



Seiberg-Witten Equations on Pseudo-Riemannian Spin^c Manifolds With Neutral Signature

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Abstract

Pseudo-Riemannian spin^c manifolds were introduced by Ikemakhen in [7]. In the present work we consider pseudo-Riemannian 4-manifolds with neutral signature whose structure groups are $SO_+(2, 2)$. We prove that such manifolds have pseudo-Riemannian spin^c structure. We construct spinor bundle S and half-spinor bundles S^+ and S^- on these manifolds. For the first Seiberg-Witten equation we define Dirac operator on these bundles. Due to the neutral metric self-duality of a 2-form is meaningful and it enables us to write down second Seiberg-Witten equation. Lastly we write down the explicit forms of these equations on 4-dimensional flat space.

1 Introduction

Spinors are geometric objects living around manifolds. They are important for the investigation of manifolds (see [6, 9]). Seiberg-Witten Monopole equations were defined by E. Witten on 4-dimensional Riemannian manifolds by using the spinors [16]. The solution space of these equations provides new invariants for 4-manifolds, namely Seiberg-Witten invariants ([1, 12, 13]). Similar equations were written down on 4-dimensional Lorentzian manifolds [3]. Pseudo-Riemannian 4-manifolds with neutral signature are being studied by various authors from different point of view (see [2, 4, 8, 10, 11]).

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Pseudo-Riemannian spin^c spinors are introduced by Ikemakhen in [7] recently. The aim of this article is to write down similar equations to Seiberg-Witten equations on 4-dimensional Pseudo-Riemannian spin^c manifolds with neutral signature.

2 Some Preliminaries

On \mathbb{R}^4 , we consider the pseudo-Riemannian metric $g(x, y) = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$, where $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. When this metric considered on the 4-dimensional space is denoted by $\mathbb{R}^{2,2}$. The isometry group of this space is denoted by $O(2, 2)$, that is

$$O(2, 2) = \{A \in GL(4, \mathbb{R}) : g(A(x), A(y)) = g(x, y), \text{ where } x, y \in \mathbb{R}^{2,2}\}.$$

The group $O(2, 2)$ has four connected components. The special orthogonal subgroup of $O(2, 2)$ is denoted by

$$SO(2, 2) = \{A \in O(2, 2) : \det A = 1\}.$$

The subgroup $SO(2, 2)$ has two connected components and the connected component to the identity of $SO(2, 2)$ is denoted by $SO_+(2, 2)$. In this work we mainly deal with the group $SO_+(2, 2)$. $Spin_+(2, 2)$ lives in the Clifford algebra $Cl_{2,2} = Cl(\mathbb{R}^4, g)$ and it is isomorphic to $SU(1, 1) \times SU(1, 1)$ (see [11]).

The covering map $\lambda : Spin_+(2, 2) \rightarrow SO_+(2, 2)$ is a 2 : 1 group homomorphism given by $\lambda(g)(x) = g \cdot x \cdot g^{-1}$ for any $x \in \mathbb{R}^4$, $g \in Spin_+(2, 2)$.

Remark 1. *Contrary to the Euclidean and Lorentzian cases the fundamental group of $SO_+(2, 2)$ is not \mathbb{Z}_2 and $Spin_+(2, 2)$ is not simply connected.*

One can define a new group which lies in the complex Clifford algebra $Cl_{2,2} \cong Cl_4$ by

$$Spin_+^c(2, 2) = Spin_+(2, 2) \times S^1/\mathbb{Z}_2.$$

The elements of $Spin_+^c(2, 2)$ are the equivalence classes $[g, z]$ of pairs $(g, z) \in Spin_+(2, 2) \times S^1$ under the equivalence relation $(g, z) \sim (-g, -z)$. From the definitions of $Spin_+(2, 2)$ and $Spin_+^c(2, 2)$ the following sequences are exact:

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin_+(2, 2) \xrightarrow{\lambda} SO_+(2, 2) \rightarrow 1,$$

$$1 \rightarrow \mathbb{Z}_2 \rightarrow Spin_+^c(2, 2) \xrightarrow{\xi} SO_+(2, 2) \times S^1 \rightarrow 1,$$

where $\xi([g, z]) = (\lambda(g), z^2)$.

Since the complex Clifford algebra $Cl_{2,2}$ is isomorphic to the endomorphism algebra $End(\mathbb{C}^4)$, there is a natural representation $\kappa : Cl_{2,2} \rightarrow End(\mathbb{C}^4)$. For example, we can define κ on the basis elements as follows:

$$\begin{aligned}\kappa(e_1) &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, & \kappa(e_2) &= \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \\ \kappa(e_3) &= \begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, & \kappa(e_4) &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\end{aligned}$$

where I_2 is 2×2 unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

are Pauli-spin matrices.

The complex vector space \mathbb{C}^4 is called the space of spinors and denoted by $\Delta_{2,2}$. The spinor space $\Delta_{2,2}$ carries a non-degenerate indefinite Hermitian inner product $\langle \cdot, \cdot \rangle_{\Delta_{2,2}}$ which is invariant under the action of $\text{Spin}_+^c(2,2)$ is given by

$$\langle \Psi_1, \Psi_2 \rangle_{\Delta_{2,2}} = \langle \kappa(e_1) \kappa(e_2) \Psi_1, \Psi_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the positive definite Hermitian inner product on \mathbb{C}^4 (see [7]). We can restrict the map κ to $\text{Spin}_+^c(2,2)$ and we obtain a group representation

$$\kappa : \text{Spin}_+^c(2,2) \rightarrow \text{Aut}(\Delta_{2,2}).$$

The restricted map κ is called spinor representation of $\text{Spin}_+^c(2,2)$. The spinor space $\Delta_{2,2}$ decomposes two parts

$$\Delta_{2,2} = \Delta_{2,2}^+ \oplus \Delta_{2,2}^-,$$

where $\Delta_{2,2}^\pm$ are the eigenspaces of $f = \kappa(e_1 e_2 e_3 e_4)$, f^2 is the identity map. The elements of $\Delta_{2,2}^+$ are called the positive spinors. Since the spinor representation of $\text{Spin}_+^c(2,2)$ preserves these eigenspaces, we obtain the following representations by restriction

$$\kappa^\pm : \text{Spin}_+^c(2,2) \rightarrow \text{Aut}(\Delta_{2,2}^\pm).$$

The spinor representation κ has the following properties:

Proposition 1.

- i) $\kappa(\text{Spin}_+(2,2)) \cong SU(1,1) \times SU(1,1)$.
- ii) $\kappa^+(\text{Spin}_+(2,2)) \cong SU(1,1)$
- iii) $\kappa^-(\text{Spin}_+(2,2)) \cong SU(1,1)$

- iv) $\kappa(\text{Spin}_+^c(2, 2)) \cong \{(A, B) \in U(1, 1) \times U(1, 1) : \det(A) = \det(B)\}$
v) $\kappa^+(\text{Spin}_+^c(2, 2)) \subset U(1, 1)$
vi) $\kappa(v)$ maps $\Delta_{2,2}^-$ to $\Delta_{2,2}^+$ and $\Delta_{2,2}^+$ to $\Delta_{2,2}^-$ for each $v \in \mathbb{R}^4$.
vii) $\kappa(v)^2 = -g(v, v)I_4$ for each $v \in \mathbb{R}^4$, where I_4 is 4×4 identity matrix.

The Lie algebras of the groups $\text{Spin}_+(2, 2)$ and $\text{Spin}_+^c(2, 2)$ are

$$\text{spin}_+(2, 2) = \{e_i \cdot e_j ; 1 \leq i < j \leq 4\}$$

and

$$\text{spin}_+^c(2, 2) = \text{spin}_+(2, 2) \oplus i\mathbb{R},$$

respectively, where $e_i \cdot e_j$ is the second order element of the $\mathbb{C}l_{2,2}$. The derivative of $\xi = \lambda \times l$ is a Lie algebra isomorphism and given by

$$\xi_*(e_i \cdot e_j, it) = (\lambda_*(e_i \cdot e_j), l_*(it)) = (2E_{ij}, 2it),$$

where E_{ij} denotes the basis elements of the Lie algebra $so_+(2, 2)$ and

$$\lambda : \text{Spin}_+^c(2, 2) \rightarrow SO_+(2, 2), \quad \lambda([g, z]) = \lambda(g)$$

and $l : \text{Spin}_+^c(2, 2) \rightarrow S^1, \quad l([g, z]) = z^2$ are group homomorphisms.

3 Pseudo-Riemannian Manifolds of Metric Signature $(++--)$

3.1 Existence of Neutral Metric

Let M be a 4-dimensional space and time oriented smooth manifold with the pseudo-Riemannian metric g of signature $(2, 2)$ (that is of type $(+, +, -, -)$). Such a metric is called neutral metric. Existence conditions of neutral metric on a 4-dimensional differentiable manifold M were given in [11] in detail form. In the present work we focus on the completely orientable case, i.e., the structure group of the tangent bundle TM is $SO_+(2, 2)$. It is pointed out in [11] that the structure group of M is $SO_+(2, 2)$ if and only if it admits a fields of orientable tangent 2-planes. Following theorem will be useful for our discussion on the existence of pseudo-Riemannian spin^c -structure.

Theorem 1. *Existence of neutral metric on a compact manifold M with structure group $SO_+(2, 2)$ is equivalent to the existence of a pair (J, J') of an almost complex structure J and an opposite almost complex structure J' on the manifold, where J and J' are orthogonal with respect to the metric g and they commutes, that is; $JJ' = J'J$ [11].*

The family of manifolds which have neutral metric is rather large, for example; K3 surfaces, Enriques surfaces, Kodaria surfaces, Ruled surfaces of genus $g \geq 1$ and see [11] for others.

3.2 Self-Duality

Neutral metric shares some properties of the Riemannian metric. For example, the Hodge star operator $*$ is an involution on the space of two forms $\Lambda^2(M)$. Since $*^2 = id$, $*$ induces a splitting of $\Lambda^2(M) = \Lambda^+ \oplus \Lambda^-$, where Λ^+ and Λ^- denote the space of *self-dual* and *anti-self-dual* 2-forms

$$\Lambda^+ = \{\eta \in \Lambda^2(M) : *\eta = \eta\}, \quad \Lambda^- = \{\eta \in \Lambda^2(M) : *\eta = -\eta\}.$$

The projection of a 2-form $\eta \in \Lambda^2(M)$ onto the subspace Λ^+ is called the self-dual part of η and we denote it by η^+ , similarly the projection of η onto the subspace Λ^- is called the anti-self-dual part of η and we denote it by η^- . Note that $\eta = \eta^+ + \eta^-$ and the self-dual and anti-self-dual parts can be expressed in terms of the Hodge star operator $*$ by the following way:

$$\eta^+ = \frac{1}{2}(\eta + *\eta) \quad \text{and} \quad \eta^- = \frac{1}{2}(\eta - *\eta).$$

Let $\{e_1, e_2, e_3, e_4\}$ be a local pseudo-orthonormal frame on the open set $U \subset M$ and $\{e^1, e^2, e^3, e^4\}$ be the corresponding dual frame. Then the vectors $f_1 = e^1 \wedge e^2 + e^3 \wedge e^4$, $f_2 = e^1 \wedge e^3 + e^2 \wedge e^4$, $f_3 = e^1 \wedge e^4 - e^2 \wedge e^3$ form a basis for self-dual 2-forms, that is

$$\Lambda^+ = \text{span} \{f_1, f_2, f_3\}.$$

Similarly the vectors $g_1 = e^1 \wedge e^2 - e^3 \wedge e^4$, $g_2 = e^1 \wedge e^3 - e^2 \wedge e^4$, $g_3 = e^1 \wedge e^4 + e^2 \wedge e^3$ form a basis for anti-self-dual 2-forms, that is

$$\Lambda^- = \text{span} \{g_1, g_2, g_3\}.$$

The componentwise expression of these two parts is given by

$$\eta^+ = \frac{1}{2} [((\eta_{12} + \eta_{34}) f_1 + (-\eta_{13} - \eta_{24}) f_2 + (-\eta_{14} + \eta_{23}) f_3]$$

$$\eta^- = \frac{1}{2} [((\eta_{12} - \eta_{34}) f_1 + (-\eta_{13} + \eta_{24}) f_2 + (-\eta_{14} - \eta_{23}) f_3].$$

Similar to the Riemannian case self-duality and anti-self-duality of a neutral metric can be defined in terms of the Weyl tensor. Such structures are also related with the geometry of underlying manifolds (see [4, 8]).

3.3 Pseudo-Riemannian spin^c Structure

The definitions of a pseudo-Riemannian spin and spin^c structures on M can be given similar to the Riemannian cases as follows:

Since the structure group of M is $SO_+(2, 2)$, there are an open covering $\{U_\alpha\}_{\alpha \in A}$ and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow SO_+(2, 2)$ for M . If there exists another collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}_+(2, 2)$$

such that the following diagram commutes

$$\begin{array}{ccc} & \text{Spin}_+(2, 2) & \\ \tilde{g}_{\alpha\beta} \nearrow & & \downarrow \lambda \text{ 2:1} \\ U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & SO_+(2, 2) \end{array}$$

that is, $\lambda \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied, then M is called a pseudo-Riemannian spin manifold. Then one can construct a principal $\text{Spin}_+(2, 2)$ -bundle $P_{\text{Spin}_+(2, 2)}$ on M and a 2 : 1 bundle map $\Lambda : P_{\text{Spin}_+(2, 2)} \rightarrow P_{SO_+(2, 2)}$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_{\text{Spin}_+(2, 2)} \times \text{Spin}_+(2, 2) & \longrightarrow & P_{\text{Spin}_+(2, 2)} & & \\ (\Lambda, \lambda) \downarrow & & \downarrow \Lambda & \searrow & \\ P_{SO_+(2, 2)} \times SO_+(2, 2) & \longrightarrow & P_{SO_+(2, 2)} & \longrightarrow & M \end{array}$$

Similarly pseudo-Riemannian spin^c structures on M can be defined by a collection of transition functions. There are an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow SO_+(2, 2)$ for M . If there exists another collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Spin}_+^c(2, 2)$$

such that the following diagram commutes

$$\begin{array}{ccc} & \text{Spin}_+^c(2, 2) & \\ \tilde{g}_{\alpha\beta} \nearrow & & \downarrow \lambda \\ U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & SO_+(2, 2) \end{array}$$

that is, $\lambda \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is

satisfied then M is called a pseudo-Riemannian spin^c manifold. Then one can construct a principal $\text{Spin}_+^c(2, 2)$ -bundle $P_{\text{Spin}_+^c(2,2)}$ on M and a $2 : 1$ bundle map $\Lambda : P_{\text{Spin}_+^c(2,2)} \rightarrow P_{SO_+(2,2)}$ such that the following diagram commutes:

$$\begin{array}{ccc} P_{\text{Spin}_+^c(2,2)} \times \text{Spin}_+^c(2, 2) & \longrightarrow & P_{\text{Spin}_+^c(2,2)} \\ \downarrow (\Lambda, \lambda) & & \downarrow \Lambda \\ P_{SO_+(2,2)} \times SO_+(2, 2) & \longrightarrow & P_{SO_+(2,2)} \longrightarrow M \end{array}$$

Remark 2. Since $\text{Spin}_+^c(2, 2)$ is isomorphic to the group

$$H = \{(A, B) \in U(1, 1) \times U(1, 1) : \det(A) = \det(B)\},$$

one can define spin^c structure on M by the existence of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H$$

such that $\text{Ad} \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma} = \tilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied. The covering map $\text{Ad} : H \rightarrow SO_+(2, 2)$ is defined by $\text{Ad}(A, B)(V) = AVB^{-1}$ for each $(A, B) \in H$ and $V \in \mathbb{C}^2 \cong \mathbb{R}^4$. In the definition of the map Ad we use the one to one correspondence between the vectors $V = (v_1, v_2, v_3, v_4)$ in $\mathbb{R}^{2,2}$ and the 2 by 2 complex matrices by the following way

$$V = (v_1, v_2, v_3, v_4) = v_1I + v_2i\sigma_3 - v_3\sigma_2 + v_4\sigma_1 = \begin{pmatrix} v_1 + iv_2 & v_4 + iv_3 \\ v_4 - iv_3 & v_1 - iv_2 \end{pmatrix}.$$

Note that the equality $\det(V) = v_1^2 + v_2^2 - v_3^2 - v_4^2 = g(V, V)$ holds. From this equality we obtain $g(\text{Ad}(A, B)(V), \text{Ad}(A, B)(V)) = \det(AVB^{-1}) = \det(V) = g(V, V)$, so $\text{Ad}(A, B)$ belongs to the group $SO_+(2, 2)$ for each $(A, B) \in H$.

If M has a pseudo-Riemannian spin (spin^c) structure, then M is called pseudo-Riemannian spin (spin^c) manifold. It is known that each pseudo-Riemannian spin structure on M induces a pseudo-Riemannian spin^c structure, hence every pseudo-Riemannian spin manifold is a pseudo-Riemannian spin^c manifold.

Theorem 2. If M is a 4-dimensional compact differentiable manifold with structure group $SO_+(2, 2)$, then M is a pseudo-Riemannian spin^c manifold.

Proof. By Theorem 1 there is a g -orthogonal almost complex structure J on M . Then the structure group of M can be reduced from $SO_+(2, 2)$ to $U(1, 1)$. That is, there are an open covering $\{U_\alpha\}_{\alpha \in A}$ and transition functions

$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow U(1, 1)$ for M . The canonical action of $U(1, 1)$ on $\mathbb{R}^{2,2} \cong \mathbb{C}^2$ is given by ordinary matrix product AV for each $A \in U(1, 1)$ and $V = (v_1 + iv_2, v_4 - iv_3)$. This action can also be interpreted as follows: Think the vector V as the following 2 by 2 matrix

$$\begin{pmatrix} v_1 + iv_2 & \dots \\ v_4 - iv_3 & \dots \end{pmatrix}$$

whose first column is the components of V and second column may be anything. Multiply A with this matrix, consider the first column of the resulting matrix. The map $j : SU(1, 1) \rightarrow H$ by $j(A) = (A, B)$ is an injective group homomorphism, where $B = \begin{pmatrix} 1 & 0 \\ 0 & \det(A) \end{pmatrix}$.

Define new transition functions $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H$ by $\tilde{g}_{\alpha\beta} = j \circ g_{\alpha\beta}$. It is clear that these functions satisfy cocycle condition. On the other hand, let $x \in U_\alpha \cap U_\beta$ be any point and say $A = g_{\alpha\beta}(x)$. We obtain following identity

$$Ad(j(A))(V) = A \begin{pmatrix} v_1 + iv_2 & v_3 + iv_4 \\ v_4 - iv_3 & v_1 + iv_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \det(A) \end{pmatrix}^{-1} = A \begin{pmatrix} v_1 + iv_2 & \dots \\ v_3 - iv_4 & \dots \end{pmatrix}.$$

It means that the commuting relation $Ad \circ \tilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x)$ holds for each $x \in U_\alpha \cap U_\beta$. This completes the proof. \square

For the Riemannian analogy of this theorem and other concepts see [13].

3.4 Connection 1-Form on $P_{Spin_+^c(2,2)}$

If M is a pseudo-Riemannian $spin^c$ manifold, then by using the map

$$l : Spin_+^c(2, 2) \rightarrow S^1, l([g, z]) = z^2,$$

we can construct an associated principal S^1 -bundle

$$P_{S^1} = P_{Spin_+^c(2,2)} \times_l S^1.$$

Let ∇ be the Levi-Civita covariant derivative associated to g on M . Then it is known that the Levi-Civita covariant derivative ∇ determines an $so(2, 2)$ -valued connection 1-form ω on the principal bundle $P_{SO_+(2,2)}$. The connection 1-form ω can be expressed locally

$$\omega_U = \sum_{i < j} \omega_{ij} E_{ij},$$

where $\{e_1, e_2, e_3, e_4\}$ is a local orthonormal frame on open set $U \subset M$ and $\omega_{ij} = \varepsilon_{ij} g(\nabla e_i, e_j)$. Take an $i\mathbb{R}$ -valued connection 1-form A on S^1 -principal

bundle P_{S^1} . Now we can define an $so(2, 2) \oplus i\mathbb{R}$ -valued connection 1-form on the principal bundle $P_{SO_+(2,2)} \tilde{\times} P_{S^1}$ (the fibre product bundle):

$$\omega \times A : T(P_{SO_+(2,2)} \tilde{\times} P_{S^1}) \rightarrow so(2, 2) \oplus i\mathbb{R}.$$

This connection can be lift to a connection 1-form Z^A in the principal bundle $P_{Spin_+^c(2,2)}$ via the 2-fold covering $\pi : P_{Spin_+^c(2,2)} \rightarrow P_{SO_+(2,2)} \tilde{\times} P_{S^1}$ and the following diagram commutes:

$$\begin{array}{ccc} T(P_{Spin_+^c(2,2)}) & \xrightarrow{Z^A} & Lie(Spin_+^c(2,2)) \cong spin(2,2) \oplus i\mathbb{R} \\ d\pi \downarrow & & \downarrow \xi_* \\ T(P_{SO_+(2,2)} \tilde{\times} P_{S^1}) & \xrightarrow{\omega \times A} & so(2, 2) \oplus i\mathbb{R} \end{array}$$

where $\xi_* : Lie(Spin_+^c(2,2)) \rightarrow so(2, 2) \oplus i\mathbb{R}$ is the differential of the 2-fold covering

$$\xi = (\lambda, l) : Spin_+^c(2, 2) \rightarrow SO_+(2, 2) \times S^1.$$

4 Spinor bundle

Let $(P_{Spin_+^c(2,2)}, \Lambda)$ be a pseudo-Riemannian spin^c structure on M . If we consider the $Spin_+^c(2, 2)$ representation

$$\kappa : Spin_+^c(2, 2) \rightarrow Aut(\Delta_{2,2})$$

then we can construct a new associated complex vector bundle

$$S = P_{Spin_+^c(2,2)} \times_{\kappa} \Delta_{2,2}.$$

This complex vector bundle is called spinor bundle for a given spin^c structure on M and sections of S are called spinor fields. One can obtain a covariant derivative operator ∇^A on S by using the connection 1-form Z^A and a local expression of ∇^A is

$$\nabla^A \Psi = d\Psi + \frac{1}{2} \sum_{i < j} \varepsilon_i \varepsilon_j w_{ij} \kappa(e_i e_j) \Psi + \frac{1}{2} A \Psi,$$

where $\varepsilon_i = g(e_i, e_i)$ and Ψ is a local section of S over the open set $U \subset M$ (see [5, 7]).

The composite map $\tau \circ \lambda : Spin_+^c(2, 2) \rightarrow Aut(\mathbb{R}^4)$ is a representation of $Spin_+^c(2, 2)$ on \mathbb{R}^4 and gives

$$TM \cong P_{Spin_+^c(2,2)} \times_{\tau \circ \lambda} \mathbb{R}^4,$$

where $\tau : SO_+(2, 2) \rightarrow Aut(\mathbb{R}^4)$ is the canonical representation. Such interpretations of tangent bundle enable us to product the elements of spinor bundle with tangent vectors by the formula

$$[p, v] \cdot [p, \psi] = [p, \kappa(v)\psi]$$

where $p \in P_{Spin_+^c(2,2)}$, $v \in \mathbb{R}^4$, $\psi \in \mathbb{C}^4$. This product is bilinear and we extend it to the tensor product space

$$\begin{aligned} TM \otimes S &\rightarrow S \\ [p, v] \otimes [p, \psi] &\mapsto [p, \kappa(v)\psi], \end{aligned}$$

and denote it as a map $\kappa : TM \otimes S \rightarrow S$ and call it Clifford multiplication. Also we obtain a bundle map

$$\kappa : TM \rightarrow End(S).$$

Some authors call the bundle map κ as the $spin^c$ structure ([15]). Generally the Clifford multiplication $\kappa(X)(\Psi)$ is denoted by $X \cdot \Psi$. One can endow S with an indefinite Hermitian inner product by using the inner product on $\Delta_{2,2}$ and denote it again by $\langle \cdot, \cdot \rangle_{\Delta_{2,2}}$. The covariant derivative operator ∇^A is compatible with $\langle \cdot, \cdot \rangle_{\Delta_{2,2}}$ and Clifford multiplication in the following sense (see [7]):

Proposition 2. For all $X, Y \in \Gamma(TM)$ and $\Psi, \Psi_1, \Psi_2 \in \Gamma(S)$,

1. $\langle X \cdot \Psi_1, \Psi_2 \rangle_{\Delta_{2,2}} = (-1) \langle \Psi_1, X \cdot \Psi_2 \rangle_{\Delta_{2,2}}$,
2. $\nabla_Y^A(X \cdot \Psi) = X \cdot \nabla_Y^A(\Psi) + (\nabla_Y X) \cdot \Psi$,
3. $X \langle \Psi_1, \Psi_2 \rangle_{\Delta_{2,2}} = \langle \nabla_X^A \Psi_1, \Psi_2 \rangle_{\Delta_{2,2}} + \langle \Psi_1, \nabla_X^A \Psi_2 \rangle_{\Delta_{2,2}}$.

5 Seiberg-Witten Like Equations on pseudo-Riemannian $spin^c$ manifolds

The spinor bundle S splits into the sum of subbundles S^+, S^- :

$$S = S^+ \oplus S^-, \quad S^\pm = P_{Spin_+^c(2,2)} \times_{\kappa^\pm} \Delta_{2,2}^\pm.$$

The subbundles S^\pm can be endowed with indefinite Hermitian inner product by Proposition 2. The indefinite Hermitian inner product on

$$S^+ = P_{\text{Spin}_+^c(2,2)} \times_{\kappa^+} \Delta_{2,2}^+$$

is crucial for the interpretation of second Seiberg-Witten equation on M . Since κ^+ takes value in $U(1,1)$, we can endow S^+ with an indefinite Hermitian inner product of type $(1,1)$ and we denote it by $\langle, \rangle_{1,1}$.

Moreover the covariant derivative operator ∇^A on S preserves the subbundles S^+ and S^- . So ∇^A induces covariant derivative operators on these subbundles and we denote both of them with same symbol ∇^A .

5.1 The Dirac Equation

Now we want to define a Dirac operator on S . Note that the covariant derivative operator ∇^A can be thought as a linear map

$$\nabla^A : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)$$

satisfying the Leibnitz rule:

$$\nabla^A(f\Psi) = (df) \otimes \Psi + f\nabla^A\Psi.$$

Definition 1. *The composite map*

$$D_A = \kappa \circ \nabla^A : \Gamma(S) \xrightarrow{\nabla^A} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow{\kappa} \Gamma(S)$$

is called *Dirac operator on pseudo-Riemannian spin^c manifold M* .

In a space and time oriented local orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the covariant derivative operator ∇^A can be written as

$$\nabla^A\Psi = \sum_{i=1}^4 \varepsilon_i e_i^* \otimes \nabla_{e_i}^A \Psi.$$

Then a local expression of D_A is

$$D_A\Psi = \sum_{i=1}^4 \varepsilon_i e_i \cdot \nabla_{e_i}^A \Psi.$$

Obviously the operator D_A is first order differential operator. The Dirac operator splits into two pieces $D_A = D_A^+ \oplus D_A^-$ with respect to the decomposition $S = S^+ \oplus S^-$, where $D_A^+ : \Gamma(S^+) \rightarrow \Gamma(S^-)$ and $D_A^- : \Gamma(S^-) \rightarrow \Gamma(S^+)$.

We are ready to express the first Seiberg-Witten equation, the Dirac equation, on a pseudo-Riemannian manifold with neutral metric. The first Seiberg-Witten equation associated to the pair (A, Ψ) is

$$D_A^+ \Psi = 0 \quad (2)$$

where A is an $i\mathbb{R}$ -valued connection 1-form on the principal bundle P_{S^1} and Ψ is a positive spinor field on M , i.e. a section of S^+ .

5.2 The Curvature Equation

We need some other concepts for the second Seiberg-Witten equation. We consider the situation in local form firstly. We can define an action of the space of 2-forms $\Lambda^2(\mathbb{R}^{2,2})^*$ on the spinor space S . Let C_2 be the set of the second order elements of the Clifford algebra $Cl_{2,2}$ and consider the linear map

$$\begin{aligned} \Lambda^2(\mathbb{R}^{2,2})^* &\rightarrow C_2 \\ \eta = \sum_{i < j} \eta_{ij} e^i \wedge e^j &\mapsto \sum_{i < j} \varepsilon_i \varepsilon_j \eta_{ij} e_i e_j \end{aligned}$$

where $\varepsilon_i = g(e_i, e_i)$. If we compose this map with the spinor representation κ , then we obtain a map $\rho : \Lambda^2(\mathbb{R}^{2,2})^* \rightarrow \text{End}(\mathbb{C}^4)$ by

$$\rho\left(\sum_{i < j} \eta_{ij} e^i \wedge e^j\right) = \sum_{i < j} \varepsilon_i \varepsilon_j \eta_{ij} \kappa(e_i) \kappa(e_j).$$

The half-spinor spaces $\Delta_{2,2}^\pm$ are invariant under $\rho(\eta)$ for every $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$, so we obtain the following maps by restriction

$$\rho^\pm(\eta) = \rho(\eta)|_{S^\pm}.$$

Now we calculate the explicit forms of the maps $\rho(\eta)$ and $\rho(\eta)^\pm$ for arbitrary 2-form $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$.

$$\begin{aligned} \rho(\eta) &= \rho\left(\sum_{i < j} \eta_{ij} e^i \wedge e^j\right) \\ &= \eta_{12} \kappa(e_1) \kappa(e_2) - \eta_{13} \kappa(e_1) \kappa(e_3) - \eta_{14} \kappa(e_1) \kappa(e_4) - \eta_{23} \kappa(e_2) \kappa(e_3) \\ &\quad - \eta_{24} \kappa(e_2) \kappa(e_4) + \eta_{34} \kappa(e_3) \kappa(e_4) \end{aligned}$$

The left upper block of $\rho(\eta)$ represents $\rho^+(\eta)$, so it is given by

$$\begin{aligned} \rho^+(\eta) &= (\eta_{12} + \eta_{34})(i\sigma_3) + (-\eta_{13} - \eta_{24})(-\sigma_2) + (-\eta_{14} + \eta_{23})\sigma_1 \\ &= \begin{pmatrix} i(\eta_{12} + \eta_{34}) & -\eta_{14} + \eta_{23} - i(\eta_{13} + \eta_{24}) \\ -\eta_{14} + \eta_{23} + i(\eta_{13} + \eta_{24}) & -i(\eta_{12} + \eta_{34}) \end{pmatrix}, \end{aligned}$$

similarly the right lower block of $\rho(\eta)$ represents $\rho^-(\eta)$, so it is given by

$$\begin{aligned} \rho^-(\eta) &= (-\eta_{12} + \eta_{34})(i\sigma_3) + (\eta_{13} - \eta_{24})(-\sigma_2) + (\eta_{14} + \eta_{23})\sigma_1 \\ &= \begin{pmatrix} i(-\eta_{12} + \eta_{34}) & \eta_{14} + \eta_{23} + i(\eta_{13} - \eta_{24}) \\ \eta_{14} + \eta_{23} - i(\eta_{13} - \eta_{24}) & -i(-\eta_{12} + \eta_{34}) \end{pmatrix}. \end{aligned}$$

Proposition 3. *Let $\eta \in \Lambda^2(\mathbb{R}^{2,2})^*$ be a 2-form, then*

- i) η is anti-self-dual if and only if $\rho^+(\eta) = 0$.*
- ii) η is self-dual if and only if $\rho^-(\eta) = 0$.*
- iii) The space of self-dual 2-forms Λ^+ is isomorphic to $su(1,1)$*
- iv) The space of complex valued self-dual 2-forms $\Lambda^+ \otimes \mathbb{C}$ is isomorphic to $End_0(\Delta_{2,2}^+)$*

Since M is a spin^c manifold, globalizing above concepts is possible. We pointed out the global map, a bundle map, $\kappa : TM \rightarrow End(S)$ in Section 4, similarly we can define bundle map

$$\rho : \Lambda^2(M) \rightarrow End(S)$$

and complexified map

$$\rho : \Lambda^2(M) \otimes \mathbb{C} \rightarrow End(S).$$

The restriction of this map to the complex valued self-dual 2-forms gives the following bundle map

$$\rho^+ : \Lambda^+ \otimes \mathbb{C} \rightarrow End_0(S^+)$$

where $End_0(S)$ denotes the space of traceless endomorphisms of the bundle S^+ . Now we can write down the second Seiberg-Witten equation. Let A be an $i\mathbb{R}$ -valued connection 1-form on the S^1 principal bundle P_{S^1} and F_A be its curvature 2-form, which is $i\mathbb{R}$ valued 2-form on P_{S^1} . It is known that such curvature 2-forms are in one to one correspondence with the $i\mathbb{R}$ -valued 2-forms on M (see [5]). We denote the corresponding 2-form on M with the same symbol F_A . Let F_A^+ be the self-dual part of F_A , then $\rho^+(F_A^+)$ is a traceless endomorphism of the bundle S^+ . On the other hand any positive spinor field Ψ determines an endomorphism $\Psi\Psi^*$ of S^+ by the formula

$$(\Psi\Psi^*)(\Phi) = \langle \Psi, \Phi \rangle_{1,1} \Psi$$

where $\langle, \rangle_{1,1}$ is indefinite Hermitian inner product on S^+ and Φ is a spinor field on S^+ . The traceless part of $\Psi\Psi^*$ is denoted by $(\Psi\Psi^*)_0$. Then the second Seiberg-Witten equation for the pair (A, Ψ) is

$$\rho^+(F_A^+) = (\Psi\Psi^*)_0. \tag{3}$$

5.3 Seiberg-Witten Equations on $\mathbb{R}^{2,2}$

Now we write down Seiberg-Witten equations on 4-dimensional flat space with neutral metric. Explicit interpretations of original Seiberg-Witten equations on flat Euclidean flat space \mathbb{R}^4 and some properties of them can be found in [14] and [15]). For the explicit interpretations of these equations in the neutral case we use the spinor representation κ given in Section 2. In this case $S = \mathbb{R}^{2,2} \times \Delta_{2,2}$, $S^+ = \mathbb{R}^{2,2} \times \Delta_{2,2}^+$ and $S^- = \mathbb{R}^{2,2} \times \Delta_{2,2}^-$. The sections of the subbundles S^\pm can be expressed as follows

$$\begin{aligned}\Gamma(S^+) &= \{(\psi_1, \psi_2, 0, 0) \mid \psi_1, \psi_2 \in C^\infty(\mathbb{R}^{2,2}, \mathbb{C})\}, \\ \Gamma(S^-) &= \{(0, 0, \psi_3, \psi_4) \mid \psi_3, \psi_4 \in C^\infty(\mathbb{R}^{2,2}, \mathbb{C})\}.\end{aligned}$$

Since $P_{S^1} = \mathbb{R}^{2,2} \times S^1$, the $i\mathbb{R}$ -valued connection 1-form on P_{S^1} is given by

$$A = \sum_{j=1}^4 A_j dx_j \in \Omega^1(\mathbb{R}^{2,2}, i\mathbb{R})$$

where $A_j : \mathbb{R}^{2,2} \rightarrow i\mathbb{R}$ are smooth maps. The associated spin^c connection $\nabla = \nabla^A$ on $\mathbb{R}^{2,2}$ is given by

$$\nabla_j \Psi = \frac{\partial \Psi}{\partial x_j} + A_j \Psi,$$

where $\Psi : \mathbb{R}^{2,2} \rightarrow \mathbb{C}^2$. Then the Dirac equation in flat case is given by

$$\begin{aligned}D_A \Psi &= e_1 \cdot \nabla_{e_1} \Psi + e_2 \cdot \nabla_{e_2} \Psi - e_3 \cdot \nabla_{e_3} \Psi - e_4 \cdot \nabla_{e_4} \Psi \\ &= \sum_{i=1}^4 \varepsilon_i \kappa(e_i) (\nabla_{e_i} \Psi) \\ &= \sum_{i=1}^4 \kappa(e_i) \begin{pmatrix} \frac{\partial \psi_1}{\partial x_i} + A_i \psi_1 \\ \frac{\partial \psi_2}{\partial x_i} + A_i \psi_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} + A_1 \psi_1 + i(\frac{\partial \psi_2}{\partial x_2} + A_2 \psi_2) + i(\frac{\partial \psi_2}{\partial x_3} + A_3 \psi_2) + \frac{\partial \psi_2}{\partial x_4} + A_4 \psi_2 \\ \frac{\partial \psi_2}{\partial x_1} + A_1 \psi_2 - i(\frac{\partial \psi_1}{\partial x_2} + A_2 \psi_1) - i(\frac{\partial \psi_1}{\partial x_3} + A_3 \psi_1) + \frac{\partial \psi_1}{\partial x_4} + A_4 \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

The explicit form of Dirac equation:

$$\begin{aligned}\frac{\partial \psi_1}{\partial x_1} + A_1 \psi_1 + i(\frac{\partial \psi_2}{\partial x_2} + A_2 \psi_2) + i(\frac{\partial \psi_2}{\partial x_3} + A_3 \psi_2) + \frac{\partial \psi_2}{\partial x_4} + A_4 \psi_2 &= 0 \\ \frac{\partial \psi_2}{\partial x_1} + A_1 \psi_2 - i(\frac{\partial \psi_1}{\partial x_2} + A_2 \psi_1) - i(\frac{\partial \psi_1}{\partial x_3} + A_3 \psi_1) + \frac{\partial \psi_1}{\partial x_4} + A_4 \psi_1 &= 0.\end{aligned}$$

The curvature 2-form F_A is given by

$$F_A = dA = \sum_{i < j} F_{ij} e^i \wedge e^j \in \Omega^2(\mathbb{R}^{2,2}, i\mathbb{R}),$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ for $i, j = 1, \dots, 4$.

The matrix form of $\Psi\Psi^*$ with respect to the frame $E_1 = (1, 0)$, $E_2 = (0, 1)$ is given by

$$(\Psi\Psi^*) = \begin{pmatrix} |\psi_1|^2 & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -|\psi_2|^2 \end{pmatrix}.$$

The traceless part of $\Psi\Psi^*$ is

$$\begin{aligned} (\Psi\Psi^*)_0 &= \begin{pmatrix} |\psi_1|^2 & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -|\psi_2|^2 \end{pmatrix} - \frac{1}{2}|\psi|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) & -\psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & -\frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) \end{pmatrix} \end{aligned}$$

Now we can interpret the curvature equation by $\rho^+(F^+) = (\Psi\Psi^*)_0$. Then we obtain following set of equations

$$\begin{aligned} F_{12} + F_{34} &= -\frac{i}{2}(|\psi_1|^2 + |\psi_2|^2) \\ F_{23} - F_{14} &= \frac{1}{2}(\psi_1\psi_2 - \psi_1\overline{\psi_2}) \\ F_{13} + F_{34} &= -\frac{i}{2}(\psi_1\psi_2 + \psi_1\overline{\psi_2}) \end{aligned} \quad (4)$$

which are consistent and similar to the classical Seiberg-Witten equations on \mathbb{R}^4 with Euclidean metric.

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