



# Stability of a Class of Nonlinear Neutral Stochastic Differential Equations with Variable Time Delays

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## Abstract

In this paper, we study the mean square asymptotic stability of a class of generalized nonlinear neutral stochastic differential equations with variable time delays by using fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved which improves and generalizes some well-known results. Finally, two examples are given to illustrate our results.

## 1 Introduction

It is well known that stochastic differential equation plays a very important role in formulation and analysis in mechanical, electrical, control engineering, neural network, economic and social sciences. Stochastic delay differential equation, also known as stochastic functional differential equation, is a natural generalization of stochastic ordinary differential equation by allowing the coefficients to depend on the past values. Recently, the studies of stochastic differential equations have attracted the considerable attentions of scholars. Many interesting results concerned with stochastic differential equations have

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been obtained over the last few years (see, for example, [13, 19] and the references therein).

Liapunov's direct method has been successfully used to investigate stability problems in deterministic/stochastic differential equations and functional differential equations for more than one hundred years. However, there are many difficulties encountered in the study of stability by means of Liapunov's direct method. Recently, Burton and other scholars studied the stability for deterministic systems by using fixed point theory which overcame the difficulties encountered in the study of stability by means of Liapunov's direct method (see, for example, [2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 20, 23, 24] and the references therein).

Very recently, many scholars have begun to deal with the stability of stochastic delay differential equations by using fixed point theory (see, for example, [1, 2, 14, 15, 16, 17, 18, 21, 22]). More precisely, Appleby in [1] (also see [2], pp.315-328) considered the almost sure stability for some classical equations by splitting the stochastic differential equation into two equations, one is a fixed stochastic problem and the other is a deterministic stability problem with forcing function. Luo [14] studied the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of fixed point theory. Furthermore, Luo [15, 16], Luo and Taniguchi [18] used fixed point theory to study the exponential stability of mild solutions of stochastic partial differential equations with bounded delays and with infinite delays. Wu et al. [21, 22] applied fixed point theory to study the stability of a general linear neutral stochastic differential equation and a half-linear neutral stochastic differential equation with variable delays respectively. Luo [17] investigated the exponential stability for the classical stochastic Volterra-Levin equations by using fixed point theory.

Motivated by the previous works mentioned above, in this paper, we study the mean square asymptotic stability of a nonlinear neutral stochastic differential equation with variable delays by applying fixed point theory. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved. Two examples are given to illustrate our results. The results presented in this paper improve and generalize the main results in [3], [14], [23] and [24].

## 2 Main Results

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  be a complete filtered probability space and  $W(t)$  denote a one-dimensional standard Brownian motion defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  such that  $\{\mathcal{F}_t\}_{t \geq 0}$  is the natural filtration of  $W(t)$ . Let  $a(t), b(t), \bar{b}(t), c(t), e(t),$

$q(t) \in C(R^+, R)$  and  $\tau(t), \delta(t) \in C(R^+, R^+)$  with  $t - \tau(t) \rightarrow \infty$  and  $t - \delta(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Here  $C(S_1, S_2)$  denotes the set of all continuous function  $\phi : S_1 \rightarrow S_2$  with the supremum norm  $\|\cdot\|$ .

Burton [3] and Zhang [23] studied the equation

$$x'(t) = -\bar{b}(t)x(t - \tau(t)) \tag{1}$$

and proved the following theorems.

**Theorem A.** (Burton [3]) Suppose that  $\tau(t) = r$  and there exists a constant  $\alpha < 1$  such that

$$\int_{t-r}^t |\bar{b}(s+r)| ds + \int_0^t |\bar{b}(s+r)| e^{-\int_s^t \bar{b}(u+r) du} \int_{s-r}^s |\bar{b}(u+r)| duds \leq \alpha$$

for all  $t \geq 0$  and  $\int_0^\infty \bar{b}(s) ds = \infty$ . Then, for every continuous initial function  $\phi : [-r, 0] \rightarrow R$ , the solution  $x(t) = x(t, 0, \phi)$  of (1) is bounded and tends to zero as  $t \rightarrow \infty$ .

**Theorem B.** (Zhang [23]) Suppose that  $\tau$  is differentiable, the inverse function  $g(t)$  of  $t - \tau(t)$  exists, and there exists a constant  $\alpha \in (0, 1)$  such that for  $t \geq 0$ ,  $\liminf_{t \rightarrow \infty} \int_0^t \bar{b}(g(s)) ds > -\infty$  and

$$\begin{aligned} & \int_{t-\tau(t)}^t |\bar{b}(g(s))| ds + \int_0^t e^{-\int_s^t \bar{b}(g(u)) du} |\bar{b}(s)| |\tau'(s)| ds \\ & + \int_0^t e^{-\int_s^t \bar{b}(g(u)) du} |\bar{b}(g(s))| \int_{s-\tau(s)}^s |\bar{b}(g(v))| dv ds \leq \alpha < 1. \end{aligned} \tag{2}$$

Then the zero solution of (1) is asymptotically stable if and only if  $\int_0^t \bar{b}(g(s)) ds \rightarrow \infty$ , as  $t \rightarrow \infty$ .

Obviously, Theorem B improves Theorem A. Recently, Zhang [24] studied the following half-linear equation

$$x'(t) = -a(t)x(t) + b(t)g(x(t - \tau(t))) \tag{3}$$

where  $g : R \rightarrow R$  is continuous and obtained Theorem C.

**Theorem C.** (Zhang [24]) Suppose that  $\tau(t) \geq 0$  such that for  $t \geq 0$ ,  $t - \tau(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , and there exists a constant  $L > 0$ , for  $|x|, |y| \leq L$ ,  $|g(x) - g(y)| \leq |x - y|$  and  $q(0) = 0$ . For  $t > 0$ ,  $\liminf_{t \rightarrow \infty} \int_0^t a(s) ds > -\infty$  and

$$\sup_{t \geq 0} \int_0^t e^{-\int_s^t a(u) du} |b(s)| ds < 1. \tag{4}$$

Then the zero solution of (3) is asymptotically stable if and only if  $\int_0^t a(s) ds \rightarrow \infty$ , as  $t \rightarrow \infty$ .

Recently, Luo [14] considered a linear neutral stochastic differential equation

$$d[x(t) - q(t)x(t - \tau(t))] = [a(t)x(t) + b(t)x(t - \tau(t))]dt + [c(t)x(t) + e(t)x(t - \delta(t))]dW(t) \quad (5)$$

and obtained Theorem D.

**Theorem D.** (Luo [14]) Let  $\tau(t)$  be differentiable. Assume that there exists a constant  $\alpha \in (0, 1)$  and a continuous function  $h(t) : [0, \infty) \rightarrow R$  such that for  $t \geq 0$ ,  $\liminf_{t \rightarrow \infty} \int_0^t h(s)ds > -\infty$  and

$$\begin{aligned} & |q(t)| + \int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)| du ds \\ & + \int_0^t e^{-\int_s^t h(u)du} |(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s)) \\ & + b(s) - q(s)h(s)| ds + \int_{t-\tau(t)}^t |a(s) + h(s)| ds \\ & + \left( \int_0^t e^{-2\int_s^t h(u)du} (|c(s)| + |e(s)|)^2 ds \right)^{\frac{1}{2}} \leq \alpha < 1. \end{aligned}$$

Then the zero solution of (5) is mean square asymptotically stable if and only if  $\int_0^t h(s)ds \rightarrow \infty$ , as  $t \rightarrow \infty$ .

Very recently, Wu et al. [21, 22] generalized Theorems B, C, and D to a general linear neutral stochastic differential equation and to a half-linear neutral stochastic differential equation, respectively. In general, time delay and system uncertainty are commonly encountered and are often sources of instability (see [12]). Thus, it should be interesting to consider the nonlinear stochastic differential equation and study the stability of nonlinear stochastic differential equation with variable time delays.

In this paper, we consider a class of nonlinear neutral stochastic differential equations,

$$d[x(t) - k(t, x(t - \tau(t)))] = [a(t)x(t) + f(t, x(t), x(t - \tau(t)))]dt + [c(t)x(t) + g(t, x(t), x(t - \delta(t)))]dW(t) \quad (6)$$

with the initial condition

$$x(s) = \phi(s) \quad \text{for } s \in [m(0), 0],$$

where  $f, g : [0, \infty) \times R \times R \rightarrow R$  and  $k : [0, \infty) \times R \rightarrow R$  are continuous,  $\phi \in C([m(0), 0], R)$ ,  $x : [m(0), \infty) \times \Omega \rightarrow R$  and

$$m(0) = \min \{ \inf \{ s - \tau(s), s \geq 0 \}, \inf \{ s - \delta(s), s \geq 0 \} \} \leq 0.$$

If  $f(t, x(t), x(t - \tau(t))) = b(t)x(t - \tau(t))$ ,  $g(t, x(t), x(t - \delta(t))) = e(t)x(t - \delta(t))$ ,  $k(t, x(t - \tau(t))) = q(t)x(t - \tau(t))$ , then it is easy to see that (6) reduces to (5). Thus, (6) includes (1), (3) and (5) as special cases. And so, the main aim of this paper is to generalize Theorems B, C and D to apply to (6).

For any  $\phi \in C([m(0), 0], R)$ , we define  $\|\phi\| = \sup_{s \in [m(0), 0]} |\phi(s)|$ . For each  $\lambda > 0$ , we define  $C(\lambda) := \{\phi \in C([m(0), 0], R) : \|\phi\| \leq \lambda\}$ . Denote by  $F$  the Banach space of all  $\mathcal{F}$ -adapted processes  $\psi(t, \omega) : [m(0), \infty) \times \Omega \rightarrow R$  which are almost surely continuous in  $t$  with norm

$$\|\psi\|_F = \left\{ E \left( \sup_{s \geq m(0)} |\psi(s, \omega)|^2 \right) \right\}^{\frac{1}{2}}.$$

Further, we define  $F(\lambda) = \{\psi \in F : \|\psi\|_F \leq \lambda\}$  for each  $\lambda > 0$ . Let  $\|\psi\|_F^{[r, t]} = \{E(\sup_{s \in [r, t]} |\psi(s, \omega)|^2)\}^{\frac{1}{2}}$  for  $r < t$ . Then  $\|\psi\|_F^{[0, \infty)} = \{E(\sup_{s \geq 0} |\psi(s, \omega)|^2)\}^{\frac{1}{2}}$ .

**Theorem 2.1.** *Suppose that  $\tau$  is differentiable, and there exist continuous functions  $h(t) : [0, \infty) \rightarrow R$ ,  $l(t), m(t), n(t) : [0, \infty) \rightarrow R^+$  and constants  $L > 0$ ,  $\alpha \in (0, 1)$  such that*

(i)  $\liminf_{t \rightarrow \infty} \int_0^t h(s) ds > -\infty;$

(ii) for any  $t \geq 0$ ,

$$\begin{aligned} n(t) &+ \int_0^t e^{-\int_s^t h(u) du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)| du ds \\ &+ \int_0^t e^{-\int_s^t h(u) du} \left( |(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s))| \right. \\ &\quad \left. + l(s) + m(s) + |h(s)|n(s) \right) ds + \int_{t-\tau(t)}^t |a(s) + h(s)| ds \\ &+ 2 \left( \int_0^t e^{-2\int_s^t h(u) du} (|c(s)| + l(s) + m(s))^2 ds \right)^{\frac{1}{2}} \leq \alpha < 1; \end{aligned}$$

(iii) for any  $t \geq 0$ ,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \vee |g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq l(t)|x - \bar{x}| + m(t)|y - \bar{y}|$$

and

$$|k(t, x) - k(t, \bar{x})| \leq n(t)|x - \bar{x}|$$

for all  $x, \bar{x}, y, \bar{y} \in F(L)$  with  $f(t, 0, 0) = g(t, 0, 0) = k(t, 0) = 0$ .

Then the zero solution of (6) is mean square asymptotically stable if and only if

$$\int_0^t h(s)ds \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (7)$$

*Proof.* At first, we suppose that (7) holds. Choose  $\delta > 0$ ,  $\delta < L$  such that  $2\delta K + \alpha L \leq L$ , where  $K = \sup_{t \geq 0} \{e^{-\int_0^t h(s)ds}\}$ . Let  $\phi \in C(\delta)$  and set

$$S = \{x : [m(0), \infty) \times \Omega \rightarrow R \mid x(t, \omega) = \phi(t) \text{ for } t \in [m(0), 0], \\ x(t, \omega) \in F(L) \text{ for } t \geq 0, E|x(t, \omega)|^2 \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

Then it is easy to check that  $S$  is a closed subset of  $F$ . From the definitions of  $\|\cdot\|$  and  $\|\cdot\|_F$ , for any  $x \in S$  and  $t > 0$ ,

$$\|x\|_F = \max\{\|\phi\|, \|x\|_F^{[0, \infty)}\} \leq L. \quad (8)$$

Define an operator  $P : S \rightarrow S$  by  $(Px)(t) = \phi(t)$  for  $t \in [m(0), 0]$  and for  $t \geq 0$ ,

$$\begin{aligned} & (Px)(t) \\ = & \left( \phi(0) - k(0, \phi(-\tau(0))) - \int_{-\tau(0)}^0 (a(s) + h(s))\phi(s)ds \right) e^{-\int_0^t h(s)ds} \\ & + k(t, x(t - \tau(t))) + \int_{t-\tau(t)}^t (a(s) + h(s))x(s)ds \\ & + \int_0^t e^{-\int_s^t h(u)du} \left( (a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s))x(s - \tau(s)) \right. \\ & \quad \left. + f(s, x(s), x(s - \tau(s))) - h(s)k(s, x(s - \tau(s))) \right) ds \\ & - \int_0^t e^{-\int_s^t h(u)du} h(s) \left( \int_{s-\tau(s)}^s (a(u) + h(u))x(u)du \right) ds \\ & + \int_0^t e^{-\int_s^t h(u)du} \left( c(s)x(s) + g(s, x(s), x(s - \delta(s))) \right) dW(s) \\ := & \sum_{i=1}^5 I_i(t). \end{aligned} \quad (9)$$

Now, we show the mean square continuity of  $P$  on  $[0, \infty)$ . Let  $x \in S$ ,  $T_1 > 0$  and  $|r|$  be sufficiently small. Then

$$E|(Px)(T_1 + r) - (Px)(T_1)|^2 \leq 5 \sum_{i=1}^5 E|I_i(T_1 + r) - I_i(T_1)|^2.$$

It is easy to verify that

$$E|I_i(T_1 + r) - I_i(T_1)|^2 \rightarrow 0, \quad \text{as } r \rightarrow 0, \quad i = 1, 2, 3, 4.$$

From the last term  $I_5$  in (9), we have

$$\begin{aligned} & E|I_5(T_1 + r) - I_5(T_1)|^2 \\ = & E \left| \int_0^{T_1} e^{-\int_s^{T_1} h(u)du} (e^{-\int_{T_1}^{T_1+r} h(u)du} - 1) \right. \\ & \cdot (c(s)x(s) + g(s, x(s), x(s - \delta(s)))) dW(s) \\ & \left. + \int_{T_1}^{T_1+r} e^{-\int_s^{T_1+r} h(u)du} (c(s)x(s) + g(s, x(s), x(s - \delta(s)))) dW(s) \right|^2 \\ \leq & 2E \int_0^{T_1} e^{-2\int_s^{T_1} h(u)du} |e^{-\int_{T_1}^{T_1+r} h(u)du} - 1|^2 \\ & \cdot |c(s)x(s) + g(s, x(s), x(s - \delta(s)))|^2 ds \\ & + 2E \int_{T_1}^{T_1+r} e^{-2\int_s^{T_1+r} h(u)du} |c(s)x(s) + g(s, x(s), x(s - \delta(s)))|^2 ds \\ \rightarrow & 0, \text{ as } r \rightarrow 0. \end{aligned}$$

Therefore,  $P$  is mean square continuous on  $[0, \infty)$ .

Next, we verify that  $\|Px\|_F \leq L$ . As  $\phi \in C(\delta)$  and  $x \in S$ ,

$$\begin{aligned} \|Px\|_F^{[0, \infty)} &= \left\{ E \left( \sup_{s \geq 0} |Px(s)|^2 \right) \right\}^{\frac{1}{2}} = \left\{ E \left( \sup_{s \geq 0} \left| \sum_{i=1}^5 I_i(s) \right|^2 \right) \right\}^{\frac{1}{2}} \\ &\leq \sum_{i=1}^5 \left\{ E \left( \sup_{s \geq 0} |I_i(s)|^2 \right) \right\}^{\frac{1}{2}}. \end{aligned} \tag{10}$$

By condition (iii), we have

$$|f(t, x, y)| \vee |g(t, x, y)| \leq l(t)|x| + m(t)|y| \quad \text{and} \quad |k(t, x)| \leq n(t)|x| \tag{11}$$

for all  $x, y \in F(L)$ . It follows from (9), (10), (11), condition (ii) and Doob's

$L^p$ -inequality (see [11]) that

$$\begin{aligned}
& \|Px\|_F^{[0,\infty)} \\
\leq & \sup_{s \geq 0} \{e^{-\int_0^s h(v)dv}\} \cdot (|\phi(0)| + n(0) \cdot |\phi(-\tau(0))| \\
& + \int_{-\tau(0)}^0 |a(v) + h(v)| \cdot |\phi(v)|dv) \\
& + \left( E \sup_{s \geq 0} \left( n(s) \cdot |x(s - \tau(s))| + \int_{s-\tau(s)}^s |a(v) + h(v)| \cdot |x(v)|dv \right)^2 \right)^{\frac{1}{2}} \\
& + \left( E \sup_{s \geq 0} \left( \int_0^s e^{-\int_v^s h(u)du} (|(a(v - \tau(v)) + h(v - \tau(v)))(1 - \tau'(v))| \right. \right. \\
& \cdot |x(v - \tau(v))| + |f(v, x(v), x(v - \delta(v)))| + |h(v)||k(v, x(v - \tau(v)))|)dv)^2 \Big)^{\frac{1}{2}} \\
& + \left( E \sup_{s \geq 0} \left( \int_0^s e^{-\int_v^s h(u)du} |h(v)| \left( \int_{v-\tau(v)}^v |a(u) + h(u)| \cdot |x(u)|du \right) dv \right)^2 \right)^{\frac{1}{2}} \\
& + 2 \sup_{s \geq 0} \left( E \int_0^s e^{-2\int_v^s h(u)du} (|c(v)| \cdot |x(v)| + |g(v, x(v), x(v - \delta(v)))|)^2 dv \right)^{\frac{1}{2}} \\
\leq & \delta K \left( 1 + n(0) + \int_{-\tau(0)}^0 |a(v) + h(v)|dv \right) \\
& + \|x\|_F \cdot \sup_{s \geq 0} \left\{ n(s) + \int_{s-\tau(s)}^s |a(v) + h(v)|dv + \int_0^s e^{-\int_v^s h(u)du} \right. \\
& \cdot (|(a(v - \tau(v)) + h(v - \tau(v)))(1 - \tau'(v))| + l(v) + m(v) + |h(v)|n(v))dv \\
& + \int_0^s e^{-\int_v^s h(u)du} |h(v)| \left( \int_{v-\tau(v)}^v |a(u) + h(u)|du \right) dv \\
& \left. + 2 \left( \int_0^s e^{-2\int_v^s h(u)du} (|c(v)| + l(v) + m(v))^2 dv \right)^{\frac{1}{2}} \right\} \\
\leq & 2\delta K + \alpha L \leq L.
\end{aligned}$$

Further, from (8), we have

$$\|Px\|_F = \max\{\|\phi\|, \|Px\|_F^{[0,\infty)}\} \leq L.$$

Thirdly, we verify that  $E|(Px)(t)|^2 \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $E|x(t)|^2 \rightarrow 0$ ,  $t - \delta(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , for each  $\epsilon > 0$ , there exists a  $T_1 > 0$  such that  $s \geq T_1$  implies  $E|x(s)|^2 < \epsilon$  and  $E|x(s - \delta(s))|^2 < \epsilon$ . By condition (ii), for  $t \geq T_1$ ,



the last term  $I_5$  in (9) satisfies

$$\begin{aligned} & E|I_5(t)|^2 \\ \leq & E \int_0^{T_1} e^{-2 \int_s^t h(u)du} \left( |c(s)| \cdot |x(s)| + l(s)|x(s)| + m(s)|x(s - \delta(s))| \right)^2 ds \\ & + E \int_{T_1}^t e^{-2 \int_s^t h(u)du} \left( |c(s)| \cdot |x(s)| + l(s)|x(s)| + m(s)|x(s - \delta(s))| \right)^2 ds \\ \leq & (\|x\|_F^{[m(0), T_1]})^2 \int_0^{T_1} e^{-2 \int_s^t h(u)du} \left( |c(s)| + l(s) + m(s) \right)^2 ds + \alpha\epsilon \\ \leq & \|x\|_F^2 \cdot e^{-2 \int_{T_1}^t h(u)du} \int_0^{T_1} e^{-2 \int_s^{T_1} h(u)du} \left( |c(s)| + l(s) + m(s) \right)^2 ds + \alpha\epsilon \\ \leq & L^2 \alpha^2 e^{-2 \int_{T_1}^t h(u)du} + \alpha\epsilon. \end{aligned}$$

From (7), there exists  $T_2 > T_1$  such that  $L^2 \alpha^2 e^{-2 \int_{T_1}^t h(u)du} < \epsilon$  for  $t \geq T_2$ . Thus, for  $t \geq T_2$ ,  $E|I_5(t)|^2 < \epsilon + \alpha\epsilon$ . This proves that  $E|I_5(t)|^2 \rightarrow 0$ , as  $t \rightarrow \infty$ . Similarly, we can show that  $E|I_i(t)|^2 \rightarrow 0$ ,  $i = 1, 2, 3, 4$ , as  $t \rightarrow \infty$ . Thus,  $E|(Px)(t)|^2 \rightarrow 0$  as  $t \rightarrow \infty$ . Hence  $Px \in S$ .

Now we show that  $P : S \rightarrow S$  is a contraction mapping. For any  $x, y \in S$ , we have

$$\begin{aligned} & \|Px - Py\|_F \\ = & \left( E \sup_{s \geq m(0)} |(Px)(s) - (Py)(s)|^2 \right)^{\frac{1}{2}} \\ = & \left( E \sup_{s \geq 0} \left| k(s, x(s - \tau(s))) - k(s, y(s - \tau(s))) \right. \right. \\ & + \int_{s-\tau(s)}^s (a(v) + h(v))(x(v) - y(v))dv + \int_0^s e^{-\int_v^s h(u)du} \left( (a(v - \tau(v)) \right. \\ & \quad + h(v - \tau(v)))(1 - \tau'(v))(x(v - \tau(v)) - y(v - \tau(v))) \\ & \quad + f(v, x(v), x(v - \tau(v))) - f(v, y(v), y(v - \tau(v))) \\ & \quad \left. \left. - h(v) \left( k(v, x(v - \tau(v))) - k(v, y(v - \tau(v))) \right) \right) dv \right. \\ & - \int_0^s e^{-\int_v^s h(u)du} h(v) \left( \int_{v-\tau(v)}^v (a(u) + h(u))(x(u) - y(u))du \right) dv \\ & \left. + \int_0^s e^{-\int_v^s h(u)du} (c(v)(x(v) - y(v)) + g(v, x(v), x(v - \tau(v))) \right. \\ & \quad \left. - g(v, y(v), y(v - \tau(v)))) dW(v) \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \|x - y\|_F \cdot \sup_{s \geq 0} \left\{ n(s) + \int_{s-\tau(s)}^s |a(v) + h(v)| dv \right. \\
&\quad + \int_0^s e^{-\int_v^s h(u) du} |h(v)| \int_{v-\tau(v)}^v |a(u) + h(u)| du dv + \int_0^s e^{-\int_v^s h(u) du} \\
&\quad \cdot \left( |(a(v - \tau(v)) + h(v - \tau(v)))(1 - \tau'(v))| + l(v) + m(v) + |h(v)|n(v) \right) dv \\
&\quad \left. + 2 \left( \int_0^s e^{-2\int_v^s h(u) du} (|c(v)| + l(v) + m(v))^2 dv \right)^{\frac{1}{2}} \right\} \\
&\leq \alpha \|x - y\|_F.
\end{aligned}$$

Therefore,  $P : S \rightarrow S$  is contraction mapping and so  $P$  has a fixed point  $x \in S$ , which is a solution of (6) with  $x(s) = \phi(s)$  on  $[m(0), 0]$  and  $E|x(t)|^2 \rightarrow 0$  as  $t \rightarrow \infty$ .

To obtain the mean square asymptotic stability, we need to show that the zero solution of (6) is mean square stable. From (ii), we can choose  $\Delta > 0$  such that  $\alpha^2 + \Delta < 1$ . Thus, we can find a constant  $N > 0$  such that

$$\begin{aligned}
&\left( 1 + \frac{1}{N} \right) \left( n(t) + \int_0^t e^{-\int_s^t h(u) du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)| du ds \right. \\
&\quad + \int_{t-\tau(t)}^t |a(s) + h(s)| ds + \int_0^t e^{-\int_s^t h(u) du} \\
&\quad \cdot \left( |(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s))| + l(s) + m(s) + |h(s)|n(s) \right) ds \Big)^2 \\
&\quad + 4(1 + N) \int_0^t e^{-2\int_s^t h(u) du} (|c(s)| + l(s) + m(s))^2 ds \leq \alpha^2 + \Delta < 1. \quad (12)
\end{aligned}$$

Let  $\epsilon > 0$  and  $\epsilon < L$  be given and choose  $\delta_0 > 0$  and  $\delta_0 < \epsilon$  satisfying the following condition

$$4(1 + N)\delta_0^2 K^2 + (\alpha^2 + \Delta)\epsilon < \epsilon,$$

where  $N$  is defined in (12). If  $x(t) = x(t, 0, \phi)$  is a solution of (6) with  $\|\phi\| < \delta_0$ , then  $x(t) = (Px)(t)$  which is defined in (9). We claim that  $E|x(t)|^2 < \epsilon$  for all  $t \geq 0$ . Notice that  $x(t) = \phi(t)$  for  $t \in [m(0), 0]$ , so  $E|x(t)|^2, \|\phi(t)\|^2 < \epsilon$  for  $t \in [m(0), 0]$ . If there exists  $t^* > 0$  such that  $E|x(t^*)|^2 = \epsilon$  and  $E|x(t)|^2 < \epsilon$

for  $t \in [m(0), t^*]$ , then (9) and (12) imply that

$$\begin{aligned}
 & E|x(t^*)|^2 \\
 \leq & (1+N)\|\phi\|^2 \left(1+n(0) + \int_{-\tau(0)}^0 |a(s)+h(s)|ds\right)^2 e^{-2\int_0^{t^*} h(u)du} \\
 & + \epsilon \left(1 + \frac{1}{N}\right) \left(|q(t^*)| + \int_0^{t^*} e^{-\int_s^{t^*} h(u)du} \left(\int_{s-\tau(s)}^s |a(u)+h(u)|du\right) |h(s)|ds\right. \\
 & + \int_{t^*-\tau(t^*)}^{t^*} |a(s)+h(s)|ds + \int_0^{t^*} e^{-\int_s^{t^*} h(u)du} \\
 & \cdot \left. \left(|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))| + l(s) + m(s) + |h(s)|n(s)\right) ds\right)^2 \\
 & + \epsilon \int_0^{t^*} e^{-2\int_s^{t^*} h(u)du} (|c(s)| + l(s) + m(s))^2 ds \\
 \leq & (1+N)\delta_0^2 \left(1+n(0) + \int_{-\tau(0)}^0 |a(s)+h(s)|ds\right)^2 e^{-2\int_0^{t^*} h(u)du} + (\alpha^2 + \Delta)\epsilon \\
 < & \epsilon, \tag{13}
 \end{aligned}$$

which contradicts the definition of  $t^*$ . Thus, the zero solution of (6) is mean square asymptotically stable if (7) holds.

Conversely, we suppose that (7) fails. From condition (i), there exists a sequence  $\{t_n\}$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} \int_0^{t_n} h(u)du = \zeta$  for some  $\zeta \in R$ . Then, we can choose a constant  $J > 0$  satisfying  $\int_0^{t_n} h(u)du \in [-J, J]$  for all  $n \geq 1$ . Denote

$$\begin{aligned}
 \omega(s) := & |(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))| \\
 & + l(s) + m(s) + |h(s)|n(s) + |h(s)| \int_{s-\tau(s)}^s |a(u)+h(u)|du,
 \end{aligned}$$

for all  $s \geq 0$ . From condition (ii), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} h(u)du} \omega(s)ds \leq \alpha,$$

which implies

$$\int_0^{t_n} e^{\int_0^s h(u)du} \omega(s)ds \leq \alpha e^{\int_0^{t_n} h(u)du} \leq e^J.$$

Therefore, the sequence  $\{\int_0^{t_n} e^{\int_0^s h(u)du} \omega(s)ds\}$  has a convergent subsequence.

Without loss of generality, we can assume that  $\lim_{n \rightarrow \infty} \int_0^{t_n} e^{\int_0^s h(u) du} \omega(s) ds = \gamma$  for some  $\gamma > 0$ . Let  $m$  be an integer such that

$$\int_{t_m}^{t_n} e^{\int_0^s h(u) du} \omega(s) ds < \frac{\delta_1}{8K} \quad (14)$$

for all  $n \geq m$ , where  $0 < \delta_1 < 1$  satisfies  $8\delta_1^2 K^2 e^{2J} + (\alpha^2 + \Delta) < 1$ .

Now, we consider the solution  $x(t) = x(t, t_m, \phi)$  of (6) with  $\|\phi(t_m)\| = \delta_1$  and  $\|\phi(t)\| < \delta_1$  for  $t < t_m$ . By the similar method in (13), we have  $E|x(t)|^2 < 1$  for  $t \geq t_m$ . We may choose  $\phi$  so that

$$\begin{aligned} G(t_m) &:= \phi(t_m) - k(t_m, \phi(t_m - \tau(t_m))) \\ &\quad - \int_{t_m - \tau(t_m)}^{t_m} (a(s) + h(s))\phi(s) ds \geq \frac{\delta_1}{2}. \end{aligned} \quad (15)$$

It follows from (9), (14) and (15) with  $x(t) = (Px)(t)$  that for  $n \geq m$ ,

$$\begin{aligned} & E \left| x(t_n) - k(t_n, x(t_n - \tau(t_n))) - \int_{t_n - \tau(t_n)}^{t_n} (a(s) + h(s))x(s) ds \right|^2 \\ & \geq G^2(t_m) e^{-2 \int_{t_m}^{t_n} h(u) du} - 2G(t_m) e^{-\int_{t_m}^{t_n} h(u) du} \int_{t_m}^{t_n} e^{-\int_s^{t_n} h(u) du} \omega(s) ds \\ & \geq G(t_m) e^{-2 \int_{t_m}^{t_n} h(u) du} \left( G(t_m) - 2e^{-\int_0^{t_m} h(u) du} \int_{t_m}^{t_n} e^{\int_0^s h(u) du} \omega(s) ds \right) \\ & \geq \frac{\delta_1}{2} e^{-2 \int_{t_m}^{t_n} h(u) du} \left( \frac{\delta_1}{2} - 2K \int_{t_m}^{t_n} e^{\int_0^s h(u) du} \omega(s) ds \right) \geq \frac{\delta_1^2}{8} e^{-2J} > 0. \end{aligned} \quad (16)$$

If the zero solution of (6) is mean square asymptotic stable, then  $E|x(t)|^2 = E|x(t, t_m, \phi)|^2 \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $t_n - \tau(t_n) \rightarrow \infty$ ,  $t_n - \delta(t_n) \rightarrow \infty$  as  $n \rightarrow \infty$  and conditions (ii) and (iii) hold, then

$$E \left| x(t_n) - k(t_n, x(t_n - \tau(t_n))) - \int_{t_n - \tau(t_n)}^{t_n} (a(s) + h(s))x(s) ds \right|^2 \rightarrow 0,$$

as  $n \rightarrow \infty$  which contradicts (16). Thus, (7) is necessary for Theorem 2.1. This completes the proof.  $\square$

**Remark 2.1.** *Theorem 2.1 is still true if condition (ii) is satisfied for  $t \geq t_a$  with some  $t_a \in R^+$ .*

**Remark 2.2.** *The method in this paper can be extended to the following non-linear neutral stochastic differential equation with several variable delays:*

$$\begin{aligned} & d\left(x(t) - \sum_{i=1}^n k_i(t, x(t - \tau_i(t)))\right) \\ = & \left(a(t)x(t) + \sum_{i=1}^n f_i(t, x(t), x(t - \tau_i(t)))\right) dt \\ & + \left(c(t)x(t) + \sum_{j=1}^m g_j(t, x(t), x(t - \delta_j(t)))\right) dW(t). \end{aligned}$$

Choosing  $h(t) \equiv -a(t)$  in Theorem 2.1, we have the following result.

**Corollary 2.1.** *Suppose that  $\tau$  is differential, and there exist continuous functions  $l(t), m(t), n(t) : [0, \infty) \rightarrow R^+$  and constants  $L > 0, \alpha \in (0, 1)$  such that*

(i')  $\liminf_{t \rightarrow \infty} \int_0^t -a(s) ds > -\infty;$

(ii') for any  $t \geq 0,$

$$\begin{aligned} n(t) + \int_0^t e^{\int_s^t a(u) du} (l(s) + m(s) + |a(s)|n(s)) ds \\ + 2 \left( \int_0^t e^{2 \int_s^t a(u) du} (|c(s)| + l(s) + m(s))^2 ds \right)^{\frac{1}{2}} \leq \alpha < 1; \end{aligned}$$

(iii') for any  $t \geq 0,$

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \vee |g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq l(t)|x - \bar{x}| + m(t)|y - \bar{y}|$$

and

$$|k(t, x) - k(t, \bar{x})| \leq n(t)|x - \bar{x}|$$

for all  $x, \bar{x}, y, \bar{y} \in F(L)$  with  $f(t, 0, 0) = g(t, 0, 0) = k(t, 0) = 0.$

Then the zero solution of (6) is mean square asymptotically stable if and only if  $\int_0^t a(s) ds \rightarrow \infty$  as  $t \rightarrow \infty.$

Now, we consider a special case of nonlinear neutral stochastic differential equation (6) that

$$\begin{aligned} & d[x(t) - k(t, x(t - \tau(t)))] \\ = & [a(t)x(t) + b(t)x(t - \tau(t)) + f(t, x(t), x(t - \tau(t)))] dt \\ & + [c(t)x(t) + e(t)x(t - \delta(t)) + g(t, x(t), x(t - \delta(t)))] dW(t), \quad (17) \end{aligned}$$

Note that (17) reduces to (5) when  $f(t, x(t), x(t - \tau(t))) \equiv g(t, x(t), x(t - \delta(t))) \equiv 0$  and  $k(t, x(t - \tau(t))) = q(t)x(t - \tau(t))$ . Then, we have the following results.

**Theorem 2.2.** *Suppose that  $\tau$  is differentiable, and there exist continuous functions  $h(t) : [0, \infty) \rightarrow R$ ,  $l(t), m(t), n(t) : [0, \infty) \rightarrow R^+$  and constants  $L > 0$ ,  $\alpha \in (0, 1)$  such that*

$$(i^\circ) \liminf_{t \rightarrow \infty} \int_0^t h(s) ds > -\infty;$$

(ii $^\circ$ ) for any  $t \geq 0$ ,

$$\begin{aligned} n(t) + \int_0^t e^{-\int_s^t h(u) du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)| du ds \\ + \int_{t-\tau(t)}^t |a(s) + h(s)| ds + \int_0^t e^{-\int_s^t h(u) du} (l(s) + m(s) + |h(s)|n(s) \\ + |(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s)) + b(s)|) ds \\ + 2 \left( \int_0^t e^{-2\int_s^t h(u) du} (|c(s)| + |e(s)| + l(s) + m(s))^2 ds \right)^{\frac{1}{2}} \leq \alpha < 1; \end{aligned}$$

(iii $^\circ$ ) for any  $t \geq 0$ ,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \sqrt{|g(t, x, y) - g(t, \bar{x}, \bar{y})|} \leq l(t)|x - \bar{x}| + m(t)|y - \bar{y}|$$

and

$$|k(t, x) - k(t, \bar{x})| \leq n(t)|x - \bar{x}|$$

for all  $x, \bar{x}, y, \bar{y} \in F(L)$  with  $f(t, 0, 0) = g(t, 0, 0) = k(t, 0) = 0$ .

Then the zero solution of (17) is mean square asymptotically stable if and only if  $\int_0^t h(s) ds \rightarrow \infty$  as  $t \rightarrow \infty$ .

The proof is analogous to that of Theorem 2.1 and so we omit it here.

**Remark 2.3.** *Theorem 2.2 improves Theorem D under different conditions.*

Let  $h(t) \equiv -b(p(t))$  in Theorem 2.2. Then we have the following corollary.

**Corollary 2.2.** *Suppose that  $\tau$  is differential, the inverse function  $p(t)$  of  $t - \tau(t)$  exists, and there exist continuous functions  $l(t), m(t), n(t) : [0, \infty) \rightarrow R^+$  and constants  $L > 0$ ,  $\alpha \in (0, 1)$  such that*

$$(i^*) \liminf_{t \rightarrow \infty} \int_0^t -b(p(s)) ds > -\infty;$$

(ii\*) for any  $t \geq 0$ ,

$$\begin{aligned} & n(t) + \int_0^t e^{\int_s^t b(p(u))du} |b(p(s))| \int_{s-\tau(s)}^s |a(u) - b(p(u))| dud s \\ & + \int_{t-\tau(t)}^t |a(s) - b(p(s))| ds + \int_0^t e^{\int_s^t b(p(u))du} \cdot (l(s) + m(s) \\ & + |b(p(s))|n(s) + |(a(s - \tau(s)) - b(s))(1 - \tau'(s)) + b(s)|) ds \\ & + 2 \left( \int_0^t e^{2 \int_s^t b(p(u))du} (|c(s)| + |e(s)| + l(s) + m(s))^2 ds \right)^{\frac{1}{2}} \leq \alpha < 1; \end{aligned}$$

(iii\*) for any  $t \geq 0$ ,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \vee |g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq l(t)|x - \bar{x}| + m(t)|y - \bar{y}|$$

and

$$|k(t, x) - k(t, \bar{x})| \leq n(t)|x - \bar{x}|$$

for all  $x, \bar{x}, y, \bar{y} \in F(L)$  with  $f(t, 0, 0) = g(t, 0, 0) = k(t, 0) = 0$ .

Then the zero solution of (17) is mean square asymptotically stable if and only if  $\int_0^t b(p(s))ds \rightarrow \infty$  as  $t \rightarrow \infty$ .

**Remark 2.4.** When  $k(t, x(t - \tau(t))) \equiv f(t, x(t), x(t - \tau(t))) \equiv g(t, x(t), x(t - \delta(t))) \equiv a(t) \equiv c(t) \equiv e(t) \equiv 0$  and  $b(t) \equiv -\bar{b}(t)$ , we know that Corollary 2.2 still holds if the condition (ii\*) is replaced by (2). Therefore, Corollary 2.2 is a generalization of Theorem B.

Now, we consider another special case of nonlinear neutral stochastic differential equation (6) that

$$dx(t) = -a(t)x(t) + f(t, x(t), x(t - \tau(t)))dt, \tag{18}$$

Note that (18) reduces to (3) when  $f(t, x(t), x(t - \tau(t))) \equiv b(t)g(x(t - \tau(t)))$ . Then, we have the following result.

**Theorem 2.3.** Suppose that  $\tau$  is differential, and there exist continuous functions  $h(t) : [0, \infty) \rightarrow R$ ,  $l(t), m(t) : [0, \infty) \rightarrow R^+$  and constants  $L > 0$ ,  $\alpha \in (0, 1)$  such that

$$(i^*) \liminf_{t \rightarrow \infty} \int_0^t h(s)ds > -\infty;$$

(ii\*) for any  $t \geq 0$ ,

$$\begin{aligned} & \int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-\tau(s)}^s |h(u) - a(u)| du ds \\ & + \int_{t-\tau(t)}^t |h(s) - a(s)| ds + \int_0^t e^{-\int_s^t h(u)du} (l(s) + m(s) \\ & + |(h(s - \tau(s)) - a(s - \tau(s)))(1 - \tau'(s))|) ds \leq \alpha < 1; \end{aligned}$$

(iii\*) for any  $t \geq 0$ ,

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq l(t)|x - \bar{x}| + m(t)|y - \bar{y}|$$

for all  $x, \bar{x}, y, \bar{y} \in F(L)$  with  $f(t, 0, 0) = 0$ .

Then the zero solution of (18) is asymptotically stable if and only if  $\int_0^t h(s) ds \rightarrow \infty$  as  $t \rightarrow \infty$ .

The proof is analogous to that of Theorem 2.1 and so we omit it here.

**Remark 2.5.** When  $f(t, x(t), x(t-\tau(t))) \equiv b(t)g(x(t-\tau(t)))$ , choosing  $h(t) \equiv a(t)$ ,  $l(t) \equiv 0$  and  $m(t) \equiv |b(t)|$ , Corollary 2.3 reduces to Theorem C.

### 3 Two Examples

In this section, we give two examples to illustrate the applications of our main results.

**Example 3.1.** Consider the following nonlinear neutral stochastic delay differential equation

$$\begin{aligned} & d\left(x(t) - \frac{1}{8}x^2\left(\frac{3t}{4}\right)\right) \\ & = \left(-2x(t) + \frac{e^{-2t}}{8} \sin\left(x(t) + x\left(\frac{3t}{4}\right)\right) \cdot \cos\left(x(t) - x\left(\frac{3t}{4}\right)\right)\right) dt \\ & + \left(\frac{1}{9}x(t) + \frac{e^{-t}}{7} \cos\left(x(t) + x\left(\frac{t}{2}\right)\right) \cdot \sin\left(x(t) - x\left(\frac{t}{2}\right)\right)\right) dW(t). \quad (19) \end{aligned}$$

Then the zero solution of (19) is mean square asymptotically stable.



*Proof.* Let

$$f(t, x(t), x(t - \tau(t))) := \frac{e^{-2t}}{8} \sin\left(x(t) + x\left(t - \frac{t}{4}\right)\right) \cdot \cos\left(x(t) - x\left(t - \frac{t}{4}\right)\right),$$

$$g(t, x(t), x(t - \delta(t))) := \frac{e^{-t}}{7} \cos\left(x(t) + x\left(t - \frac{t}{2}\right)\right) \cdot \sin\left(x(t) - x\left(t - \frac{t}{2}\right)\right).$$

Since  $|\sin x| \leq |x|$  for  $x \in R$ , then we get

$$\begin{aligned} &|f(t, x(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t)))| \\ &\leq \frac{e^{-2t}}{8} |x(t) - y(t)| + \frac{e^{-2t}}{8} \left| x\left(t - \frac{t}{4}\right) - y\left(t - \frac{t}{4}\right) \right|, \\ &|g(t, x(t), x(t - \delta(t))) - g(t, y(t), y(t - \delta(t)))| \\ &\leq \frac{e^{-t}}{7} |x(t) - y(t)| + \frac{e^{-t}}{7} \left| x\left(t - \frac{t}{2}\right) - y\left(t - \frac{t}{2}\right) \right|. \end{aligned}$$

As  $|x^2| \leq |x|$  when  $|x| \leq 1$ , we can choose  $L = 1/2$ ,  $l(t) = m(t) = e^{-t}/7$  and  $n(t) = 1/8$  such that the condition (iii) of Theorem 2.1 holds. Moreover, it is easy to verify that  $t - \tau(t) = t - t/4 \rightarrow \infty$  and  $t - \delta(t) = t - t/2 \rightarrow \infty$  as  $t \rightarrow \infty$ .

Choosing  $h(t) = 2$  in Theorem 2.1, we have

$$\begin{aligned} &\int_0^t e^{-\int_s^t h(u)du} (|(a(s - \tau(s)) + h(s - \tau(s)))(1 - \tau'(s))| + l(s) + m(s) \\ &\quad + |h(s)|n(s)) ds = \int_0^t e^{-2(t-s)} \left(\frac{2e^{-s}}{7} + \frac{1}{4}\right) ds \leq 0.18, \\ &2 \left( \int_0^t e^{-2\int_s^t h(u)du} (|c(s)| + l(s) + m(s))^2 ds \right)^{\frac{1}{2}} \\ &= 2 \left( \int_0^t e^{-2(t-s)} \left(\frac{2e^{-s}}{7} + \frac{1}{9}\right) ds \right)^{\frac{1}{2}} \leq 0.68, \end{aligned}$$

and

$$\int_{t-\tau(t)}^t |a(s) + h(s)| ds = \int_0^t e^{-\int_s^t h(u)du} |h(s)| \int_{s-\tau(s)}^s |a(u) + h(u)| du ds = 0.$$

It is easy to check that  $\int_0^t h(s) ds \rightarrow \infty$  as  $t \rightarrow \infty$ . Let  $\alpha = 1/8 + 0.18 + 0.68$ . Then  $\alpha = 0.985 < 1$  and so the zero solution of (19) is mean square asymptotically stable.  $\square$

**Example 3.2.** Consider the following delay differential equation

$$x'(t) = -\frac{1}{5}x(t) + 2e^{-t} \sin\left(\frac{1}{10}x(t - e^{-t})\right). \tag{20}$$

Then the zero solution of (20) is asymptotically stable.

*Proof.* Let

$$f(t, x(t), x(t - \tau(t))) := 2e^{-t} \sin \left( \frac{1}{10} x(t - e^{-t}) \right).$$

Since  $|\sin \frac{x}{10}| \leq \frac{1}{10}|x|$  for  $x \in R$ , we have

$$|f(t, x(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t)))| \leq \frac{e^{-t}}{5} |x(t - e^{-t}) - y(t - e^{-t})|.$$

Therefore, we can choose  $l(t) \equiv 0$ ,  $m(t) = e^{-t}/5$  and  $L$  for any positive constant such that the condition (iii\*) of Theorem 2.3 holds. Moreover, it is easy to verify that  $t - \tau(t) = t - e^{-t} \rightarrow \infty$  as  $t \rightarrow \infty$ . Choosing  $h(t) \equiv 0.3$  in Theorem 2.3, we have

$$\begin{aligned} \int_{t-\tau(t)}^t |h(s) - a(s)| ds &= \int_{t-e^{-t}}^t (0.3 - 0.2) ds \rightarrow 0.1e^{-t} \leq 0.1, \\ \int_0^t e^{-\int_s^t h(u) du} |h(s)| \int_{s-\tau(s)}^s |h(u) - a(u)| du ds &\leq \int_0^t \frac{0.03}{e^{0.3(t-s)}} ds \leq 0.1, \end{aligned}$$

and

$$\begin{aligned} &\int_0^t e^{-\int_s^t h(u) du} (|(h(s - \tau(s)) - a(s - \tau(s)))(1 - \tau'(s))| + l(s) + m(s)) ds \\ &= \int_0^t e^{-0.3(t-s)} [(0.3 - 0.2)(1 + e^{-s}) + \frac{e^{-s}}{5}] ds \leq 0.36. \end{aligned}$$

It is easy to see that all conditions of Theorem 2.3 hold for  $\alpha = 0.1 + 0.1 + 0.36 = 0.56 < 1$ . Thus, Theorem 2.3 implies that the zero solution of (20) is asymptotically stable.

However, Theorem C can not be used to verify that the zero solution of (20) is asymptotically stable. In fact, noticing that  $|\sin \frac{x}{10} - \sin \frac{y}{10}| \leq |x - y|$  for all  $x, y \in R$ ,  $b(t) \equiv 2e^{-t}$ ,  $a(t) \equiv 1/5$  and

$$\int_0^t e^{-\int_s^t a(u) du} |b(s)| ds = \int_0^t e^{-0.2(t-s)} \cdot 2e^{-s} ds < 1.33.$$

Obviously, the condition (4) of Theorem C does not hold with  $\alpha = 1.33 > 1$ . □

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