An. Şt. Univ. Ovidius Constanţa **Vol. 20(1), 2012, 173–188**

Stability and superstability of homomorphisms on C^*−ternary algebras

M. Eshaghi Gordji and A.Fazeli

Abstract

In this paper, we investigate the stability and superstability of homomorphisms on C^*−ternary algebras associated with the functional equation

$$f(\frac{x+2y+2z}{5})+f(\frac{2x+y-z}{5})+f(\frac{2x-3y-z}{5})=f(x).$$

1 Introduction

The *stability problem* of functional equations started with the following question concerning stability of group homomorphisms proposed by S.M. Ulam [40] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940:

Let $(G_1, .)$ be a group and $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that, if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [18] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

If E and $E^{'}$ are Banach spaces and $f : E \longrightarrow E^{'}$ is a mapping for which there is $\varepsilon > 0$ such that

Key Words: stability; suprestability ; C^*−ternary algebra; homomorphism
2010 Mathematics Subject Classification: 46L99, 17C65, 46L05, 47B48, 39B82, 39B52.
Received: February, 2011.
Accepted: February, 2012.

$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$ for all $x, y \in E$, then there is a unique additive mapping $L: E \longrightarrow E'$ such that $\|f(x)-L(x)\| \leq \varepsilon$ for all $x \in E$.

Hyers' Theorem was generalized by Rassias [36] for linear mappings by considering an unbounded Cauchy difference.

The paper of Rassias [37] has provided a lot of influence in the development of what we now call the *generalized Hyers–Ulam stability* or as *Hyers–Ulam–Rassias stability* of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [3, 4, 5, 11, 16, 25, 13, 15, 19, 20, 21, 22] and [29]–[34] and [7, 38] .

Ternary algebraic operations have propounded originally in 19th century in Cayley [2] and J.J.Silvester's paper [39]. The application of ternary algebra in supersymmetry is presented in [23] and in Yang-Baxter equation in [27]. Cubic analogue of Laplace and d'alembert equations have been considered for first order by Himbert in [17],[24]. The previous definition of C^*-ternary algebras has been propounded by H.Zettle in [41]. In relation to homomorphisms and isomorphisms between various spaces we refer readers to [28]–[35], [1, 6, 8, 12, 9, 10].

2 Prelimiaries

Let A be a linear space over a complex field equipped with a mapping $[\,]: A^3 = A \times A \times A \to A$ with $(x, y, z) \to [x, y, z]$ that is linear in variables x, y, z and satisfy the associative identity, i.e. $[x, y, [z, u, v]] = [x, [y, z, u], v] = [[x, y, z], u, v]$ for all $x, y, z, u, v \in A$. The pair $(A, [\,])$ is called a ternary algebra. The ternary algebra $(A, [\ ])$ is called unital if it has an identity element, i.e. an element $e \in A$ such that $[x, e, e] = [e, e, x] = x$ for every $x \in A$. A $* - ternary$ algebra is a ternary algebra together with a mapping $*: A \to A$ which satisfies $(x^*)^* = x$, $(\lambda x)^* = \bar{\lambda} x^*$, $(x+y)^* = x^* + y^*$, $[x, y, z]^* = [z^*, y^*, x^*]$ for all $x, y, z \in A$ and all $\lambda \in \mathbb{C}$. In the case that A is unital and e is its unit, we assume that $e^* = e$.

A is normed ternary algebra if A is a ternary algebra and there exists a norm $\|.\|$ on A which satisfies $\|[x, y, z]\| \leq \|x\|\ \|y\|\ \|z\|$ for all $x, y, z \in A$. Whenever the ternary algebra A is unital with unit element e, we repute $\|e\| = 1$. A normed ternary algebra A is called a Banach ternary algebra, if $(A, \|\ \|)$ is a Banach space. If A is a ternary algebra, $x \in A$ is called central if $[x, y, z] = [z, x, y] = [y, z, x]$ for all y, z in A.
The set of central elements of A is called the center of A and is shown by $Z(A)$.

In case A is $*-$normed ternary algebra and $Z(A) = 0$ we grant $||x^*|| = ||x||$. A C^*-ternary algebra is a Banach $*-$*ternary algebra* if $\|[x^*, y, x]\| = \|x\|^2\|y\|$ for all x in A and y in $Z(A)$.

Let A,B be two C^*-ternary algebras. A linear mapping $h : A \to B$ is called a homomorphism if $h([x, y, z]) = [h(x), h(y), h(z)]$ for all $x, y, z \in A$ and a homomorphism $h : A \to B$ is called a $*-$homomorphism if $h(a^*) = h(a)^*$ for all $a \in A$.

Comments : If A is a unital (binary) C^*-algebra with unit e, we define $[x, y, z] := (xy)z$ for all x, y, z in A. Then we have $[x, y, z]^* = ((xy)z)^* = z^*(xy)^* = z^*(y^*x^*) = (z^*y^*)x^* = [z^*, y^*, x^*]$. Now if y in ternary algebra A belongs to $Z(A)$, then we have $yy^* = (yy^*)e = [y, y^*, e] = [y^*, e, y] = (y^*e)y = y^*y$. Thus y is normal in C^*-algebra A. On the other hand, for every normal element x in a C^*-algebra, we have $\|x\| = \rho(x)$ in which $\rho(x)$ is spectral radius of x. Theorem 1.3.4 of [26] expresses that if A is a unital and commutative Banach algebra and $\Omega(A)$ is its maximal ideal space, then for every a in A $\sigma(a) = \{h(a)\ ;\ h \in \Omega(A)\}$. Now, if z belongs to C^*-algebra A and $z = z^*$ and $x \in A$ is normal and $xz = zx$, then zx is a normal element of C^*-algebra and if B is the C^*-algebra generated by x,z,e then B is unital and commutative and so

$$||zx|| = \sup_{h\in\Omega(A)} |h(zx)| = \sup_{h\in\Omega(A)} |h(z)| \sup_{h\in\Omega(A)} |h(x)| = \|z\|\|x\|$$

Now let $y \in A$ be a central element of ternary algebra A and let x belongs to A. Then

$$\|[x^*, y, x]\| = \|[x, x^*, y]\| = \|(xx^*)y\| = \|xx^*\|\|y\| = \|x\|^2\|y\|$$

Thus A is a unital C^*-ternary algebra.

3 Solution

We start our work with solution of functional equation

$$f(\frac{x + 2y + 2z}{5}) + f(\frac{2x + y - z}{5}) + f(\frac{2x - 3y - z}{5}) = f(x).$$

Theorem 3.1. Let X and Y be linear spaces and $f : X \to Y$ be a mapping. Then f is additive if and only if

$$f(\frac{x + 2y + 2z}{5}) + f(\frac{2x + y - z}{5}) + f(\frac{2x - 3y - z}{5}) = f(x) \tag{1}$$

for all x,y,z in $X-\{0\}$.

Proof. If f is additive, it is obvious that f satisfies (1). Conversely suppose that f satisfies (1).
Letting $x=y=z$ in (1), we have $f(\frac{2x}{5})+f(\frac{-2x}{5})=0$.
Replacing x by $\frac{5x}{2}$ to get $f(-x)=-f(x)$.
Putting $z=-y$ and $x=y$ in (1) we get

$$f(\frac{y}{5})+f(\frac{4y}{5})+f(0)=f(y) \tag{2}$$

Laying $z=3x$ and $y=x$ in (1) we infer that

$$f(x)=f(\frac{9x}{5})+f(0)-f(\frac{4x}{5}). \tag{3}$$

Letting $x=2z$ and $y=-2z$ in (1) we conclude that

$$f(2z)=f(0)+f(\frac{z}{5})+f(\frac{9z}{5}). \tag{4}$$

Using (3) and (4) we see that

$$f(2x)-f(x)=f(\frac{x}{5})+f(\frac{4x}{5}). \tag{5}$$

By use of (5) and (2) we obtain

$$f(2x)=2f(x)-f(0). \tag{6}$$

It follows from (6) that

$$f(\frac{4y}{5})=4f(\frac{y}{5})-3f(0). \tag{7}$$

We deduce from (2) and (7) that

$$5f(\frac{y}{5})=f(y)+2f(0). \tag{8}$$

Multiplying by 5 both sides of (3) with (8) we lead to

$$f(9x)=5f(x)+f(4x)-5f(0). \tag{9}$$

It follows from (6) and (9) that

$$f(9x)=9f(x)-8f(0). \tag{10}$$

Multiply by 5 both sides of (4), together with (6) and (8) one gets

$$f(9x) = 9f(x) - 14f(0). \tag{11}$$

We infer from (11) and (10) that $f(0) = 0$. Hence by (6) and (8) and (10), we have

$$f(2x) = 2f(x),\ f(9x) = 9f(x),\ f(y) = 5f(\frac{y}{5}). \tag{12}$$

Replacing y by 5y in (12) we get

$$f(2x) = 2f(x),\ f(9x) = 9f(x),\ f(5y) = 5f(y). \tag{13}$$

Substituting x with 5x and y with 5y and z with 5z in (1) together (13) we have

$$f(x+2y+2z) + f(2x+y-z) + f(2x-3y-z) = 5f(x). \tag{14}$$

Laying $y = -z$ in (14) with (13) one gets

$$f(x-z) + f(x+z) = f(2x). \tag{15}$$

We replace $r = x-z$ and $s = x+z$ in (15), then we have $f(r)+f(s) = f(r+s)$. Hence f is additive. $\square$

We need the following theorem in our main results.

Theorem 3.2. Let $n_0 \in \mathbb{N}$ be a fixed positive integer number and X and Y be linear spaces and $f : X \to Y$ be an additive function. Then f is linear if and only if $f(\mu x) = \mu f(x)$ for all x in X and μ in $T^1_{\frac{1}{n_o}} = \{e^{i\theta}\ ;\ 0 \le \theta \le \frac{2\pi}{n_o}\}$.

Proof. Suppose that f is additive and $f(\mu x) = \mu f(x)$ for all x in X and μ in $T^1_{\frac{1}{n_o}}$.
Let μ be in T^1, then $\mu = e^{i\theta}$ that $0 \le \theta \le 2\pi$.
We set $\mu_1 = e^{\frac{i\theta}{n_o}}$, thus μ_1 is in $T^1_{\frac{1}{n_o}}$ and $f(\mu x) = f(\mu_1^{n_o} x) = \mu_1^{n_o} f(x) = \mu f(x)$ for all x in X. If μ belongs to $nT^1 = \{nz\ ;\ z \in T^1\}$ then by additivity of f, $f(\mu x) = \mu f(x)$
for all x in X and μ in nT^1. If $t \in (0, \infty)$ then by archimedean property there exists a natural number n such that the point $(t, 0)$ lies in the interior of circle with center at origin and radius n.
Let $t_1 = t + \sqrt{n^2 - t^2}\ i \in nT^1$ and $t_2 = t - \sqrt{n^2 - t^2}\ i \in nT^1$.
We have $t = \frac{t_1 + t_2}{2}$ and $f(tx) = f(\frac{t_1 + t_2}{2} x) = \frac{t_1 + t_2}{2} f(x) = tf(x)$ for all x in X.
If $\mu \in \mathbb{C}$, then $\mu = |\mu| e^{i\mu_1}$ so $f(\mu x) = f(|\mu| e^{i\mu_1} x) = |\mu| e^{i\mu_1} f(x) = \mu f(x)$ for all x in X.
The converse is clear. $\square$

Theorem 3.3. Let X and Y be linear spaces and $f : X \to Y$ be a mapping. Then f is $\mathbb{C}-$linear if and only if

$$f(\frac{\mu x + 2y + 2z}{5}) + f(\frac{2\mu x + y - z}{5}) + f(\frac{2\mu x - 3y - z}{5}) = \mu f(x) \tag{16}$$

for all x,y,z in $X - \{0\}$ and μ in $T^1_{\frac{1}{n_o}}$.

Proof. If f is $\mathbb{C}-$linear, it is clear that f satisfies (16). Conversely, let f satisfies (16). We set $\mu = 1$ in (16), then by Theorem 3.1, f is an additive mapping. Letting $y = z = 0$ in (16) we have $f(\frac{\mu x}{5}) + 2f(\frac{2\mu x}{5}) = \mu f(x)$. By additivity of f we get $f(\mu x) = \mu f(x)$ for all x in X and μ in $T^1_{\frac{1}{n_o}}$.

So by Theorem 3.2 f is a $\mathbb{C}-$linear. □

Notation 3.4. Let X and Y be linear spaces and $f : X \to Y$ be a mapping. Then we set

$$E_\mu f(x, y, z) = f(\frac{\mu x + 2y + 2z}{5}) + f(\frac{2\mu x + y - z}{5}) + f(\frac{2\mu x - 3y - z}{5}) - \mu f(x)$$

for all x,y,z in X and μ in $\mathbb{C}$.

4 Stability

In this section we investigate the Stability of $*-$homomorphisms between C^*-ternary algebras.

Theorem 4.1. Let A and B be two C^*-ternary algebras and $\varphi, \psi : A^3 \to [0, \infty)$ be functions such that

$$\tilde{\varphi}(x) = \sum_{n=1}^{\infty} 3^{n-1}\varphi(\frac{x}{3^{n-1}}, 0, \frac{x}{3^n}) < \infty \quad [\tilde{\varphi}(x) = \sum_{n=1}^{\infty} \frac{1}{3^n}\varphi(3^n x, 0, 3^{n-1}x) < \infty], \tag{17}$$

$$\lim_{n\to\infty} 3^n \varphi(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}) = 0 \quad [\lim_{n\to\infty} \frac{1}{3^n}\varphi(3^n x, 3^n y, 3^n z) = 0], \tag{18}$$

$$\lim_{n\to\infty} 3^{3n} \psi(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}) = 0 \quad [\lim_{n\to\infty} \frac{1}{3^{3n}}\psi(3^n x, 3^n y, 3^n z) = 0], \tag{19}$$

for all x,y,z in A. Suppose that $f : A \to B$ is a mapping such that

$$\|E_\mu f(x, y, z)\| \le \varphi(x, y, z) \tag{20}$$

$$\|f([x,y,z]) - [f(x),f(y),f(z)]\| \le \psi(x,y,z), \qquad \|f(a^*) - f(a)^*\| \le \psi(a,0,0) \tag{21}$$

for all x,y,z,a in A and μ in $T^1_{\frac{1}{n_o}}$.
Then there exists a unique $*-$homomorphism $T : A \to B$ such that

$$\|T(x) - f(x)\| \le \tilde{\varphi}(x) \tag{22}$$

and we have

$$T(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \qquad [T(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)], \tag{23}$$

for all x in A.

Proof. Letting $\mu = 1$ and $z = \dfrac{x}{3}$ and $y = 0$ in (20), we have

$$\|3f(\frac{x}{3}) - f(x)\| \le \varphi(x, 0, \frac{x}{3}). \tag{24}$$

Replacing x by $\dfrac{x}{3}$ in (24) and multiplying by 3 both sides of (24), we get

$$\|3^2 f(\frac{x}{3^2}) - 3f(\frac{x}{3})\| \le 3\varphi(\frac{x}{3}, 0, \frac{x}{3^2}). \tag{25}$$

Using (24) and (25) we get

$$\|3^2 f(\frac{x}{3^2}) - f(x)\| \le \varphi(x, 0, \frac{x}{3}) + 3\varphi(\frac{x}{3}, o, \frac{x}{3^2})$$

By use of the above method, by induction, we infer that

$$\|3^n f(\frac{x}{3^n}) - f(x)\| \le \sum_{i=1}^{n} 3^{i-1} \varphi(\frac{x}{3^{i-1}}, 0, \frac{x}{3^i}). \tag{26}$$

Substitute x with $\dfrac{x}{3^m}$ in (26) and multiply by 3^m its both parties of inequality, we lead to

$$\|3^{n+m} f(\frac{x}{3^{n+m}}) - 3^m f(\frac{x}{3^m})\| \le \sum_{i=m+1}^{n+m} 3^{i-1} \varphi(\frac{x}{3^{i-1}}, 0, \frac{x}{3^i}) \le \sum_{i=m+1}^{\infty} 3^{i-1} \varphi(\frac{x}{3^{i-1}}, 0, \frac{x}{3^i}). \tag{27}$$

The right expression of (27) by (17) tends to zero as m tends to infinity. So the sequence $\{3^n f(\dfrac{x}{3^n})\}$ is a Cauchy sequence in complete space B. Hence, one can define $T : A \to B$ by $T(x) = \lim\limits_{n\to\infty} 3^n f(\frac{x}{3^n})$. From (20) and (18) we arrive at

$$\|E_\mu T(x,y,z)\| = \lim_{n\to\infty} 3^n \|E_\mu f(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})\| \le \lim_{n\to\infty} 3^n \varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}) = 0.$$

So $E_\mu T(x,y,z) = 0$ for all x,y,z in A and μ in $T^1_{\frac{1}{n_o}}$.
By Theorem 3.3, T is $\mathbb{C}-$linear. (21) and (19) imply that

$$\|T([x,y,z]) - [T(x),T(y),T(z)]\| =$$
$$\lim_{n\to\infty} 3^{3n}\|f([\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}]) - [f(\frac{x}{3^n}), f(\frac{y}{3^n}), f(\frac{z}{3^n})]\| \le$$
$$\lim_{n\to\infty} 3^{3n}\psi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}) = 0.$$

Thus $T([x,y,z]) = [T(x),T(y),T(z)]$ for all x,y,z in A. By a same method as above, we can show that $T(a^*) = T(a)^*$ for all a in A. Therefore, T is a $*-$homomorphism.
Now let $T' : A \to B$ be another $*-$homomorphism satisfying
$\|T'(x) - f(x)\| \le \tilde{\varphi}(x)$ for all x in A. Then from linearity of T' we see that

$$\|T(x) - T'(x)\| = \lim_{n\to\infty} \|3^n f(\frac{x}{3^n}) - T'(x)\| = \lim_{n\to\infty} 3^n \|f(\frac{x}{3^n}) - T'(\frac{x}{3^n})\| \le$$

$$\lim_{n\to\infty} 3^n \tilde{\varphi}(\frac{x}{3^n})) = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} 3^{i-1}\varphi(\frac{x}{3^{i-1}}, 0, \frac{x}{3^i}) = 0.$$

Therefore $T(x) = T'(x)$ for all x in A. □

Corollary 4.2. Let $\theta, p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3$ be real numbers such that $\theta, p_2 > 0$,
$p_1, p_2, p_3 > 1$ $[p_1, p_2, p_3 < 1]$, $p_4 + p_5 > 1$ $[p_4 + p_5 < 1]$, $q_1, q_2, q_3 > 3$ $[q_1, q_2, q_3 < 3]$
and A,B be two C^*-ternary algebras and $f : A \to B$ be a mapping satisfying

$$\|E_\mu f(x,y,z)\| \le \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3} + \|x\|^{p_4}\|z\|^{p_5}),$$

$\|f([x,y,z]) - [f(x),f(y),f(z)] \le \theta(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}),\quad \|f(a^*) - f(a)^*\| \le \theta\|a\|^{q_1}$
for all x,y,z,a in A and all μ in $T^1_{\frac{1}{n_o}}$. Then there exists a unique $*-$homomorphism $T : A \to B$ such that

$$\|f(x) - T(x)\| \le \theta(\frac{3^{p_1}}{|3^{p_1} - 3|}\|x\|^{p_1} + \frac{1}{|3^{p_3} - 3|}\|x\|^{p_3} + \frac{3^{p_4}}{|3^{p_4+p_5} - 3|}\|x\|^{p_4+p_5})$$

and

$$T(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) \quad [T(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)]$$

for all x in A.

Proof. Putting

$$\varphi(x,y,z) = \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3} + \|x\|^{p_4}\|z\|^{p_5})$$

and

$$\psi(x,y,z) = \theta(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3})$$

in Theorem 4.1. □

5 Superstability

Theorem 5.1. Let A and B be two C^*-ternary algebras and $\varphi, \psi : A^3 \to [0,\infty)$ be functions such that

$$\varphi(x,0,z) = 0,$$

$$\lim_{n\to\infty} 3^n\varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}) = 0 \qquad [\lim_{n\to\infty}\frac{1}{3^n}\varphi(3^nx,3^ny,3^nz) = 0],$$

$$\lim_{n\to\infty} 3^{3n}\psi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}) = 0 \qquad [\lim_{n\to\infty}\frac{1}{3^{3n}}\psi(3^nx,3^ny,3^nz) = 0],$$

for all x,y,z in A. Suppose that $f : A \to B$ is a mapping such that

$$\|E_\mu f(x,y,z)\| \le \varphi(x,y,z)$$

$$\|f([x,y,z]) - [f(x),f(y),f(z)]\| \le \psi(x,y,z), \quad \|f(a^*) - f(a)^*\| \le \psi(a,0,0)$$

for all x,y,z,a in A and μ in $T^1_{\frac{1}{n_o}}$. Then f is a $*-$homomorphism.

Proof. Because $\varphi(x,o,z) = 0$ for all x,z in A, like the proof of Theorem 4.1, we have $3f(\frac{x}{3}) = f(x)$ and by induction we infer that $3^nf(\frac{x}{3^n}) = f(x)$. Therefore $T(x) = f(x)$ for all x in A. Thus f is a $*-$homomorphism between C^*-ternary algebras. The other case is similar. □

Corollary 5.2. Let $\theta, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, q_1, q_2, q_3$ be real numbers such that $\theta \ge 0$, $p_1 > 1$ $[p_1 < 1]$, $p_2 + p_3 + p_4 > 1$ $[p_2 + p_3 + p_4 < 1]$, $p_5 + p_6 > 1$ $[p_5 + p_6 < 1]$, $p_7 + p_8 > 1$ $[p_7 + p_8 < 1]$, $q_1 + q_2 + q_3 > 3$ $[q_1 + q_2 + q_3 < 3]$ and let A,B be two C^*-ternary algebras. Let $f : A \to B$ be a mapping such that

$$\|E_\mu f(x,y,z)\| \le \theta(\|y\|^{p_1} + \|x\|^{p_2}\|y\|^{p_3}\|z\|^{p_4} + \|x\|^{p_5}\|y\|^{p_6} + \|y\|^{p_7}\|z\|^{p_8})$$

$$\|f([x,y,z]) - [f(x),f(y),f(z)]\| \le \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3})$$

for all x,y,z in A and μ in $T^1_{\frac{1}{n_o}}$. Then f is a homomorphism.

Proof. It follows by Theorem 5.1 by putting

$$\varphi(x,y,z)=\theta(\|y\|^{p_1}+\|x\|^{p_2}\|y\|^{p_3}\|z\|^{p_4}+\|x\|^{p_5}\|y\|^{p_6}+\|y\|^{p_7}\|z\|^{p_8})$$
$$\psi(x,y,z)=\theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}).$$

□

Theorem 5.3. Let A and B be two C^*-ternary algebras and let B be unital with unit e' and let $\varphi,\psi : A^3 \to [0,\infty)$ be functions such that

$$\tilde{\varphi}(x)=\sum_{n=1}^{\infty}3^{n-1}\varphi(\frac{x}{3^{n-1}},0,\frac{x}{3^n})<\infty \quad [\tilde{\varphi}(x)=\sum_{n=1}^{\infty}\frac{1}{3^n}\varphi(3^nx,0,3^{n-1}x)<\infty],$$

$$\lim_{n\to\infty}3^n\varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})=0 \quad [\lim_{n\to\infty}\frac{1}{3^n}\varphi(3^nx,3^ny,3^nz)=0],$$
$$\lim_{n\to\infty}3^{3n}\psi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})=0 \quad [\lim_{n\to\infty}\frac{1}{3^{3n}}\psi(3^nx,3^ny,3^nz)=0],$$
$$\lim_{n\to\infty}3^{2n}\psi(\frac{x}{3^n},\frac{y}{3^n},z)=0 \quad [\lim_{n\to\infty}\frac{1}{3^{2n}}\psi(3^nx,3^ny,z)=0],$$

for all x,y,z in A. Suppose that $f : A \to B$ is a mapping satisfying

$$\|E_\mu f(x,y,z)\|\le\varphi(x,y,z)$$
$$\|f([x,y,z])-[f(x),f(y),f(z)]\le\theta(\|x\|^{q_1}+\|y\|^{q_2}+\|z\|^{q_3}),\quad \|f(a^*)-f(a)^*\|\le\theta\|a\|^{q_1}$$

for all x,y,z,a in A and μ in $T^1_{\frac{1}{n_o}}$ and there exists a x_0 in A such that $e'=\lim_{n\to\infty}3^nf(\frac{x_0}{3^n})$
$[e'=\lim_{n\to\infty}\frac{1}{3^n}f(3^nx)]$. Then f is a $*-$homomorphism.

Proof. By Theorem 4.1 there exists a $*-$homomorphism $T : A \to B$ such that

$$\|T(x)-f(x)\|\le\tilde{\varphi}(x) \quad and \quad T(x)=\lim_{n\to\infty}3^nf(\frac{x}{3^n})$$

for all x in A. Now observe that

$$\|[T(x),T(y),T(z)]-[T(x),T(y),f(z)]\|=\|T([x,y,z])-[T(x),T(y),f(z)]\|=$$
$$\lim_{n\to\infty}3^{2n}\|f([\frac{x}{3^n},\frac{y}{3^n},z])-[f(\frac{x}{3^n},f(\frac{y}{3^n}),f(z)]\|\le\lim_{n\to\infty}3^{2n}\psi(\frac{x}{3^n},\frac{y}{3^n},z)=0$$

for all x,y,z in A. So $[T(x),T(y),T(z)-f(z)]=0$ for all x,y,z in A.
By hypothesis of theorem, we get $T(x_0)=e'$. Replacing x,y by x_0 in the last bracket, we have $[e',e',T(z)-f(z)]=0$ for all z in A. Hence $f(z)=T(z)$ for all z in A.
Therefore f is a $*-$homomorphism between C^*-ternary algebras A and B. □

Theorem 5.4. Let A and B be two C^*-ternary algebras and $\varphi, \psi : A^3 \to [0, \infty)$ be functions such that

$$\lim_{n\to\infty} 3^n \psi(\frac{x}{3^n}, y, z) = 0, \quad [\lim_{n\to\infty} \frac{1}{3^n}\psi(3^n x, y, z) = 0], \tag{28}$$

and satisfy $(17), (18), (19)$.
Suppose that $f : A \to B$ is a mapping that satisfies $(20), (21)$ for all $x, y, z \in A$ and all $\mu \in T^1_{\frac{1}{n_o}}$.
Assume that $S(B)$ be the set of all self adjoint elements of B and there exists an element y_o in A such that $0 \neq \lim_{n\to\infty} 3^n f(\frac{y_o}{3^n}) \in S(B)$ $[0 \neq \lim_{n\to\infty} \frac{1}{3^n} f(3^n y_o) \in S(B)]$ and $\{f(3x) - 3f(x)\ ;\ x \in A\} \subseteq Z(B)$.
Then f is a $*-$homomorphism between C^*-ternary algebras.

Proof. By Theorem 4.1 there exists a $*-$homomorphism T such that $T(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ for all x in A. Also by (21) we get

$$||T([x_1, x_2, x_3]) - [T(x_1), f(x_2), f(x_3)]|| =$$

$$\lim_{n\to\infty} 3^n ||f([\frac{x_1}{3^n}, x_2, x_3]) - [f(\frac{x_1}{3^n}), f(x_2), f(x_3)]|| \leq \lim_{n\to\infty} 3^n \psi(\frac{x_1}{3^n}, x_2, x_3) = 0.$$

So $T([x_1, x_2, x_3]) = [T(x_1), f(x_2), f(x_3)]$ for all x_1, x_2, x_3 in A. Now assume n belongs to $\mathbb{N}$ and let $x \in A$. We set $x_1 = y_o, x_2 = x, x_3 = \frac{y_o}{3^n}$. So

$$T([y_o, x, \frac{y_o}{3^n}]) = [T(y_o), f(x), f(\frac{y_o}{3^n})] \tag{29}$$

Replacing x by 3x in (29), we obtain

$$3T([y_o, x, \frac{y_o}{3^n}]) = [T(y_o), f(3x), f(\frac{y_o}{3^n})]. \tag{30}$$

Multiply both sides of (29) by 3, we conclude that

$$3T([y_o, x, \frac{y_o}{3^n}]) = [T(y_o), 3f(x), f(\frac{y_o}{3^n})]. \tag{31}$$

It follows from (30) and (31) that

$$[T(y_o), f(3x) - 3f(x), f(\frac{y_o}{3^n})] = 0. \tag{32}$$

Multiply both sides of (32) by 3^n and letting $n \to \infty$ we arrive at $[T(y_o), f(3x) - 3f(x), T(y_o)] = 0$.

By assumption, we have $0 \neq T(y_o) \in S(B)$ and $f(3x) - 3f(x) \in Z(B)$. According to the property of C^*-norm we obtain $f(3x) - 3f(x) = 0$ for all x in A. By induction, we find out that $3^n f(\frac{x}{3^n}) = f(x)$ for all x in A and n in $\mathbb{N}$. Taking the limit we have $T(x) = f(x)$ for all x in A. Hence f is a $*-$homomorphism. □

References

[1] M. Bavand Savadkouhi, M. E. Gordji, J. M. Rassias and N. Ghobadipour, *Approximate ternary Jordan derivations on Banach ternary algebras*, J. Math. Phys. 50, 042303 (2009), 9 pages.

[2] A. Cayley, cambridge Math.4, P.1 (1845).

[3] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequat. Math. 27 (1984), 76–86.

[4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg 62 (1992), 59–64.

[5] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, London, 2002.

[6] A. Ebadian, N. Ghobadipour, M. E. Gordji, *A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras*, J. Math. Phys. 51, 1, 2010, 10 pages, doi:10.1063/1.3496391.

[7] A. Ebadian, A. Najati and M. E. Gordji, *On approximate additive–quartic and quadratic–cubic functional equations in two variables on abelian groups*, Results. Math. DOI 10.1007/s00025-010-0018-4 (2010)

[8] M. Eshaghi Gordji, M. B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, A. Ebadian, *On the stability of J^*-derivations*, Journal of Geometry and Physics 60 (3) (2010) 454–459.

[9] M. Eshaghi Gordji, N. Ghobadipour, *Stability of $(\alpha, \beta, \gamma)-$derivations on Lie C^*-algebras*, International Journal of Geometric Methods in Modern Physics, Vol. 7, No. 7 (2010) 1-10, DOI: 10.1142/S0219887810004737.

[10] M. Eshaghi Gordji, H. Khodaei and R. Khodabakhsh, *On approximate $n-$ary derivations*, Int. J. Geom. Methods Mod. Phys., Vol.8, No 3, (2011), 485–500.

[11] M. Eshaghi Gordji and H. Khodaei, Stability of Functional Equations, LAP LAMBERT Academic Publishing, 2010.

[12] M. Eshaghi Gordji and A. Najati, *Approximately J^*-homomorphisms: A fixed point approach*, Journal of Geometry and Physics **60** (2010), 809–814.

[13] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. 14 (1991), 431–434.

[14] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994), 431–436.

[15] P. Găvruta and L. Găvruta, *A new method for the generalized Hyers-Ulam-Rassias stability*, Int. J. Nonlinear Anal. Appl. 1 (2010), 11–18.

[16] M. E. Gordji and H. Khodaei, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi–Banach spaces*, Nonlinear Anal. 71 (2009), 5629–5643.

[17] A. Himbert, Comptes Rendus de l'Acad. Sci. Paris, (1985).

[18] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27 (1941), 222–224.

[19] D. H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[20] S. M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.

[21] S. M. Jung, *Hyers–Ulam–Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. 126 (1998), 3137–3143.

[22] S. M. Jung, *Stability of the quadratic equation of Pexider type*, Abh. Math. Sem. Univ. Hamburg 70 (2000), 175–190.

[23] R. Kerner, *Z_3-graded algebras and the cubic root of the supersymmetry translations*, *J. Math. phys.* 33(1), 403-411 (1992).

[24] R. Kerner, *The cubic chessboard: Geometry and physics, Class. Quantum Grav*, 14, A203 (1997).

[25] H. Khodaei and Th. M. Rassias, *Approximately generalized additive functions in several variables*, Int. J. Nonlinear Anal. Appl. 1 (2010), 22–41.

[26] Gerald J. Murphy, C^*-Algebras and operator theory, *Academic Press.Inc.* 1990.

[27] S. Okabo, *Triple products and Yang-Baxter equation I& II. Octonionic and quaternionic triple systems*, J. Math. phys. 34(7), 3273-3291 (1993) and 34(7), 3292-3215.

[28] C. Park, *Approximate homomorphims on JB*-tribles* , J. Math. Anal. Appl. 306 (2005) 375-381. MR2132907 (2005m:46113).

[29] C. Park, *Isomorphisms between unital C*-algebras*, J. Math. Anal. Appl. 307 (2005) 753-762. MR2142458 (2005k:46140).

[30] C. Park, *Lie *-homomorphisms between Lie C*-algebras and Lie *-derivations on Lie C*-algebras*, J. Math. Anal. Appl. 293 (2004) 419-434. MR2053888 (2005f:47092).

[31] C. Park, *Homomorphisms between Lie JC*-algebras and Cauchy-Rassias stability of Lie JC*-algebra derivations*, J. Lie theory 15 (2005) 393-414. MR2147435 (2006b:17033).

[32] C. Park, *Homomorphisms between poisson JC^*-algebras*, Bull. Braz. math. Soc. 36(2005).

[33] C. Park and M. E. Gordji, *Comment on Approximate ternary Jordan derivations on Banach ternary algebras [Bavand Savadkouhi et al. J. Math. Phys. 50, 042303 (2009)]*, J. Math. Phys. 51, 044102 (2010), 7 pages.

[34] C. Park, A. Najati, *Homomorphisms and derivations in C*-algebras*, Abst. Appl. Anal. 2007(2007), Article ID 80630. MR2302193 (2007m:46100).

[35] J. M. Rassias and H. M. Kim, *Approximate homomorphisms and derivations between C^*-ternary algebras*, J. Math. Phys. 49 (2008), no. 6, 063507, 10 pp. 46Lxx (39B82).

[36] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297–300.

[37] Th. M. Rassias, *New characterization of inner product spaces*, Bull. Sci. Math. 108 (1984), 95–99.

[38] Th. M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. 114 (1992), 989–993.

[39] J. J. Sylvester, Johns Hopkins Circ. Journ., 3 P.7(1883).

[40] S. M. Ulam, *Problems in Modern Mathematics*, *Chapter VI, science ed. Wiley, New York,* 1940.

[41] H. Zettl, *A characterization of ternary rings of operators*, *Adv. Math.* 48 (1983) 117-143. MR0700979 (84h:46093).

M. Eshaghi Gordji,
Department of Mathematics,
Semnan University,
P. O. Box 35195-363, Semnan, Iran
Email: madjid.eshaghi@gmail.com

A. Fazeli,
Science and Research Branch,
Islamic AZAD University (IAU),
Tehran, Iran.
Email: asad.fazeli@gmail.com