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Singularities of the eta function of first-order differential operators

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Abstract

We report on a particular case of the paper [7], joint with Raphaël Ponge, showing that generically, the eta function of a first-order differential operator over a closed manifold of dimension n has first-order poles at *all* positive integers of the form $n-1, n-3, n-5, \ldots$.

Version française abrégée

Soit $\mathscr{D}$ la classe des operateurs différentiels elliptiques symmétriques d'ordre 1 sur une variété Riemannienne fermée M, agissant sur les sections d'un fibré vectoriel Hermitien E. Il est connu que pour un opérateur $D \in \mathscr{D}$, le spectre de D en tant qu'opérateur non-borné dans $L^2(M,E)$ est discret. Les fonctions êta et zêta associées à D sont défines par les séries

$$\eta(D,s) := \sum_{\lambda\in\mathrm{Spec}(D)\setminus\{0\}} |\lambda|^{-s}\mathrm{sign}(\lambda), \qquad \zeta(D,s) := \sum_{\lambda\in\mathrm{Spec}(D)\setminus\{0\}} |\lambda|^{-s}.$$

La fonction zêta de Riemann peut être ainsi obtenue (modulo un facteur de 2) à partir de l'opérateur $D = i\frac{d}{dt}$ sur le cercle unité. Quoique les séries ci-dessus ne sont absolument convergentes que pour $\Re(s) > n = \dim(M)$, les fonctions $\eta(D,s)$ et $\zeta(D,s)$ se prolongent méromorphiquement à $\mathbb{C}$, possiblement avec des pôles simples en $s = n-1, n-3, \ldots$ pour la fonction êta, respectivement

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en $s = n, n-2, \dots$ pour la fonction zêta. Ces pôles sont liés aux coefficients du developpement asymptotique de la trace de l'opérateur de la chaleur par les équations (2) et (3). Le point $s = 0$ est toujours régulier pour ces deux fonctions [2, 4], [10]. Si D est un opérateur Dirac compatible (voir [3]), alors $\eta(D, s)$ est reguliere dans le demi-plan $\Re(s) > 0$ et si de surcôut n est pair, alors la fonction êta est entiere. Le but de cette note est de décider quels pôles apparaissent génériquement pour des opérateurs dans la classe $\mathscr{D}$ des operateurs différentiels elliptiques auto-adjoints d'ordre 1. Nous partons de l'observation élémentaire que si $D \in \mathscr{D}$ alors pour tout $t \in \mathbb{R}$, $D + t \in \mathscr{D}$, où t agit sur E par multiplication.

Théorème. *Pour chaque $D \in \mathscr{D}$ et $k \in \mathbb{N} = \{0, 1, \dots\}$ tel que $n - 1 - 2k > 0$, la fonction*

$$t \mapsto E_k(t) := \operatorname{Res}_{s=n-1-2k} \eta(D + t, s)$$

est un polynôme de degré $2k + 1$ dont le coéfficient dominant est strictement négatif. Pour $k \in \mathbb{N}$ tel que $n - 2k > 0$, la fonction

$$t \mapsto Z_k(t) := \operatorname{Res}_{s=n-2k} \zeta(D + t, s)$$

est un polynôme de degré $2k$ dont le coéfficient dominant est strictement positif.

Il s'en suit que pour tout t en dehors d'un ensemble fini de $\mathbb{R}$, les fonctions êta et zêta correspondant à $D + t$ ont des vrais pôles aux entiers positifs de la forme $n - 1, n - 3, n - 5, \dots$ (pour êta), réspectivement aux entiers positifs de la forme $n, n-2, \dots$ (pour zêta). En particulier, le sous-ensemble d'opérateurs D de $\mathscr{D}$ pour lesquels au moins un des coefficients du théorème s'annule est d'interieur vide.

La preuve s'appuie sur la formule de premiere variation des fonctions êta et zêta par rapport à D (Lemme 6). Le fait que $D + t$ commute avec D pour chaque t nous permet de calculer dans (5) les derivées d'ordre supérieur. Nous déduisons ainsi les formules de variation (6) reliant les résidus des fonctions êta et zêta. Nous utilisons ensuite la positivité bien connue du residu de $\zeta(D, s)$ en $s = n$ pour tout $D \in \mathscr{D}$. Ces considerations sont a priori valables pour t en dehors du spectre de D, mais grâce au Lemme 7 nous les étendons à $\mathbb{R}$. Ceci achève la preuve du Théorème.

1 Statement of results

Let D be an elliptic first order differential operator acting in the fibers of a Hermitian vector bundle E over a closed Riemannian manifold M. Assume that the square of the principal symbol of D is positive definite when evaluated on nonzero covectors. Then the spectrum of D as an unbounded self-adjoint operator in $L^2(M,E)$ is discrete and concentrated near the real line. The eta and zeta functions of D are defined as follows:

$$\eta(D,s) := \sum_{\lambda\in\mathrm{Spec}(D)\setminus\{i\mathbb{R}\}} \lambda\,(\lambda^2)^{-\frac{s+1}{2}}, \quad \zeta(D,s) := \sum_{\lambda\in\mathrm{Spec}(D)\setminus\{i\mathbb{R}\}} (\lambda^2)^{-\frac{s}{2}}.$$

Here, for $\lambda\in\mathbb{C}\setminus i\mathbb{R}$ we define $(\lambda^2)^{-\frac{s}{2}} := \exp(-\frac{s}{2}\log(\lambda^2))$ using the cut along the negative real axis for the log function. When λ is purely imaginary (there may be finitely many such eigenvalues under our assumptions), its contribution to the eta function vanishes. The eta function was first introduced in [1] and the zeta function in [8] for the Laplacian and in [10] in great generality. The series defining the eta and zeta functions are absolutely convergent for $\Re(s) > n$ where n is the dimension of M. Modulo a harmless factor of 2, the Riemann zeta function is the simplest example of such a zeta function, obtained by taking D to be the operator $i\frac{d}{dt}$ on the unit circle.

Because of their importance in spectral geometry, much effort has gone into understanding the meromorphic properties of these spectral functions. For example, it is well-known (see Section 2) that $\eta(D,s)$ and $\zeta(D,s)$ extend meromorphically to $\mathbb{C}$, with possible simple poles at $s\in n-1-2\mathbb{N}$, respectively at $s\in n-2\mathbb{N}$ where $\mathbb{N}=\{0,1,\ldots\}$. It is fairly straightforward (for the zeta function [10]) and rather complicated (for the eta function [2, 4]) to show that $s=0$ is always a regular point for these two functions. In particular cases some of the poles besides $s=0$ may disappear. For instance, in the case of the Riemann zeta function, only $s=1=\dim(\mathbb{S}^1)$ is a pole, while the corresponding eta function vanishes identically. If D is a compatible Dirac operator, the poles of $\eta(D,s)$ may only be located at negative integers in $n-1-2\mathbb{N}$ and in addition, if the dimension of M is even, the eta function is entire [3]. Deciding which poles occur *generically* is the goal of the present note. Specifically, as a corollary of Theorem 2 below, for general first order elliptic differential operators, the eta (and zeta) functions generically have poles (independent of the parity of the dimension). This was proved by Branson and Gilkey [3] in the case the operator D is of Dirac-type, meaning that the square of the principal symbol of D defines a Riemannian metric.

To state our results precisely, let us fix some terminology .

Definition 1. Let $\mathscr{D}$ denote the set of elliptic first order differential operators on (M,E) such that the square of their principal symbol is positive definite on non-zero covectors.

Following [3], an operator in $\mathscr{D}$ is called of *Dirac type* if the square of its symbol is scalar.

There is a natural free action of $\mathbb{R}$ on $\mathscr{D}$ given by translations:

$$t.D := D + t$$

where t acts by multiplication in the fibers of E. We will denote $t.D$ by D_t. We call *admissible* poles for the eta, respectively the zeta function of an operator in $\mathscr{D}$, those nonzero integers in $n - 1 - 2\mathbb{N}$, respectively $n - 2\mathbb{N}$, where recall that $n = \dim(M)$; we make no assumptions on the parity of n.

Theorem 2. *For every $D \in \mathscr{D}$ and $k \in \mathbb{N}$ such that $n - 1 - 2k > 0$, the function*

$$t \mapsto E_k(t) := \mathrm{Res}_{s=n-1-2k}\, \eta(D_t, s)$$

is a polynomial of degree $2k+1$ with strictly negative leading term. For $k \in \mathbb{N}$ such that $n - 2k > 0$, the function

$$t \mapsto Z_k(t) := \mathrm{Res}_{s=n-2k}\, \zeta(D_t, s)$$

is a polynomial of degree $2k$ with strictly positive leading term.

This theorem is a special case of more general results in [7] dealing with pseudodifferential operators, joint with Raphaël Ponge. The proofs in [7] use the noncommutative residue trace of Guillemin [6] and Wodzicki [11]. In contrast, the proof of Theorem 2 presented in this note for the special case of differential operators uses only elementary facts concerning eta and zeta functions known from their inception [2, 10].

As a consequence of Theorem 2, for t outside a finite subset of $\mathbb{R}$, the eta and zeta functions of D_t have nonzero residues at *all* admissible positive integers. One can put a natural Fréchet topology on $\mathscr{D}$ by taking a partition of unity of M with respect to a coordinate cover and writing an element $D \in \mathscr{D}$ in local coordinates, then taking the C^∞ topology induced by the coefficients of the partial derivatives in the presentation of D in the coordinates. Then Theorem 2 implies that the set of operators in $\mathscr{D}$ with an entire eta function forms a nowhere dense set and hence generically eta functions of first order differential operators have non-trivial poles in the sense of Baire Category. Moreover, Theorem 2 implies generic non-vanishing results for any subclass of $\mathscr{D}$ closed under the $\mathbb{R}$ action; we summarize below five such classes.

Corollary 3. *The residues of the eta function at positive admissible integers are generically nonzero for operators of the following types:*

1. *Operators in the class $\mathscr{D}$.*

2. *Operators of Dirac-type.*

3. *Self-adjoint elliptic first-order differential operators.*
4. *Elliptic first-order differential operators with a self-adjoint symbol.*
5. *Self-adjoint operators of Dirac-type.*

In particular, (2) provides a simple proof of the generic non-vanishing theorem of Branson and Gilkey for Dirac-type operators ([3, Theorem 4.3.c]) at positive admissible integers. We remark that none of the generic non-vanishing results implies another one, since the sub-classes are not generic in the larger classes. It is not obvious how to adapt the invariance theory arguments of [3] to some of the above classes.

2 Review of known facts on the poles

To keep this paper self-contained we quickly review some well-known results on the meromorphic nature of the eta and zeta functions. Assume that $D \in \mathscr{D}$ is such that all eigenvalues of D^2 have positive real part; then

$$\eta(D,s) = \mathrm{Tr}\left(D(D^2)^{-\frac{s+1}{2}}\right), \qquad \zeta(D,s) = \mathrm{Tr}\left((D^2)^{-\frac{s}{2}}\right).$$

Following the seminal paper [8], using these formulas one can link the poles of the zeta and eta functions with the short-time asymptotics of the trace of the heat, respectively of the odd heat operator. The identity

$$(D^2)^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty u^{\frac{s}{2}-1} e^{-uD^2} du$$

follows easily from the definition of the Gamma function. It implies

$$D(D^2)^{-\frac{s+1}{2}} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty u^{\frac{s-1}{2}} D e^{-uD^2} du. \tag{1}$$

It follows that

$$\eta(D,s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty u^{\frac{s-1}{2}} \mathrm{Tr}(De^{-uD^2}) du \tag{2}$$

and

$$\zeta(D,s) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty u^{\frac{s}{2}-1} \mathrm{Tr}(e^{-uD^2}) du. \tag{3}$$

Now it is well-known [5, §1.9] that for any first order differential operator A, as $u \to 0$ we have an expansion

$$\mathrm{Tr}(Ae^{-uD^2}) \sim \sum_{k=0}^\infty u^{-\frac{n}{2}+k} a_k$$

for some constants a_k. Using this formula together with the formulas (2) and (3) we see that

$$\Gamma\left(\frac{s+1}{2}\right)\eta(D,s) \sim \sum_{k=0}^{\infty} \frac{2}{s+1-n+2k}\,\eta_k\,, \qquad \Gamma\left(\frac{s}{2}\right)\zeta(D,s) \sim \sum_{k=0}^{\infty} \frac{2}{s-n+2k}\,\zeta_k \tag{4}$$

for some constants η_k, ζ_k. It follows that the eta and zeta functions can only have poles at $s \in n-1-2\mathbb{N}$, respectively at $s \in n-2\mathbb{N}$ where $\mathbb{N} = \{0,1,\dots\}$. As we mentioned earlier, $s=0$ is always a regular point for these two functions [2, 4, 10].

Remark 4. It is well-known that (4) holds without the assumption on the spectrum of D^2. To see why, on the right-hand sides of (2) and (3) we subtract the finitely many terms of the form $\lambda e^{-t\lambda^2}$ for the eta function and $e^{-t\lambda^2}$ for the zeta function corresponding to eigenvalues λ with $\Re(\lambda^2) \leq 0$. It is easy to check that these extra terms give rise to entire functions of s and hence do not contribute to the poles of the eta and zeta functions.

Remark 5. When n is even we see from (4) that the only possible poles of the eta and zeta functions occur at positive points, due to the Gamma factors. Our main result implies in particular that the possible poles at the positive admissible integers are generically nonzero. Furthermore, when n is odd, certain zeros appear at $s=-1,-3,\dots$ for the eta function, respectively at $s=-2,-4,\dots$ for the zeta function. At the same time, possible negative poles of eta may appear $s=-2,-4,\dots$, a subject we'll return to at the end of this note.

3 Proof of the main result

For real $t \notin -\Re(\mathrm{Spec}(D))$, the operator $D_t = D+t$ does not have eigenvalues in $i\mathbb{R}$. It follows easily that for such t, $\eta(D_t,s)$ and $\zeta(D_t,s)$ are smooth functions with values in the meromorphic functions on $\mathbb{C}$. The following lemma was first proven in [2], Propositions 2.9 and 2.10; since the argument is so simple we provide a quick proof.

Lemma 6. *For real $t \notin -\Re(\mathrm{Spec}(D))$, we have*

$$\partial_t \eta(D_t,s) = -s\,\zeta(D_t,s+1) \quad \text{and} \quad \partial_t \zeta(D_t,s) = -s\,\eta(D_t,s+1).$$

Proof. Since $\partial_t D_t = 1$, by an easy computation we obtain

$$\begin{aligned}\partial_t \eta(D_t, s) &= \partial_t \mathrm{Tr}\left(D_t (D_t^2)^{-\frac{s+1}{2}}\right)\\ &= \mathrm{Tr}\left((D_t^2)^{-\frac{s+1}{2}}\right) - \frac{s+1}{2}\mathrm{Tr}\left(D_t (D_t^2)^{-\frac{s+3}{2}}(2D_t)\right)\\ &= -s\mathrm{Tr}\left((D_t^2)^{-\frac{s+1}{2}}\right) = -s\zeta(D_t, s+1).\end{aligned}$$

A similar argument gives the second claim. □

Setting $(s)_k := s(s+1)\dots(s+k-1)$, by iterating the above identities we get

$$\begin{aligned}\partial_t^{2k}\eta(D_t,s) &= (s)_{2k}\,\eta(D_t, s+2k)\\ \partial_t^{2k+1}\eta(D_t,s) &= -(s)_{2k+1}\,\zeta(D_t, s+2k+1)\\ \partial_t^{2k}\zeta(D_t,s) &= (s)_{2k}\,\zeta(D_t, s+2k)\\ \partial_t^{2k+1}\zeta(D_t,s) &= -(s)_{2k+1}\,\eta(D_t, s+2k+1).\end{aligned} \tag{5}$$

Recalling that $E_k(t) := \mathrm{Res}_{s=n-1-2k}\eta(D_t, s)$ and $Z_k(t) := \mathrm{Res}_{s=n-2k}\zeta(D_t, s)$, immediately from (5) we deduce that for all $k \in \mathbb{N}$ and all $t \notin \mathrm{Spec}(D)$,

$$\partial_t^{2k+1}E_k(t) = -\frac{(n-1)!}{(n-2k-2)!}Z_0(t), \quad \partial_t^{2k}Z_k(t) = \frac{(n-1)!}{(n-2k-1)!}Z_0(t), \tag{6}$$

where $Z_0(t) = \mathrm{Res}_{s=n}\zeta(D_t, s)$.

Now the zeta function *always* has a pole with strictly positive residue at $s = n = \dim(M)$. More precisely, the residue $Z_0(t)$ is independent of t and equals [10]

$$Z_0(0) = \mathrm{Res}_{s=n}\zeta(D, s) = (2\pi)^{-n}\int_{S^*M} \mathrm{tr}\left((A^2(x,\xi))^{-n/2}\right), \tag{7}$$

where A is the principal symbol of D and the integral is over the cosphere bundle of M (with respect to the canonical measure). By definition of $\mathscr{D}$, $A^2(x,\xi)$ is positive definite, so the above integral is strictly positive.

It follows that $\partial_t\eta(D_t, s)$ has a pole at $s = n-1$. In the notation of Theorem 2, for real $t \notin -\Re(\mathrm{Spec}(D))$, we have

$$\begin{aligned}\frac{\partial}{\partial t}E_0(t) &= \frac{\partial}{\partial t}\mathrm{Res}_{s=n-1}\eta(D_t, s)\\ &= -(n-1)\mathrm{Res}_{s=n}\zeta(D,s) = -(n-1)Z_0(0) < 0,\end{aligned}$$

in other words the residue $E_0(t)$ is a piecewise affine function of t with negative slope given by $-(n-1)Z_0(0) < 0$; in particular it is nonzero. Of course, this argument holds only for $n > 1$ and on the complement of $-\Re(\mathrm{Spec}(D))$ in $\mathbb{R}$. However, we have the following.

Lemma 7. *The residues of the eta and zeta functions are smooth in $t \in \mathbb{R}$.*

Proof. It is well-known [10] that the residues of the eta and zeta functions are given by integrals over the cosphere bundle of expressions that can be written using finitely terms of the local symbol of D_t, which depend smoothly on t. It follows that the residues are smooth functions of $t \in \mathbb{R}$. □

This lemma implies that $E_0(t)$ is affine on the whole real axis. Such a function vanishes exactly at one point, thus the residue is nonzero except at a single point. Similarly, for all $k \in \mathbb{N}$ such that $n - 2k - 2 \geq 0$, from (6) we deduce that $E_k(t)$ is a polynomial of degree $2k + 1$ with leading term

$$E_k(t) = -\binom{n-1}{n-2k-2} Z_0(0)t^{2k+1} + O(t^{2k})$$

while for $n - 2k - 1 \geq 0$,

$$Z_k(t) = \binom{n-1}{n-2k-1} Z_0(0)t^{2k} + O(t^{2k-1}).$$

This finishes the proof of Theorem 2.

Remark 8. The above computation is valid irrespective of the parity of $n = \dim(M)$ and, although we have been focusing on differential operators throughout this paper, is valid also for first-order *pseudodifferential* operators.

Remark 9. We know from [3] that the residues $E_k(0)$ vanish for all $k \in \mathbb{N}$ such that $n - 1 - 2k > 0$ when D is a compatible Dirac operator on a Clifford module E. In this case, from the formula (7) we have $Z_0(0) = (2\pi)^{-n} \dim(E)\mathrm{vol}(M)\mathrm{vol}(S^{n-1})$, so the residue at $s = n - 1$ of $\eta(D + t, s)$ equals tC with

$$C = -(n-1)(2\pi)^{-n} \dim(E)\mathrm{vol}(M)\mathrm{vol}(S^{n-1}).$$

In particular, for all $t \in \mathbb{R}$, the eta function of $D + t$ always has a non-trivial pole except when $t = 0$. To construct an explicit example, let us take D to be $d + d^*$ on forms over the standard 2-torus. Then

$$\mathrm{Res}_{s=1}\, \eta(D + t, s) = -8\pi t \quad \text{for all } t \in \mathbb{R}.$$

Remark 10. If n is even, using the formula $\partial_t^{2k}\zeta(D_t, s) = (s)_{2k}\, \zeta(D_t, s + 2k)$, one can show that the *values* of the zeta function at $0, -2, -4, \ldots$ are generically non-zero; in fact, the value $\zeta(D_t, -2k)$ is a polynomial in t of degree

$2k+n$ with a positive leading term (a constant multiple of $Z_0(0)$). Moreover, when n is odd, the formula $\partial_t^n \eta(D_t, s) = -(s)_n\, \zeta(D_t, s+n)$ implies that the eta invariant, $\eta(D_t, 0)$, is generically non-zero; it's a polynomial in t of degree n with negative leading term.

4 The residues at negative integers

As explained above, for n odd the zeta and eta functions may also have poles in certain negative integers (more precisely, $\zeta(s)$ may have poles in $s = -1-2k$ while $\eta(s)$ may have poles in $s = -2-2k$ for all $k = 0, 1, \ldots$.

Using (5), in order to prove generic non-vanishing of these poles, it is enough to show generic non-vanishing of $\mathrm{Res}_{s=-1}\zeta(D,s)$. Indeed, if we assume this first residue to be non-zero, the same proof as above applies to the next residues, the leading term in the polynomial being now $\mathrm{Res}_{s=-1}\zeta(D,s)$ instead of $Z_0(0)$. We focus therefore on the residue of zeta at $s = -1$.

Variation as above by a scalar will break down since we know that the eta function is regular at zero. Instead we consider variation of D by an endomorphism of E. We define $D_t := D + tA$ for some $A \in C^\infty(M, \mathrm{End}(E))$, which can be chosen self-adjoint if we work in a class of self-adjoint operators. Then as in Lemma 6,

$$\partial_t \zeta(D_t, s) = -s\, \mathrm{Tr}(AD_t(D_t^2)^{-\frac{s+2}{2}}).$$

Thus, it is enough to show generic non-vanishing of $\mathrm{Res}_{s=0}\mathrm{Tr}(AD(D^2)^{-\frac{s+1}{2}})$ when A and D vary. By using the heat kernel expansion

$$(De^{-uD^2})(x,x) \sim \sum_{k=0}^{\infty} u^{-\frac{n}{2}+k} a_k(x)$$

and the relation (1) between the heat kernel and complex powers, we get

$$\mathrm{Res}_{s=0}\mathrm{Tr}(AD(D^2)^{-\frac{s+1}{2}}) = \frac{2}{\sqrt{\pi}} \int_M \mathrm{tr}\big(Aa_{\frac{n-1}{2}}\big).$$

For generic A this is nonzero if and only if the local eta residue $a_{\frac{n-1}{2}}$ is a nonzero section in the endomorphism bundle (note that the pointwise trace of $a_{\frac{n-1}{2}}$ always integrates to 0 by the vanishing of the eta residue at 0). Thus the generic non-vanishing of the admissible negative poles of the eta and zeta functions is implied by the generic non-vanishing of the local eta residue (before taking the trace). This is true for Dirac-type operators by the work of Branson and Gilkey (Theorems 2.7 and 4.3 in [3]). Hence, we conclude the generic non-vanishing of the admissible negative poles for Dirac-type operators, which was already known for the eta function for $n \geq 3$ [3, Th. 4.3]. Although this is most likely true for first-order elliptic differential operators in each of the classes from Corollary 3, we do not have an easy argument at hand and thus we leave this issue open.

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