An. Şt. Univ. Ovidius Constanţa **Vol. 19(3), 2011, 167–174**

Roman domination perfect graphs

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Abstract

A Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex $v \in V(G)$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on G. A Roman dominating function $f : V(G) \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of $V(G)$, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. A Roman dominating function $f = (V_0, V_1, V_2)$ on a graph G is an independent Roman dominating function if $V_1 \cup V_2$ is an independent set. The independent Roman domination number $i_R(G)$ of G is the minimum weight of an independent Roman dominating function on G. In this paper, we study graphs G for which $\gamma_R(G) = i_R(G)$. In addition, we investigate so called Roman domination perfect graphs. These are graphs G with $\gamma_R(H) = i_R(H)$ for every induced subgraph H of G.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* $deg(x)$ of a vertex x denotes the number of neighbors of x in G, and $\Delta(G)$ is the *maximum degree* of G. Also $\delta(G)$ is the *minimum degree* of G. A set of vertices S in G is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. If S is a subset of $V(G)$, then we denote by $G[S]$

Key Words: Domination, Roman domination, Independent Roman domination
Mathematics Subject Classification: 05C69

the subgraph of G induced by S. We write K_n for the *complete graph* of order n. By $\overline{G}$ we denote the *complement* of the graph G. A subset S of vertices is *independent* if $G[S]$ has no edge. For notation and graph theory terminology in general we follow [5] or [9].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. A Roman dominating function $f : V(G) \to \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) of $V(G)$, where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i = 0, 1, 2$. A function $f = (V_0, V_1, V_2)$ is called a γ_R-function (or $\gamma_R(G)$-function when we want to refer f to G) if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. Roman domination has been studied, for example, in [3, 2, 6, 7].

Independent Roman dominating functions in graphs were studied by Adabi et al. in [1]. A RDF $f = (V_0, V_1, V_2)$ in a graph G is an independent RDF, or just IRDF, if $V_1 \cup V_2$ is independent. The *independent Roman domination number* $i_R(G)$ of G is the minimum weight of an IRDF of G. An IRDF with minimum weight in a graph G will be referred to as an i_R-function. The definitions imply that $\gamma_R(G) \leq i_R(G)$ for any graph G.

In this paper, we study graphs G for which $\gamma_R(G) = i_R(G)$. In addition, we investigate so-called Roman domination perfect graphs. These are graphs G with $\gamma_R(H) = i_R(H)$ for every induced subgraph H of G. We frequently use the following.

Lemma 1. *([1]) Let $f = (V_0, V_1, V_2)$ be a RDF for a graph G. If V_2 is independent, then there is an independent RDF g for G such that $w(g) \leq w(f)$.*

2 On graphs G with $\gamma_R(G) = i_R(G)$

We start with characterizations of graphs G with $i_R(G) = 2$, $i_R(G) = 3$, $i_R(G) = 4$ and $i_R(G) = 5$. The proof is straightforward, and so is omitted.

Proposition 2. *(1) For a graph G of order $n \geq 2$, $i_R(G) = 2$ if and only if $G = \overline{K_2}$ or $\Delta(G) = n - 1$.*

(2) For a graph G of order $n \geq 3$, $i_R(G) = 3$ if and only if either $G = \overline{K_3}$ or $\Delta(G) = n - 2$.

(3) For a graph G of order $n \geq 4$, $i_R(G) = 4$ if and only if one of the following conditions holds:
(i) $G = \overline{K_4}$.
(ii) $\Delta(G) = n - 3$, and G contains a vertex v of maximum degree such that

$G[V(G) - N[v]] = \overline{K_2}$.
(iii) $\Delta(G) \leq n-3$ and there are two nonadjacent vertices u, v in G such that $N_G[u] \cup N_G[v] = V(G)$.

(4) For a graph G of order $n \geq 5$, $i_R(G) = 5$ if and only if one of the following conditions hold:
(i) $G = \overline{K_5}$.
(ii) $\Delta(G) \leq n-4$ and $|N_G[x] \cup N_G[y]| \leq |V(G)|-1$ for all pairs of nonadjacent vertices $x, y \in V(G)$. In addition, there are two nonadjacent vertices u, v in G such that $|N_G[u] \cup N_G[v]| = |V(G)| - 1$ or G contains a vertex v of degree $n-4$ such that $G[V(G) - N[v]] = \overline{K_3}$.

According to Lemma 1, the following is obviously verified.

Lemma 3. *For a graph G, $\gamma_R(G) = i_R(G)$ if and only if there is a γ_R-function $f = (V_0, V_1, V_2)$ for G such that $G[V_2]$ has no edge.*

We note that a forbidden subgraph characterization for the graphs G having $\gamma_R(G) = i_R(G)$ cannot be obtained since for any graph G, the addition of a new vertex that is adjacent to all vertices of G produces a new graph H with $\gamma_R(H) = i_R(H) = 2$.

Theorem 4. *Let $k \geq 2$ be an integer. If a graph G of order $n > 1$ does not contain the star $K_{1,k+1}$ as an induced subgraph, then*

$$i_R(G) \leq (k-1)\gamma_R(G) - 2(k-2).$$

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_R-function for G. Let I be a maximal independent subset of V_2. Then I is a dominating set for V_2. Let $X = V(G) - (N[I] \cup V_1)$, and let Y be a maximal independent subset of X. Then Y is a dominating set for X. Since G is $K_{1,k+1}$-free, any vertex of $V_2 - I$ is adjacent to at most $k-1$ vertices of Y. We deduce that $|Y| \leq (k-1)|V_2 - I|$. Now define $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(v) = 2$ if $v \in I \cup Y$, $g(v) = 1$ if $v \in V_1$, and $g(v) = 0$ otherwise. Then g is a RDF for G. Now

$$\begin{aligned} w(g) &\leq 2(k-1)|V_2 - I| + 2|I| + |V_1| \\ &= 2(k-1)|V_2| - 2(k-2)|I| + |V_1| \\ &\leq 2(k-1)|V_2| - 2(k-2)|I| + (k-1)|V_1| \\ &= (k-1)(2|V_2| + |V_1|) - 2(k-2)|I| \\ &\leq (k-1)\gamma_R(G) - 2(k-2). \end{aligned}$$

Now the result follows by Lemma 1. □

Next we will list some properties of the $K_{1,k+1}$-free graphs G with $i_R(G) = (k-1)\gamma_R(G) - 2(k-2)$. Of course, we may assume that $k \geq 3$, since for $k = 2$ it is the well-known family of claw-free graphs.

If $i_R(G) = (k-1)\gamma_R(G) - 2(k-2)$, then, using the notation of the proof of Theorem 4 equality holds at each point in the above sequence of inequalities.

The equality $2(k-2)|I| = 2(k-2)$ implies that $|I| = 1$ for every choice of I, and thus $G[V_2]$ is complete.

The equality $|V_1| = (k-1)|V_1|$ leads to $|V_1| = 0$. This implies that $\gamma_R(G) = 2|V_2|$. Because of $|Y| = (k-1)|V_2 - I|$, we note (i) that every maximal independent set Y in $G[X]$ has $(k-1)(|V_2| - 1)$ vertices, with exactly $k-1$ vertices adjacent to each vertex of $V_2 - I$. Furthermore, every vertex in X is joined to exactly one vertex of $V_2 - I$, otherwise, Y can be chosen to contain a vertex joined to at least two vertices of $V_2 - I$, contradicting (i).

As a consequence of Theorem 4, we obtain the following corollary.

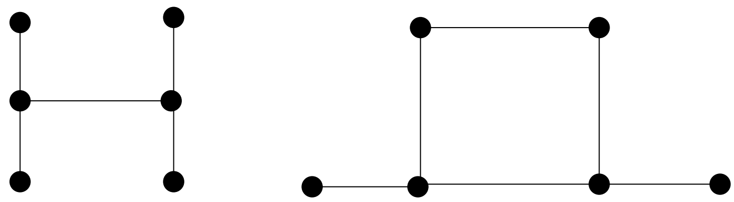
Corollary 5. *If G is a claw-free graph, then $\gamma_R(G) = i_R(G)$.*

Since any line graph is claw-free, Corollary 5 implies that $\gamma_R(L(G)) = i_R(L(G))$, where $L(G)$ is the line graph of G.

3 Roman domination perfect graphs

In 1990, Sumner [8] defines a graph G to be *domination perfect* if $\gamma(H) = i(H)$ for any induced subgraph H of G, where $i(H)$ is the independent domination number of H. Fulman [4] showed that the absence of all of the eight induced subgraphs of Figure 1 in G is sufficient for G to be domination perfect.

Theorem 6. (Fulman [4] 1993) *If a graph G does not contain any of the graphs in Figure 1 as an induced subgraph, then G is domination perfect.*



G_1 G_2

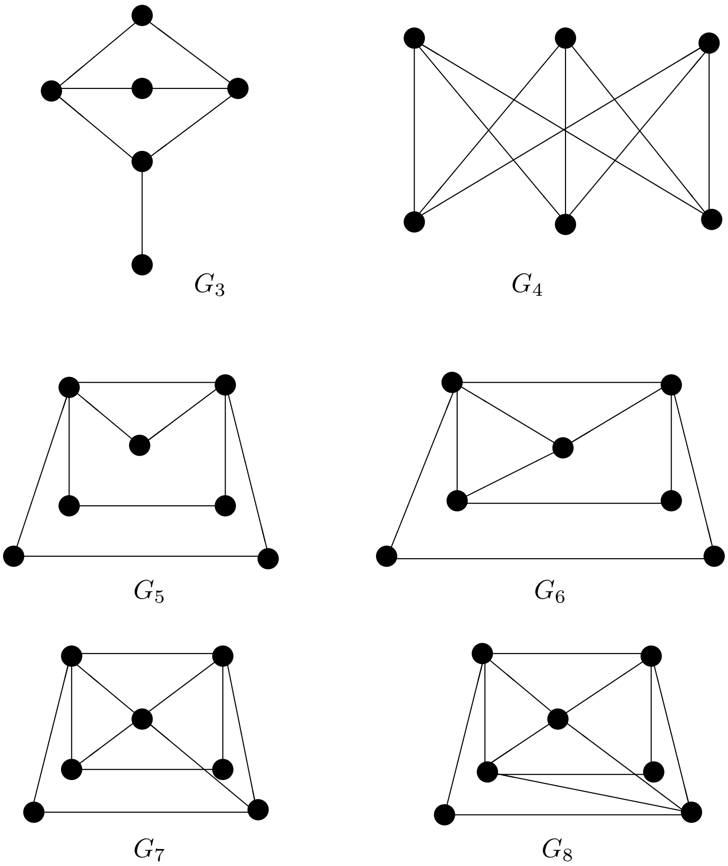


Figure 1.

We next consider a closely related concept. A graph G is called *Roman domination perfect* if $\gamma_R(H) = i_R(H)$ for any induced subgraph H of G. For $x \in X \subseteq V(G)$, we define $I(x, X) = N[x] - N[X - \{x\}]$. Note that $I(x, X)$ is the set of vertices dominated by x but not by the rest of X. Corollary 5 implies that if G has no induced subgraph isomorphic to the claw $K_{1,3}$, then G is domination perfect. Following the ideas in [4] and [10], we now prove an analogue to Theorem 6.

Theorem 7. *If a graph G does not contain any of the graphs in Figure 1 as an induced subgraph, then G is Roman domination perfect.*

Proof. It suffices to prove that if G does not contain the graphs in Figure 1 as induced subgraphs, then $\gamma_R(G) = i_R(G)$. Suppose to the contrary that $\gamma_R(G) < i_R(G)$, and let $f = (V_0, V_1, V_2)$ be a γ_R-function for G such that the number of edges of the induced subgraph $G[V_2]$ is minimum. It follows from our assumption $\gamma_R(G) < i_R(G)$ and Lemmas 1 and 3 that V_2 is not dependent. Let u, v be two adjacent vertices in V_2. Since f is a γ_R-function, $I(u, V_2)$ and $I(v, V_2)$ are disjoint sets each of cardinality at least two. Since the number of edges in $G[V_2]$ is minimum, $I(u, V_2)$ as well as $I(v, V_2)$ do not contain a dominating vertex. Thus there exist $a_1, a_2 \in I(u, V_2)$ and $b_1, b_2 \in I(v, V_2)$ such that $a_1a_2 \notin E(G)$ and $b_1b_2 \notin E(G)$. If each vertex of $I(u, V_2)$ is adjacent to each vertex of $I(v, V_2)$, then G contains an induced subgraph isomorphic to G_4, a contradiction. Hence it remains that case that there are two nonadjacent vertices $u_1 \in I(u, V_2)$ and $v_1 \in I(v, V_2)$.

If $\{u_1, v_1\}$ does not dominate the set $I = I(u, V_2) \cup I(v, V_2)$, then there exists a vertex $u_2 \in I(u, V_2) \cup I(v, V_2)$ such that $u_2u_1 \notin E(G)$ and $u_2v_1 \notin E(G)$. We assume, without loss of generality, that $u_2 \in I(u, V_2)$. As $I(v, V_2)$ does not contain a dominating vertex, we see that there is a vertex $v_2 \in I(v, V_2)$ such that $v_2v_1 \notin E(G)$. Considering the subgraph $H = G[\{u, v, u_1, v_1, u_2, v_2\}]$, it is easy to see that depending on the existence of edges u_1v_2 and u_2v_2, the subgraph H is isomorphic to one of G_1, G_2 or G_3, a contradiction. So we assume next that $\{u_1, v_1\}$ dominates the set $I = I(u, V_2) \cup I(v, V_2)$.

Since $D = (V_2 - \{u, v\}) \cup \{u_1, v_1\}$ has fewer edges than V_2, the function $(V(G) - (V_1 \cup D), V_1, D)$ is not a RDF. Thus there exists a vertex w that is not dominated by D. The definition of D shows that w must be adjacent to u or to v. Moreover, since $\{u_1, v_1\}$ dominates I, the vertex w does not belong to I. This implies that w must be adjacent to both u and v. Since $I(u, V_2)$ does not contain a dominating vertex, there is a vertex $u_2 \in I(u, V_2)$ such that $u_1u_2 \notin E(G)$. Similarly, there is a vertex $v_2 \in I(v, V_2)$ such that $v_1v_2 \notin E(G)$. As $\{u_1, v_1\}$ dominates the set I, we find that $\{u_1v_2, v_1u_2\} \subseteq E(G)$. Now consider the subgraph $H = G[\{u, v, w, u_1, v_1, u_2, v_2\}]$. The only edges in H whose existence is undetermined are u_2v_2, u_2w and v_2w. If none is present, H is isomorphic to G_5, a contradiction. If only u_2v_2 is present, then $H - v$ is isomorphic to G_2, a contradiction. If only u_2w or if only v_2w is present, then we obtain the contradiction that H is isomorphic to G_6. If only u_2v_2 and u_2w are present, then $H - u$ is isomorphic to G_3, a contradiction. The same occurs if only u_2v_2 and v_2w are present. Finally, if only u_2w and v_2w are present, H is isomorphic to G_7, and if all three edges are present, H is isomorphic to G_8. In both cases a contradiction, and the proof is complete. □

Recall that a graph is called *chordal* if every cycle of length exceeding three has an edge joining two nonadjacent vertices in the cycle.

Corollary 8. *If a chordal graph G does not contain G_1 as an induced subgraph, then G is Roman domination perfect.*

Proof. Assume that G does not contain G_1 as an induced subgraph. Note that the graphs $G_2, G_3, \ldots, G_8$ in Figure 1 are not chordal. Applying Theorem 7, we deduce that G is Roman domination perfect. □

Note that since the graph G_1 is Roman domination perfect, the converses of Theorem 7 and Corollary 8 are false.

The proofs of the next two corollaries are similar to that of Corollary 8.

Corollary 9. *If a graph G of girth at least five does not contain G_1 as an induced subgraph, then G is Roman domination perfect.*

Corollary 10. *If a bipartite graph G does not contain G_1, G_2, G_3 and G_4 as induced subgraphs, then G is Roman domination perfect.*

The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by subdividing each edge of G. A subdivision graph $S(G)$ does not contain two adjacent vertices u and v such that $deg(u) \geq 3$ and $deg(v) \geq 3$. Since each graph of $G_1, G_2, \ldots, G_8$ has two adjacent vertices of degree at least three, the next result follows from Theorem 7.

Corollary 11. *If $S(G)$ is the subdivision graph of a graph G, then $S(G)$ is Roman domination perfect.*

References

[1] M. Adabi, E. Ebrahimi Targhi, N. Jafari Rad and M. Saied Moradi, *Properties of independent Roman domination in graphs*, submitted for publication.

[2] E.W. Chambers., B. Kinnersley, N. Prince, and D.B. West, *Extremal Problems for Roman Domination*, SIAM J. Discr. Math., 23 (2009), 1575-1586.

[3] E. J. Cockayne, P. M. Dreyer Jr., S. M. Hedetniemi, and S. T. Hedetniemi, *On Roman domination in graphs*, Discrete Math. **278** (2004), 11-22.

[4] J. Fulmann, *A note on the characterization of domination perfect graphs*, J. Graph Theory, 17 (1993), 47-51.

[5] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, NewYork, 1998.

[6] C. S. ReVelle, K. E. Rosing, *Defendens imperium romanum: a classical problem in military strategy*, Amer. Math. Monthly **107** (2000), 585-594.

[7] I. Stewart, *Defend the Roman Empire!*, Sci. Amer. **281** (6) (1999), 136 – 139.

[8] D. P. Sumner, *Critial concepts in domination*, Discrete Math. **86** (1990), 33-46.

[9] D. B. West, *Introduction to graph theory*, (2nd edition), Prentice Hall, USA (2001).

[10] I. E. Zverovich and V. E. Zverovich, *A characterization of domination perfect graphs*, J. Graph Theory, 15 1991, 109-114.

Acknowledgment. This research is supported by Shahrood University of Technology

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