $+1, \dots, +t$, according to the orientation of ∂B^2_0. Then label the other t boundary points by $-1, \dots, -t$, where for each $i = 1, \dots, t$, according to the identifications. An example is shown in Figure 10.

Now we explain which diagram violations arise from condition 1)-6). As usual, conditions 1), 2) and 3) correspond to forbidden configurations V_1, V_2 and V_3 of Figure 2.

Condition 4), as in the projective case, corresponds to V_4. On the contrary, condition 5) does not behave as in the projective case. Indeed two forbidden configurations arise from it (V_5 and V_6), as Figure 11 shows. The difference between them is that V_5 involves two arcs of L' that end in the same hemisphere of ∂B^3, on the contrary V_6 involves arcs that end in different hemispheres.

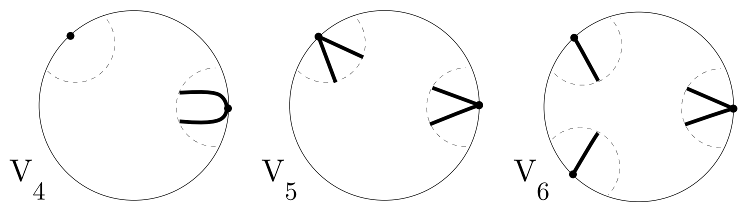

Figure 11:

Finally, condition 6) produces a family of forbidden configurations, called $V_{7,1}, \dots, V_{7,p-1}$ (see Figure 12). The difference between them is that $V_{7,1}$ has the arcs of the projection identified directly by $g_{p,q}$, while $V_{7,k}$ has the arcs identified by $g_{p,q}^k$, for $k = 2, \dots, p-1$.

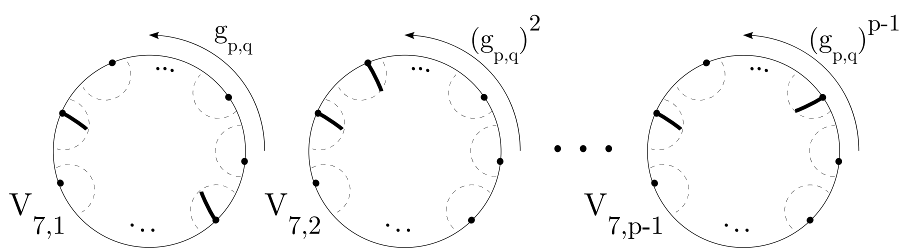

Figure 12:

It is easy to see what kind of small isotopy on L is necessary, in order to make the projection of the link regular when we deal with configurations $V_1, \dots, V_6$. Now we explain how the link can avoid to meet ∂B_0^2 up to isotopy, that is to say, avoid $V_{7,1}, \dots, V_{7,p-1}$.

We start with a link with two arcs ending on ∂B_0^2. Suppose that the endpoints B and C of the arcs are connected by $g_{p,q}$, (a $V_{7,1}$ violation), namely $C = g_{p,q}(B)$. In this case the required isotopy is the one that lift up a bit the arc ending in B and lower down the other one.

Now if we suppose that the endpoints B and C of the arcs are connected by a power of $g_{p,q}$, (a $V_{7,k}$ violation with $k > 1$), namely $C = g_{p,q}^k(B)$. In this case the required isotopy is similar to the one of the example in $L(9,1)$ of Figure 13. In lens spaces with $q \neq 1$, the new arcs end in the faces specified by the map $f_3 \circ g_{p,q}$.

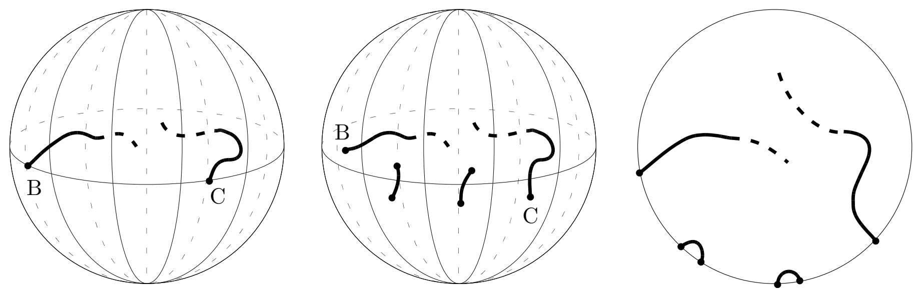

Figure 13: How to avoid ∂B_0^2 in $L(9,1)$.

4.2 Generalized Reidemeister moves

Again, with the aim of discovering when two diagrams represent equivalent links in $L(p,q)$, we generalize Reidemeister moves to this contest.

The *generalized Reidemeister moves* on a diagram of a link $L \subset L(p,q)$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5, R_6 and R_7 of Figure 14.

Theorem 4.1. *Two links L_0 and L_1 in $L(p,q)$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_7$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then there is an ambient isotopy $H : L(p,q) \times [0,1] \to L(p,q)$ such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t \circ l_0$. As before, using general position

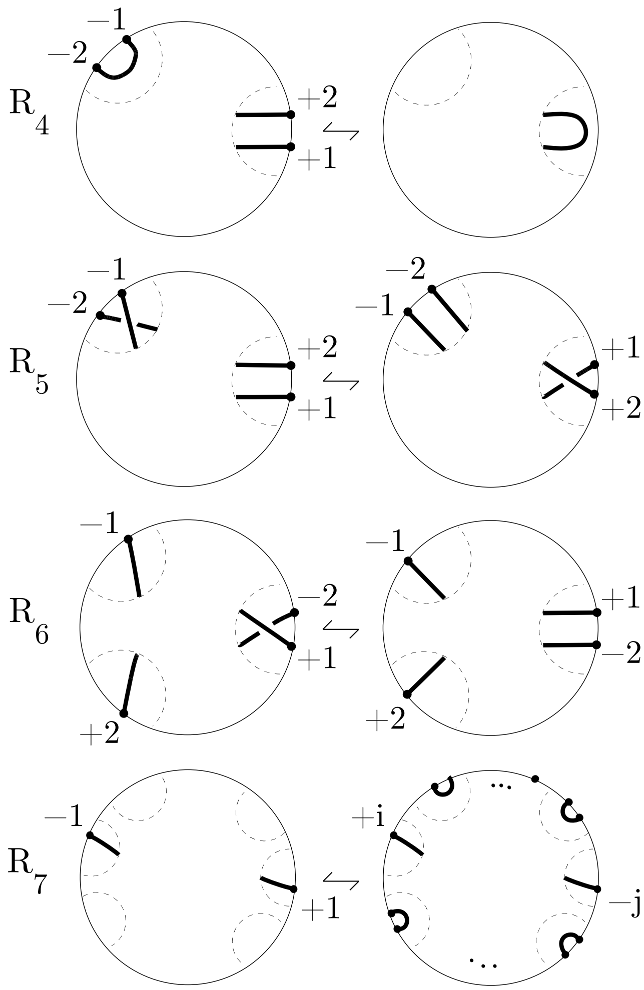

Figure 14: Generalized Reidemeister moves for links in $L(p, q)$.

arguments we can assume that the projection $\mathbf{p}(L'_t)$ is not regular only a finite number of times, and that at each of these times only one condition is violated.

From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as follows (see Figures 5 and 15):

– from V_1, V_2 and V_3 we obtain the classical Reidemeister moves R_1, R_2

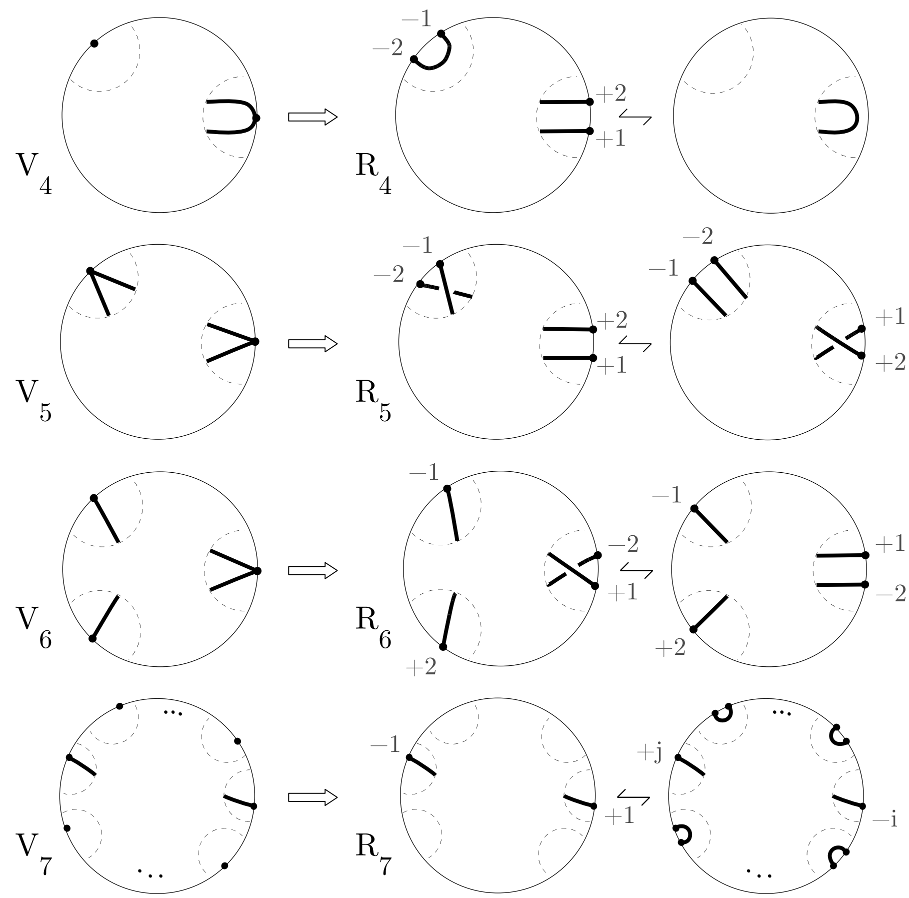

Figure 15:

and R_3;

- from V_4 we obtain the move R_4;
- from V_5, we obtain two different moves: if the endpoints of the arcs corresponding to the double point belong in the same hemisphere, then we obtain R_5; on the contrary we obtain R_6;
- from condition 6 we have a family of forbidden configurations $V_{7,1}, \dots, V_{7,p-1}$, from which we obtain the moves $R_{7,1}, \dots, R_{7,p-1}$.

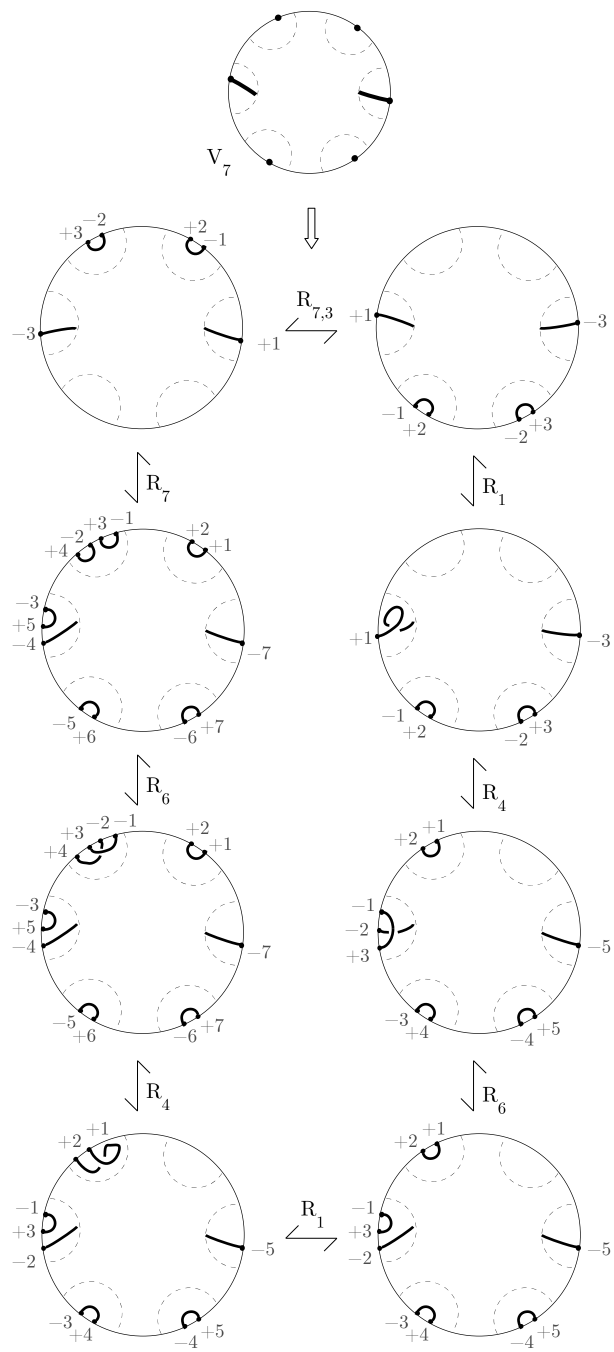

Figure 16: How to reduce a composite move.

Indeed, if an arc cross the equator during the isotopy, then we have a class of moves, $R_{7,1} = R_7, R_{7,2}, \ldots, R_{7,p-1}$. All these moves can be seen as the composition of R_7, R_6, R_4 and R_1 moves. More precisely, the move $R_{7,k}$ with $k = 2, \ldots, p-1$, can be obtained by the following sequence of moves: first we perform an R_7 move on one overpass that end on the equator and the corresponding point in a small arc, then we repeat for $k-1$ times the three moves R_6-R_4-R_1 necessary to retract the small arc with ending points having the same sign (see an example in Figure 16).

So we can drop $R_{7,2}, \ldots, R_{7,p-1}$ from the set of moves, and keep only $R_{7,1} = R_7$. As a consequence, any pair of diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_7$ and diagram isotopies. □

References

[1] Y. V. Drobotukhina, *An analogue of the Jones polynomial for links in* $\mathbb{RP}^3$ *and a generalization of the Kauffman-Murasugi theorem*, Leningrad Math. J. **2** (1991), 613–630.

[2] M. Gonzato, *Invarianti polinomiali per link in spazi lenticolari*, Degree thesis, University of Bologna, 2007.

[3] D. Roseman, *Elementary moves for higher dimensional knots*, Fund. Math. **184** (2004), 291–310.

Enrico MANFREDI,
Dipartimento di Matematica,
Università di Bologna,
Piazza di Porta San Donato, 5 - 40126 Bologna, Italia. Email: enrico.manfredi3@unibo.it

Michele MULAZZANI,
Dipartimento di Matematica,
Università di Bologna,
Piazza di Porta San Donato, 5 - 40126 Bologna, Italia.
Email: michele.mulazzani@unibo.it