On Henselian valuations and Brauer groups of primarily quasilocal fields

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Abstract

This paper finds a classification, up-to an isomorphism, of abelian torsion groups realizable as Brauer groups of major types of Henselian valued primarily quasilocal fields with totally indivisible value groups. When E is a quasilocal field with such a valuation, it shows that the Brauer group of E is divisible and embeddable in the quotient group of the additive group of rational numbers by the subgroup of integers.

1 Introduction and statement of the main result

A field K is said to be primarily quasilocal (abbr, PQL), if every cyclic extension F of K is embeddable as a subalgebra in each central division K-algebra D of Schur index $\mathrm{ind}(D)$ divisible by the degree $[F\colon K]$; we say that K is quasilocal, if its finite extensions are PQL-fields. This paper is devoted to the study of the Brauer group $\mathrm{Br}(K)$ when K is PQL and possesses a Henselian valuation v. It determines the structure of the p-component $\mathrm{Br}(K)_p$ of $\mathrm{Br}(K)$, for a given prime number p, under the hypothesis that the value group $v(K)$ of (K, v) is p-indivisible, i.e. $v(K) \neq pv(K)$. This enables us to describe the isomorphism classes of Brauer groups of Henselian PQL-fields with totally indivisible value groups (i.e. p-indivisible, for each prime p), and to do the same in the special case where the considered valued fields are quasilocal. The method of proving our main results makes it possible to establish the existence of new types of Henselian real-valued quasilocal fields, which make interest in

the context of the recently posed problem of characterizing central division algebras over finitely-generated fields F by their splitting fields of finite degree over F (see Proposition 6.5 and the comment on Remark 6.6).

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [8], I, and [9]. Throughout, Brauer and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. As usual, $\mathbb{Q}/\mathbb{Z}$ denotes the quotient group of the additive group of rational numbers by the subgroup of integers. We write $\mathbb{P}$ for the set of prime numbers, and for each $p \in \mathbb{P}$, $\mathbb{F}_p$ is a field with p elements, $\mathbb{Z}_p$ is the additive group of p-adic integers and $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group. For any profinite group G, we denote by $\mathrm{cd}(G)$ the cohomological dimension of G, and by $\mathrm{cd}_p(G)$ its cohomological p-dimension, for each $p \in \mathbb{P}$. Given a field E, E_{sep} denotes a separable closure of E, $\mathcal{G}_E = \mathcal{G}(E_{\mathrm{sep}}/E)$ is the absolute Galois group of E, $\Pi(E) = \{p \in \mathbb{P}\colon \mathrm{cd}_p(\mathcal{G}_E) \neq 0\}$ and $P(E)$ is the set of those $p \in \mathbb{P}$, for which E is properly included in its maximal p-extension $E(p)$ in E_{sep}. In what follows, for any $p \in P(E)$, $r(p)_E$ denotes the rank of $\mathcal{G}(E(p)/E)$, i.e. the cardinality of any minimal system of generators of $\mathcal{G}(E(p)/E)$ as a profinite group; we put $r(p)_E = 0$ in case $p \notin P(E)$. We write $s(E)$ for the class of finite-dimensional central simple E-algebras, $d(E)$ stands for the class of division algebras $D \in s(E)$, and for each $A \in s(E)$, $[A]$ is the similarity class of A in $\mathrm{Br}(E)$. For any field extension E'/E, we denote by $I(E'/E)$ the set of its intermediate fields, and by $\rho_{E'/E}$ the scalar extension map of $\mathrm{Br}(E)$ into $\mathrm{Br}(E')$. When E'/E is finite and separable, $\mathrm{Cor}_{E'/E}$ denotes the corestriction homomorphism of $\mathrm{Br}(E')$ into $\mathrm{Br}(E)$. For convenience of the reader, we recall that E is said to be stable, if each $D \in d(E)$ has exponent $\exp(D)$ equal to $\mathrm{ind}(D)$; we say that E is absolutely stable, if its finite extensions are stable fields. The field E is called p-quasilocal, for some $p \in \mathbb{P}$, if one of the following conditions holds: (i) $\mathrm{Br}(E)_p \neq \{0\}$ or $p \notin P(E)$; (ii) every extension of E in $E(p)$ of degree p is embeddable as an E-subalgebra in each $\Delta_p \in d(E)$ of index p. By [8], I, Theorem 4.1, E is PQL if and only if it is p-quasilocal, for each $p \in P(E)$. In this paper, we use at crucial points the following characterization of the p-quasilocal property (which is obtained as a consequence of [8], I, Theorems 3.1 (i)-(ii) and 4.1, and the general restriction-corestriction (abbr, RC) formula for Brauer groups, see, e.g., [42], Theorem 2.5):

(1.1) A field E is p-quasilocal, for some $p \in \mathbb{P}$, if and only if $\mathrm{Cor}_{M/E}$ maps $\mathrm{Br}(M)_p$ injectively into $\mathrm{Br}(E)_p$, for each finite extension M of E in $E(p)$. When this is the case and $\mathrm{Br}(E)_p$ is divisible, $\mathrm{Cor}_{M/E}$ maps $\mathrm{Br}(M)_p$ bijectively upon $\mathrm{Br}(E)_p$, for every M of the considered type.

The present research is naturally incorporated in the study of Brauer

groups of the basic types of stable fields. This problem has two major aspects. In the first place, the structure of $\mathrm{Br}(F)$ of stable fields F makes interest in the context of index-exponent relations in central simple algebras over arbitrary fields (cf. [31], Sects. 14.4 and 19.6). In the absolutely stable case, the discussed problem is also related to the study of cohomological properties of $\mathcal{G}_E$ (see [31], Sect. 14.6, and [8], I, Theorem 8.1). Secondly, the description of $\mathrm{Br}(L)$, for a given stable field L, usually reflects adequately an essential part of the specific nature of L. Note in this connection that important classes of stable fields L have been singled out by analyzing special properties of L. In particular, this applies to the absolute stability of global fields (cf. [36], (32.19), function fields of algebraic curves defined over a PAC-field [15], function fields of algebraic surfaces over an algebraically closed field of zero characteristic [23] (see also [26]), and quasilocal fields [8], I, Proposition 2.3. In these cases, $\mathrm{cd}(\mathcal{G}_{L(\sqrt{-1})}) \leq 2$ (cf. [40], Ch. II, 3.3 and 4.1, and [5], I, Sect. 4), which ensures that $\mathrm{Br}(L)$ is a divisible group unless L is formally real (see [40], Ch. II, 2.3, and [12], page 110). The study of the stability property in the class of Henselian fields (K, v) sheds new light on the considered problem. As it turns out, the residue field $\widehat{K}$ of (K, v) is PQL whenever K is stable and the value group $v(K)$ is totally indivisible [8], I, Proposition 2.1. Moreover, these conditions frequently ensure that $\widehat{K}$ is almost perfect, i.e. its finite extensions have primitive elements (cf. [4], Theorem 2.1, and [8], I, (1.8) and Proposition 2.3). The relations between $v(K)$ and $\widehat{K}$ make it possible to characterize basic classes of Henselian stable fields (see Proposition 2.2 and [4], Theorem 3.1 and Sect. 4). They also show explicitly how Brauer and character groups of PQL-fields determine the structure of Brauer groups of stable fields (see Remark 2.4 and [8], II).

Brauer groups of PQL-fields also have strong influence on the study of the norm groups of their finite abelian extensions. Specifically, this applies to the relations described by the second part of the following assertion (cf. [9], Theorem 3.1, and [8], I, Lemma 4.2 (ii)):

(1.2) Let E be a p-quasilocal field, $\Omega_p(E)$ the set of finite abelian extensions of E in $E(p)$, $\mathrm{Nr}(E)$ the set of norm groups of E, and $_p\mathrm{Br}(E) = \{b \in \mathrm{Br}(E) \colon pb = 0\}$. Then:

(i) The natural mapping of $\Omega_p(E)$ into $\mathrm{Nr}(E)$ (by the rule $M \to N(M/E)$, $M \in \Omega_p(E)$) is injective, and for each $M_1, M_2 \in \Omega_p(E)$, the norm group (over E) of the compositum $M_1 M_2$ equals the intersection $N(M_1/E) \cap N(M_2/E)$, and $N(M_1 \cap M_2/E) = N(M_1/E)N(M_2/E)$.

(ii) For each $M \in \Omega_p(E)$, the quotient group $E^*/N(M/E)$ decomposes into a direct sum $\mathcal{G}(M/E)^{d(p)}$ of isomorphic copies of the Galois group $\mathcal{G}(M/E)$, indexed by a set of cardinality $d(p)$, the dimension of $_p\mathrm{Br}(E)$ as a vector space

over $\mathbb{F}_p$. In particular, if $\mathrm{Br}(E)_p = \{0\}$, then $N(M/E) = E^*$.

When E is a PQL-field and L/E is a finite abelian extension, it follows from (1.2) (ii) and [9], Lemma 2.1, that $E^*/N(L/E)$ is isomorphic to the direct product of the groups $E^*/N(L_p/E)$: $p \in P(E), p \mid [L: E]$, where $L_p = L \cap E(p)$, for each admissible p. This is an analogue to the local reciprocity law whose form is determined by the sequence $d(p)$: $p \in P(E)$, defined in (1.2) (ii). It is therefore worth noting that an abelian torsion group is isomorphic to $\mathrm{Br}(\Phi)$, for some PQL-field $\Phi = \Phi(T)$ if and only if one of the following conditions holds (see [10], Sect. 1 and Proposition 6.4):

(1.3) (i) T is divisible; then Φ is nonreal and can be chosen among those quasilocal fields, for which the maps $\rho_{\Phi/\Phi'}$, $\Phi' \in I(\Phi_{\mathrm{sep}}/\Phi)$, are surjective;

(ii) The 2-component T_2 is of order 2 and the p-components T_p, $p \in (\mathbb{P} \setminus \{2\})$, are divisible; in this case, Φ is formally real.

Since the notion of a quasilocal field extends the one of a local field and defines a class containing such frequently used representatives as p-adically closed fields and Henselian discrete valued fields with quasifinite residue fields (cf. [41], Ch. XIII, Sect. 3, and [34], Theorem 3.1 and Lemma 2.9), these facts attract interest in the role of Henselian valuations for arbitrary quasilocal fields. The main results of this paper, stated below, enable one to evaluate this role by comparing (1.3) with the structure of $\mathrm{Br}(K)$ when (K, v) is a Henselian quasilocal field, such that $v(K)$ is totally indivisible (see also Corollaries 5.3 and 5.4):

Theorem 1.1. *Let (K, v) be a Henselian p-quasilocal field with $v(K) \neq pv(K)$, for some $p \in \mathbb{P}$. Then:*

(i) $\mathrm{Br}(K)_p$ is trivial or isomorphic to $\mathbb{Z}(p^\infty)$ except, possibly, in the case where $r(p)_K = 1$, $\mathrm{char}(K) \neq p$ and K does not contain a primitive p-th root of unity;

(ii) K is subject to the following alternative relative to v:

(α) There exists a $\mathbb{Z}_p$-extension I_∞ of K in $K(p)$, such $v_\infty(I_\infty) = v(K)$ and the residue field of (I_∞, v_∞) is separable over $\widehat{K}$, where v_∞ is the unique, up-to an equivalence, valuation of I_∞ extending v;

(β) Finite extensions of K in $K(p)$ are totally ramified;

(iii) When $p \in P(K)$, $\mathrm{Br}(K)_p = \{0\}$ if and only if finite extensions of K in $K(p)$ are totally ramified and the group $v(K)/pv(K)$ is of order p.

When $p \neq \mathrm{char}(\widehat{K})$ and $\mathrm{Br}(K)_p \neq \{0\}$, the isomorphism $\mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$ is established in Section 4 by proving the following assertion:

(1.4) If $p \neq \operatorname{char}(\widehat{K})$ and K contains a primitive p-th root of unity, then $\mathcal{G}(K(p)/K)$ is a Demushkin group (in the sense of [40]) with $r(p)_K = 2$ or is isomorphic to $\mathbb{Z}_p$ depending on whether or not $\operatorname{Br}(K)_p \neq \{0\}$.

The proof of Theorem 1.1 (i) in the case where $\operatorname{char}(\widehat{K}) = p$ and there exists an immediate cyclic extension I/K of degree p is presented in Section 3 (the realizability of this special case is demonstrated by Proposition 6.2). This part of the proof is based on the divisibility of $\operatorname{Br}(K)_p$ (see Lemma 3.3 (i)) as well as on (1.2) (i) and general properties of Henselian valuations and isolated subgroups of their value groups. The rest of the proof of Theorem 1.1 is contained in Section 4. When $p = \operatorname{char}(\widehat{K})$, we adapt to our setting the proof of [43], Theorem 3.1. Our argument also relies on (1.2) and on the method of proving the main results of [5], I. The remaining part of the paper presents consequences of the main result. In Section 5, we describe the isomorphism classes of Brauer groups of Henselian PQL-fields (K, v) such that $v(K)$ is totally indivisible (see Corollary 5.5, (5.2) and (5.3)). When K is quasilocal, we also prove the cyclicity of every $D \in d(K)$ (see Corollary 5.3). In Section 6, we complete the characterization of the quasilocal property in the class of Henselian fields with totally indivisible value groups, started in [5], I; also, we give a criterion for divisibility of Brauer groups of quasilocal fields, and for defectlessness of their finite separable extensions.

2   Preliminaries on Henselian valuations and completions

Let K be a field with a nontrivial (Krull) valuation v, $O_v(K) = \{a \in K : v(a) \geq 0\}$ the valuation ring of (K, v), $M_v(K) = \{\mu \in K : v(\mu) > 0\}$ the unique maximal ideal of O_v, $v(K)$ and $\widehat{K}$ the value group and the residue field of (K, v), respectively, $\operatorname{Is}'_v(K)$ the set of isolated subgroups of $v(K)$ and $\operatorname{Is}_v(K) = \operatorname{Is}'_v(K) \setminus \{v(K)\}$. It is well-known that, for each $H \in \operatorname{Is}_v(K)$, the ordering of $v(K)$ induces canonically on $v(K)/H$ a structure of an ordered group, and one can naturally associate with v and H a valuation v_H of K with $v_h(K) = v(K)/H$. Unless specified otherwise, K_H will denote the residue field of (K, v_H), η_H the natural projection $O_{v_H}(K) \to K_H$, and $\hat{v}_H$ the valuation of K_H induced canonically by v and H. The valuations v, v_H and $\hat{v}_H$ are related as follows (see [16], Proposition 5.2.1):

(2.1) (i) $\hat{v}_H(K_H) = H$, $\widehat{K}_H$ is isomorphic to $\widehat{K}$ and η_H induces a surjective homomorphism of $O_v(K)$ upon $O_{\hat{v}_H}(K_H)$; when H is divisible, $v(K)$ is isomorphic to the lexicographically ordered direct sum $v_H(K) \oplus \hat{v}_H(\widehat{K}_H)$;

(ii) If $v(K)$ properly includes the union $H(K)$ of the groups from $\operatorname{Is}_v(K)$, then $v_{H(K)}$ is real-valued.

Recall further that the topology of K induced by v_H does not depend on the choice of H and the mapping of $\mathrm{Is}_v(K)$ on the set V_v of subrings of K including O_v, defined by the rule $X \to O_{v_X}(K)$, $X \in \mathrm{Is}_v(K)$, is an inclusion-preserving bijection. By Hölder's theorem (cf. [16], Theorem 2.5.2), $\mathrm{Is}_v(K) = \{0\}$ if and only if $v(K)$ is Archimedean, i.e. it embeds as an ordered subgroup in the additive group $\mathbb{R}$ of real numbers. When this is the case, we identify $v(K)$ with its isomorphic copy in $\mathbb{R}$.

We say that (K, v) is Henselian, if the valuation v is Henselian, i.e. v is uniquely, up-to an equivalence, extendable to a valuation v_L on each algebraic field extension L/K. In order that v is Henselian, it is necessary and sufficient that the Hensel-Rychlik condition holds (cf. [16], Sect. 18.1):

(2.2) Given a polynomial $f(X) \in O_v(K)[X]$, and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the formal derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c - a) = v(f(a)/f'(a))$.

When $v(K)$ is not Archimedean, the Henselian property can be also characterized as follows (see, e.g., [11], Sect. 2):

Proposition 2.1. *Let (K, v) be a valued field, and let $H \in \mathrm{Is}_v(K)$. Then v is Henselian if and only if v_H and $\hat{v}_H$ are Henselian.*

When v is Henselian and L/K is an algebraic extension, v_L is also Henselian and extends uniquely to a valuation v_D on each $D \in d(L)$. Denote by $\widehat{D}$ the residue field of (D, v_D), put $v(D) = v_D(D)$, and let $e(D/K)$ be the ramification index of D/K, i.e. the index of $v(K)$ in $v(D)$. By the Ostrowski-Draxl theorem [13], $[D \colon K]$, $[\widehat{D} \colon \widehat{K}]$ and $e(D/K)$ are related as follows:

(2.3) $[D \colon K]$ is divisible by $[\widehat{D} \colon \widehat{K}]e(D/K)$ and $[D \colon K]/([\widehat{D} \colon \widehat{K}]e(D/K))$ is not divisible by any $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$.

The K-algebra D is said to be defectless, if $[D \colon K] = [\widehat{D} \colon \widehat{K}]e(D/K)$, and it is called totally ramified, if $e(D/K) = [D \colon K]$. The following lemma, proved in [11], Sect. 6, characterizes the case in which $v(K) \neq pv(K)$ and $I(K(p)/K)$ does not contain totally ramified extensions of K, for a given $p \in \mathbb{P}$.

Lemma 2.2. *Let (K, v) be a Henselian field with $v(K) \neq pv(K)$, for some $p \in \mathbb{P}$. Then K is subject to the following alternative relative to v:*
   (i) There exists a field $\Phi \in I(K(p)/K)$, such that $[\Phi \colon K] = p$ and Φ/K is totally ramified;
   (ii) $\mathrm{char}(K) = 0$, K does not contain a primitive p-th root of unity and the minimal group from $\mathrm{Is}'_v(K)$ containing $v(p)$ is p-divisible.

Let v be Henselian and $\mathrm{In}(K)$ be the class of inertial K-algebras (i.e. those division K-algebras D, for which $[D\colon K] = [\widehat{D}\colon \widehat{K}] \in \mathbb{N}$ and $\widehat{Z}/\widehat{K}$ is a finite separable extension, where $\widehat{Z}$ is the centre of $\widehat{D}$). Then:

(2.4) (i) For each finite-dimensional division $\widehat{K}$-algebra $\widetilde{\Delta}$ whose centre is separable over $\widehat{K}$, there exists $\Delta \in \mathrm{In}(K)$ with $\widehat{\Delta} \cong \widetilde{\Delta}$ over $\widehat{K}$; Δ is uniquely determined by $\widetilde{\Delta}$, up-to a K-isomorphism [21], Theorem 2.8 (a) (and is called an inertial lift of $\widetilde{D}$ over K).

(ii) In order that an inertial field extension Y/K is a Galois extension, it is necessary and sufficient that $\widehat{Y}/\widehat{K}$ is Galois; when this occurs, the Galois groups $\mathcal{G}(Y/K)$ and $\mathcal{G}(\widehat{Y}/\widehat{K})$ are canonically isomorphic (cf. [21], page 135).

(iii) The set $\mathrm{IBr}(K) = \{[I]\colon I \in \mathrm{In}(K)\}$ forms a subgroup of $\mathrm{Br}(K)$, which is canonically isomorphic to $\mathrm{Br}(\widehat{K})$ (cf. [21], Theorem 2.8 (b)).

Assuming as above that v is Henselian, let D be an arbitrary finite-dimensional division K-algebra, D_v a completion of D with respect to the topology induced by v, and $Z(D)$ the centre of D. It is known that there is a close relationship between finite-dimensional division K-algebras and the corresponding K_v-algebras, described in part by the following statement:

(2.5) (i) K is separably closed in K_v and the valuation $\bar{v}$ of K_v continuously extending v is Henselian;

(ii) The natural mapping of $D \otimes_K K_v$ into D_v is a K_v-isomorphism whenever $Z(D)/K$ is separable; hence, ρ_{K/K_v} is injective and preserves indices and exponents;

(iii) A finite extension L of K in K_{sep} embeds in an algebra $U \in d(K)$ if and only if L_v embeds in U_v over K_v.

Note also the following characterization of finite extensions of K_v in $K_{v,\mathrm{sep}}$, in case v is Henselian (cf. [3], Ch. VI, Sect. 8, No 2, and [21], page 135):

(2.6) (i) Every finite extension L of K_v in $K_{v,\mathrm{sep}}$ is K_v-isomorphic to $\widetilde{L} \otimes_K K_v$ and $\widetilde{L}_v$, where $\widetilde{L}$ is the separable closure of K in L; L/K_v is Galois if and only if $\widetilde{L}/K$ is Galois; when this holds, $\mathcal{G}(L/K_v) \cong \mathcal{G}(\widetilde{L}/K)$ (canonically).

(ii) $K_{\mathrm{sep}} \otimes_K K_v$ is a field, $K_{\mathrm{sep}} \otimes_K K_v \cong K_{v,\mathrm{sep}}$ and $\mathcal{G}_K \cong \mathcal{G}_{K_v}$.

Statements (2.5) reduce the study of index-exponent relations in central division algebras over a Henselian field (K, v) to the special case in which $K = K_v$. Combining, for instance, (2.5) and (2.6) (i) with [48], Proposition 2.1, and [4], Theorem 3.1, one obtains the following result:

Proposition 2.3. *Let (K, v) be a Henselian discrete valued field. Then:*
(i) K is stable, provided that $\widehat{K}$ is almost perfect, stable and PQL.
(ii) K is absolutely stable if and only if $\widehat{K}$ is quasilocal and almost perfect.

Remark 2.4. *Statement (1.3) and the method of proving it in [10], combined with Proposition 2.2 and Scharlau's generalization of Witt's decomposition theorem [38], make it possible to study effectively the structure of the Brauer groups of various types of stable fields (see also [8], II, Sect. 3 and Lemma 2.3, for the relations between $Br(E)$ and the character group of $\mathcal{G}_E$). Specifically, they enable one to find series of absolutely stable fields F with Henselian valuations and indivisible groups $Br(F)$, for infinitely many $p \in \mathbb{P}$ (see [4], Corollary 4.7, and [10], Proposition 6.8).*

Let us note that, for every Henselian field (K, v) with $\mathrm{char}(K) = p$ and $v(K) \neq pv(K)$, $r(p)_K = \infty$, i.e. $\mathcal{G}(K(p)/K)$ is not finitely-generated as a pro-p-group (cf. [11], Remark 4.2). Therefore, the following two propositions (proved in [11], and generalizing [32], (2.7)) fully characterize the case where $r(p)_K \in \mathbb{N}$. The characterization depends on whether or not the minimal group $G(K) \in \mathrm{Is}'_v(K)$ containing $v(p)$ is p-divisible.

Proposition 2.5. *Let (K, v) be a Henselian field with $\mathrm{char}(K) = 0$, $\mathrm{char}(\widehat{K}) = p > 0$ and $G(K) = pG(K)$, and let ε be a primitive p-th root of unity in K_{sep}. Then $r(p)_K \in \mathbb{N}$ if and only if $r(p)_{K_{G(K)}} \in \mathbb{N}$ and one of the following conditions holds:*

(i) $v(K) = pv(K)$ or $\varepsilon \notin K$; in this case, finite extensions of K in $K(p)$ are inertial relative to $v_{G(K)}$ and $\mathcal{G}(K(p)/K) \cong \mathcal{G}(K_{G(K)}(p)/K_{G(K)})$;

(ii) $\varepsilon \in K$ and $v(K)/pv(K)$ is of order p^τ, for some $\tau \in \mathbb{N}$; in this case, $\mathcal{G}(K(p)/K)$ is isomorphic to a topological semi-direct product $\mathbb{Z}_p^\tau \times \mathcal{G}(K_{G(K)}(p)/K_{G(K)})$.

When $r(p)_K \in \mathbb{N}$, $\widehat{K}$ is perfect, $\mathcal{G}(K_{G(K)}(p)/K_{G(K)})$ is a trivial or a free pro-p-group, and either $p \in P(\widehat{K})$ or $K_{G(K)}(p)/K_{G(K)}$ is immediate relative to $\hat{v}_{G(K)}$.

The concluding part of our next result is implied by (2.3), [43], Proposition 2.2.

Proposition 2.6. *Let (K, v) be a Henselian field with $\mathrm{char}(K) = 0$ and $\mathrm{char}(\widehat{K}) = p \neq 0$. Then the following conditions are equivalent:*

(i) $G(K) \neq pG(K)$ and $r(p)_K \in \mathbb{N}$;

(ii) $\widehat{K}$ is finite, $G(K)$ is cyclic, and in case K contains a primitive p-th root of unity, the group $v_{G(K)}(K)/pv_{G(K)}(K)$ is finite.

When these conditions hold, finite extensions of K in K_{sep} are defectless.

Corollary 2.7. *Assume that (K, v) is a Henselian p-quasilocal field, such that $\operatorname{char}(\widehat{K}) = p$, $r(p)_K \geq 2$ and $v(K)/pv(K)$ is noncyclic. Then:*

(i) $v(K)/pv(K)$ is of order p^2 and finite extensions of K in $K(p)$ are totally ramified;

(ii) K contains a primitive p^n-th root of unity, for each $n \in \mathbb{N}$, and $\mathcal{G}(K(p)/K)$ is isomorphic to the additive group $\mathbb{Z}_p^2$ with respect to its standard topology.

Proof. Denote by $\Sigma_p(K)$ the set of extensions of K in $K(p)$ of degree p, and by $V_p(K)$ the set of subgroups of $v(K)$ properly including $pv(K)$ and different from $v(K)$. Let K_1 and K_2 be different elements of $\Sigma_p(K)$. Then (1.2) (i) yields $N(K_1/K)N(K_2/K) = K^*$, which means that $v(K) = pv(K_1) + pv(K_2)$. In other words, it follows from (2.3) and the noncyclicity of $v(K)/pv(K)$ that $v(K)/pv(K)$ is of order p^2 and the mapping of $\Sigma_p(K)$ into $V_p(K)$ by the rule $L \to pv(L)$, $L \in \Sigma_p(K)$, is well-defined and bijective. This observation proves that $r(p)_K = 2$ and every $L \in \Sigma_p(K)$ is totally ramified over K. It is now easy to see from (2.4) (i) that $p \notin P(\widehat{K})$. This implies $\widehat{K}$ is infinite, so it follows from Proposition 2.6 and [11], Remark 4.2, that $\operatorname{char}(K) = 0$ and $G(K) = pG(K)$. Applying Proposition 2.5, one obtains further that $\widehat{K}$ is perfect and $p \notin P(K_{G(K)})$. Since $\operatorname{char}(K_{G(K)}) = 0$, $v_{G(K)}$ is Henselian and $p \notin P(K_{G(K)})$, this enables one to deduce from (2.3) that finite extensions of K in $K(p)$ are totally ramified relative to $v_{G(K)}$ (and because of the equality $G(K) = pG(K)$, they have the same property relative to v). In view of [6], Lemma 1.1, these observations show that K contains a primitive p^n-th root of unity, for each $n \in \mathbb{N}$. As $v(K)/pv(K)$ is of order p^2, it is now easy to see from Proposition 2.5 that $\mathcal{G}(K(p)/K) \cong \mathbb{Z}_p^2$, which completes the proof of Corollary 2.7. $\qquad\square$

Remark 2.8. *It is known that if (K, v) is a Henselian field satisfying the conditions of Proposition 2.6, then $Br(K_{G(K)}) \cong \mathbb{Q}/\mathbb{Z}$ and $Br(K_{G(K)})$ embeds in $Br(K)$ (see (2.4) (iii) and [41], Ch. XII, Sect. 3). Also, it follows from (2.1) (i), (2.4) (ii), (2.6) and [40], Ch. II, Theorems 3 and 4, that $r(p)_{K_{G(K)}} \geq 2$.*

3   On the Brauer group of a Henselian p-quasilocal field with a p-indivisible value group

In this Section we prove that if (K, v) is a Henselian p-quasilocal field satisfying the conditions of Theorem 1.1, and if K possesses an immediate extension in $K(p)$ of degree p, then $Br(K)_p$ is isomorphic to $\mathbb{Z}(p^\infty)$.

Theorem 3.1. *Under the hypotheses of Theorem 1.1, suppose that $K(p)$ contains as a subfield an immediate extension I of K of degree p. Then $\widehat{K}$ is perfect, $v(K)/pv(K)$ is of order p and $\nabla_0(K) \subset N(I/K)$.*

The proof of Theorem 3.1 relies on the following two lemmas lemma, the first of which has been proved in [11].

Lemma 3.2. *In the setting of Lemma 2.2, suppose that $\widehat{K}$ is imperfect and $W_p(K)$ is the set of those $\Lambda \in I(K(p)/K)$, for which $[\Lambda\colon K] = [\widehat{\Lambda}\colon \widehat{K}] = p$ and $\widehat{\Lambda}$ is purely inseparable over $\widehat{K}$. Then:*

(i) $W_p(K)$ is infinite except, possibly, in the case where $\mathrm{char}(K) = 0$, $v(p) \notin pv(K)$ and K does not contain a primitive p-th root of unity;

(ii) When $\mathrm{char}(K) = 0$ and $v(p) \notin pv(K)$, there exists a field $\Lambda' \in I(K(p)/K)$, such that $[\Lambda'\colon K] = p$ and $v(p) \in pv(\Lambda')$.

Lemma 3.3. *Let (K, v) be a Henselian p-quasilocal field with $v(K) \neq pv(K)$. Then:*

(i) $\mathrm{Br}(K)_p$ is divisible;

(ii) There is at most one extension of K in $K(p)$ of degree p, which is not totally ramified over K; when such an extension exists, $Br(K)_p \neq \{0\}$;

(iii) $\widehat{K}$ is perfect, provided that $p = \mathrm{char}(\widehat{K})$.

Proof. When $p > 2$, the divisibility of $\mathrm{Br}(K)_p$ is a special case of [8], I, Theorem 3.1 (ii), and in case $p = 2$, it is implied by (1.4) and the fact that $\mathcal{G}(E(2)/E)$ is a group of order 2, for every formally real 2-quasilocal field E [8], I, Lemma 3.5. The rest of our proof relies on the fact that if R is a finite extension of K in $K(p)$, which is not totally ramified, then $v(\lambda) \in pv(K)$, for every $\lambda \in N(R/K)$. At the same time, it follows from Galois theory and the normality of maximal subgroups of finite p-groups (cf. [27], Ch. I, Sect. 6; Ch. VIII) that if $R \in I(K(p)/K)$ and $[R\colon K] = p$, then R/K is cyclic. Since, by (1.2) (i), $N(R_1/K)N(R_2/K) = K^*$ whenever R_1 and R_2 lie in $I(K(p)/K)$, $R_1 \neq R_2$ and $[R_j\colon K] = p$, $j = 1, 2$, these observations prove the former part of Lemma 3.3 (ii). Combined with [31], Sect. 15.1, Proposition b, they also imply the latter assertion of Lemma 3.3 (ii). For the proof of Lemma 3.3 (iii), it suffices to note that (M, v_M) satisfies the conditions of Lemma 3.3 whenever $M \in I(K(p)/K)$ and $[M\colon K] \in \mathbb{N}$ (see (2.3) and [8], I, Theorem 4.1 (ii)), which reduces our concluding assertion to a consequence of Lemma 3.2 and the former part of Lemma 3.3 (ii). $\square$

Remark 3.4. *Let (K, v) be a Henselian p-quasilocal field with $v(K) \neq pv(K)$ and $char(\widehat{K}) \neq p$. In view of (2.3), the former part of Lemma 3.3 (ii) can be restated by saying that $r(p)_{\widehat{K}} \leq 1$. Combining (2.3) and (2.4) (i) with Lemma*

3.3 (i) and [8], I, Lemma 3.5, one also obtains that $\mathcal{G}(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}_p$ unless $p \notin P(\widehat{K})$. It is therefore clear from Lemma 3.3 (ii) and [6], Lemma 1.1 (a), that if K does not contain a primitive p-th root of unity, then $\mathcal{G}(K(p)/K) \cong \mathcal{G}(\widehat{K}(p)/\widehat{K})$ and finite extensions of K in $K(p)$ are inertial. Hence, by [31], Sect. 15.1, Proposition b (and the proof of Lemma 3.3 (ii)), $_p\mathrm{Br}(K) \cong_p \mathrm{Br}(\widehat{K})$ or $_p\mathrm{Br}(K)$ is isomorphic to the direct sum $_p\mathrm{Br}(\widehat{K}) \oplus v(K)/pv(K)$, depending on whether or not $r(p)_{\widehat{K}} = 0$.

Lemma 3.5. *In the setting of Lemma 3.3, suppose that* $\mathrm{char}(\widehat{K}) = p$ *and there exists a field* $I \in I(K(p)/K)$, *such that* $[I: K] = p$ *and* I/K *is not totally ramified. Then* (K, v) *has the following properties:*

(i) *The group* $v(K)/pv(K)$ *is of order* p, *provided that* $r(p)_K \geq 2$; *in particular, this applies to the case where* $v(K)$ *is Archimedean;*

(ii) *A finite extension* M *of* K *in* $K(p)$ *is totally ramified if and only if* $M \cap I = K$;

(iii) $p \in P(\widehat{K})$ *if and only if* I/K *is inertial; when this holds,* $\widehat{K}(p)/\widehat{K}$ *is a* $\mathbb{Z}_p$-*extension.*

Proof. Statement (2.4) (i) and Lemma 3.3 (ii) imply the former assertion of Lemma 3.5 (iii) and the inequality $r(p)_{\widehat{K}} \leq 1$. As $\widehat{K}$ is a nonreal field, this in turn enables one to deduce the latter part of Lemma 3.3 (iii) from Galois theory and [47], Theorem 2. Note further that every $L \in I(K(p)/K)$, $L \neq K$, contains as a subfield a cyclic extension L_0 of K of degree p. This well-known fact is implied by Galois theory and the subnormality of proper subgroups of finite p-groups. Let now $r(p)_K \geq 2$ or, equivalently, there is a field $T \in K(p)/K$, such that $[T: K] = p$ and $T \neq I$. By (1.2) (i), then $N(T/K)N(I/K) = K^*$, which implies that T/K is totally ramified. At the same time, it becomes clear from (2.3) that IT/T is not totally ramified. Since, by [8], I, Theorem 4.1 (ii), T is p-quasilocal, the noted properties of T and L make it easy to prove Lemma 3.5 (ii), arguing by induction on $n = \log_p([M: K])$. The former part of Lemma 3.5 (i) follows from Corollary 2.7 and the assumption on I/K, and for the proof of the latter one, it suffices to observe that the existence of T in case $v(E) \leq \mathbb{R}$ is guaranteed by Lemma 2.2 (and by Proposition 2.6 and Remark 2.8). □

Lemma 3.6. *Let* (K, v) *be a Henselian field with* $\mathrm{char}(\widehat{K}) = p > 0$, $v(K) = pv(K)$ *and* $\mathrm{Br}(K)_p \neq \{0\}$. *Suppose also that* K *is* p-*quasilocal and* $p \in P(\widehat{K})$. *Then finite extensions of* K *in* $K(p)$ *are defectless,* $\mathrm{Br}(K)_p \subseteq \mathrm{IBr}_v(K)$, $\mathrm{Br}(K)_p \cong \mathrm{Br}(\widehat{K})_p$ *and* $[\widehat{K}: \widehat{K}^p] = p$.

Proof. Statement (2.4) (i) and the assumption that $p \in P(\widehat{K})$ imply the existence of an inertial extension I_p of K in $K(p)$ of degree p. This ensures that $\nabla_0(K) \subseteq N(I_p/K)$. Note further that if $\widehat{K}$ is perfect, then $K^* = \nabla_0(K).K^{*p}$, so the noted inclusion requires that $N(I_p/K) = K^*$. As $\mathrm{Br}(K)_p \neq \{0\}$, this contradicts (1.2) (i) and thereby proves that $\widehat{K} \neq \widehat{K}^p$. We show that $[\widehat{K}:\widehat{K}^p] = p$. By Lemma 3.2, there is a field $\Lambda \in I(K(p)/K)$, such that $[\Lambda : K] = [\widehat{L}:\widehat{K}] = p$ and $\widehat{\Lambda}$ is purely inseparable over $\widehat{K}$. Let φ be a generator of $\mathcal{G}(\Lambda/K)$. It is easily seen that if $[\widehat{K}:\widehat{K}^p] \geq p^2$, then $O_v(K)$ contains an element b, such that $\hat{b} \notin \widehat{\Lambda}^p$. Therefore, the cyclic K-algebra $D_b = (\Lambda/K, \varphi, b)$ lies in $d(K)$ and $\widehat{D}_b/\widehat{K}$ is a purely inseparable field extension of degree p^2. This leads to the conclusion that I_p does not embed in D_b as a K-subalgebra. Our conclusion contradicts the assumption that K is p-quasilocal, which proves that $[\widehat{K}:\widehat{K}^p] = p$.

We show that $\mathrm{Br}(K)_p \subseteq \mathrm{IBr}(K)$ and finite extensions of K in $K(p)$ are defectless. Fix a generator σ of $\mathcal{G}(I_p/K)$ and an algebra $\Delta \in d(K)$ of exponent p. As K is p-quasilocal and, by [8], I, Theorem 3.1, $\mathrm{ind}(\Delta) = p$, it is easily seen that Δ is isomorphic to the K-algebra $(I_p/K, \sigma, a)$, for some $a \in K^* \setminus N(I_p/K)$. Moreover, it follows from the equality $v(K) = pv(K)$ that a can be chosen so that $v(a) = 0$. Hence, by the Henselian property of v and the fact that I_p/K is inertial, Δ/K is inertial too, which proves that $_p\mathrm{Br}(K) \subseteq \mathrm{IBr}(K)$. Applying (2.4) (iii) and Witt's theorem (see [12], Sect. 15, and [21], Theorem 2.8), one obtains consecutively that $\mathrm{Br}(K)_p \cap \mathrm{IBr}(K) \cong \mathrm{Br}(\widehat{K})_p$ and $\mathrm{Br}(K)_p \cap \mathrm{IBr}(K)$ is a divisible subgroup of $\mathrm{Br}(K)_p$. Therefore, by [17], Theorem 24.5, $\mathrm{Br}(K)_p \cap \mathrm{IBr}(K)$ is a direct summand in $\mathrm{Br}(K)_p$, so the inclusion $_p\mathrm{Br}(K) \subset \mathrm{IBr}(K)$ implies that $\mathrm{Br}(K)_p \subseteq \mathrm{IBr}(K)$. This indicates that the maximal subfields of $(I_p/K, \sigma, a)$ are defectless over K. As K is p-quasilocal, the obtained result proves that every $L \in I(K(p)/K)$ with $[L : K] = p$ is defectless over K. Since finite extensions of K in $K(p)$ are p-quasilocal, by [8], I, Theorem 4.1, this enables one to deduce from Galois theory and the normality of maximal subgroups of finite p-groups that finite extensions of K in $K(p)$ are defectless. $\square$

Lemma 3.7. *Under the hypotheses of Lemma 3.5, suppose that $r(p)_K \geq 2$ and there exists a p-indivisible group $H \in \mathrm{Is}_v(K)$. Then:*

(i) *K_H is p-quasilocal and $r(p)_{K_H} \geq 2$.*

(ii) *I_H/K_H is immediate and I/K is inertial relative to $\hat{v}_H$ and v_H, respectively.*

Proof. Lemma 3.5 (ii) and the inequality $r(p)_K \geq 2$ ensure that $v(K)/pv(K)$ is of order p. As $H \neq pH$, this implies that $v_H(K) = pv_H(K)$. It is therefore

clear from (2.3) that if $v(p) \in H$, i.e. $\operatorname{char}(K_H) = 0$, then finite extensions of K in $K(p)$ are inertial relative to v_H, which yields $\mathcal{G}(K(p)/K) \cong \mathcal{G}(K_H(p)/K_H)$ (cf. [21], page 135), whence $r(p)_K = r(p)_{K_H}$. Suppose now that $v(p) \notin pH$, fix an element $\pi \in M_v(K)$ so that $v(\pi) \in H \setminus pH$, and denote by J the root field in K_{sep} of the polynomial $f(X) = X^p - X - \pi^{-1}$ over K. It is easy to see that $J \in I(K(p)/K)$, $[J \colon K] = p$, and J/K is inertial relative to v_H and totally ramified relative to v. In particular, $J \neq I$ and $p \in P(K_H)$, so it follows from Lemma 3.6, applied to K, v_H and p, that K_H is p-quasilocal and $\operatorname{Br}(K)_p \cong \operatorname{Br}(K_H)_p \neq \{0\}$. In view of [22], Proposition 4.4.8, this indicates that $r(p)_{K_H} = \infty$. We show that I/K is inertial relative to v_H. This has already been established in the case where $v(p) \in H$, so we assume here that $v(p) \notin H$. It is clearly sufficient to prove that IJ/J is inertial relative to the prolongation v'_H of v_H on J. This implies that each generator ψ of $\mathcal{G}(I/K)$ is uniquely extendable to a generator ψ' of $\mathcal{G}(IJ/J)$. We show that IJ/J is inertial relative to v'_H by proving the following statement:

(3.1) There exists $r \in O_v(I)$, such that $v_I(r - \psi'(r)) \le p^{-1}v(\pi)$.

Fix a root ξ of f in J, put $\theta = \xi^{-1}$, and denote by H' the sum of H and the cyclic group $\langle v_J(\xi) \rangle$. It is easily verified that $H' \in \mathrm{Is}_{v_J}(J)$, $v'_H = v_{J,H'}$ and $v_H(\eta_H(\pi)) = p v'_H(\eta_{H'}(\theta))$. For convenience, we put $\kappa_H = \eta_H(\kappa)$ and $\kappa'_{H'} = \eta_{H'}(\kappa')$, for each $\kappa \in O_{v_H}(K)$, $\kappa' \in O_{v'_H}(J)$. Observing that $\operatorname{char}(K_H) = p$ and $\theta_{H'} = \pi_H \prod_{u=1}^{p-1}(\xi_{H'} + u)$, one obtains by direct calculations that $\hat{v}'_H(\theta_{H'}) = p^{-1}\hat{v}_H(\pi_H)$ and $\hat{v}'_H(\theta_{H'} - \tilde{\zeta}(\theta_{H'})) = (2p^{-1})\hat{v}_H(\pi_H)$, for each generator $\tilde{\zeta}$ of $\mathcal{G}(J_{H'}/K_H)$. Thus it turns out that $v_J(\theta) = p^{-1}v(\pi)$, $v_J(\theta - \zeta(\theta)) = (2p^{-1})v(\pi)$ and $v_J(1 - \zeta(\theta)\theta^{-1}) = p^{-1}v(\pi)$, provided that ζ generates $\mathcal{G}(J/K)$. Hence, by (2.3) and the choice of π, $v_J(\zeta(\theta)\theta^{-1} - 1) \notin pv(J)$. Note also that the p-quasilocal property of K is preserved by J [8], I, Theorem 4.1, which ensures that $\zeta(\lambda)\lambda^{-1} \in N(IJ/J)$, for each $\lambda \in (IJ)^*$ (see [9], Lemma 4.2). Take an element $\theta' \in IJ$ of norm $N_J^{IJ}(\theta') = \zeta(\theta)\theta^{-1}$ and put $\lambda' = \theta' - 1$. We show that $v_{IJ}(\theta' - \psi'(\theta')) \le p^{-1}v(\pi)$. It follows from the Henselian property of v and the primality of p that $v_{IJ}(\theta' - \psi'(\theta')) = v_{IJ}(\psi'^u(\theta') - \psi'^{u'}(\theta'))$, for $\psi'^u \neq \psi'^{u'}$. Therefore, the equality $N_J^{IJ}(\theta') = \zeta(\theta)\theta^{-1}$ implies that if $v_{IJ}(\theta' - \psi'(\theta')) > p^{-1}v(\pi)$, then $v_{IJ}(\lambda'^p) = v_J(\zeta(\theta)\theta^{-1} - 1)$. Since IJ/J is immediate relative to v_J, our conclusion requires that $p^{-1}v(\pi) \in pv(J)$ and $v(\pi) \in pv(K)$, a contradiction proving (3.1) (and the fact that I/K is inertial relative to v_H).

It remains to be seen that I_H/K_H is immediate relative to $\hat{v}_H$. Observing that $v(I) = v(K)$ and $v(K)/H$ is torsion-free, one obtains that $\hat{v}_H(I_H) = \hat{v}_H(K_H)$. Since, by Lemma 3.3 (iii), $\widehat{K}$ is perfect, and by (2.1) (i), it is isomorphic to the residue field of $(K_H, \hat{v}_H)$, this implies that I_H/K_H is immediate or inertial. Suppose for a moment that I_H/K_H is inertial. Then I_H/K_H pos-

sesses a primitive element $\tilde{\alpha} \in O_{\hat{v}_H}(I_H)$, such that $\hat{v}_H(d(\tilde{g})) = 0$, where $\tilde{g}$ is the minimal (monic) polynomial of $\tilde{\alpha}$ over K_H, and $d(\tilde{g})$ is the discriminant of $\tilde{g}$. The choice of $\tilde{\alpha}$ guarantees that $\tilde{g}(X) \in O_{\hat{v}_H}(K_H)[X]$, whence $\tilde{g}$ is a reduction modulo $M_v(K)$ of a monic polynomial $g(X) \in O_v(K)[X]$ (see (2.1)). Denote by $d(g)$ the discriminant of g. It is easily obtained that $v(d(g)) = 0$ and the residue class of $d(g)$ in K_H equals $d(\tilde{g})$. Observe also that, for each root $\tilde{\beta} \in O_{\hat{v}_H}(I_H)$ of $\tilde{g}$, there exists a root $\beta \in O_v(I)$ of g, such that $\hat{\beta} = \tilde{\beta}$. The obtained result leads to the conclusion that I/K is inertial. This contradicts our assumptions and thereby proves that I_H/K_H is immediate relative to $\hat{v}_H$, as claimed. $\square$

Remark 3.8. *The assertion of Lemma 3.7 (ii) can be restated by saying that $v(\sigma(\lambda) - \lambda) > 0$ whenever $\lambda \in O_v(I)$, and there exists $\alpha_H \in O_v(I)$, $v(\sigma(\alpha_H) - \alpha_H) \in H$, where σ is a generator of $\mathcal{G}(I/K)$.*

Our objective now is to prove Theorem 3.1, under the extra hypothesis that $\mathrm{Is}_v(K) \neq \{0\}$ and $H \neq pH$, for each $H \in \mathrm{Is}_v(K)$, $H \neq \{0\}$. Suppose first that $\mathrm{Is}_v(K)$ does not contain a minimal element (with respect to inclusion), and fix an arbitrary element $\beta \in \nabla_0(K)$. Then $v(\beta - 1) \notin H_\beta$, for some $H_\beta \in \mathrm{Is}_v(K)$, so it follows from (2.2), (3.1) and Lemma 3.7 that $\beta \subset N(I/K)$. It remains to be seen that $\nabla_0(K) \subseteq N(I/K)$, provided that $\mathrm{Is}'_v(K)$ contains a minimal element $\Gamma \neq \{0\}$. Applying Lemma 3.7, one sees that it suffices to consider the special case of $v(K) = \Gamma$. The minimality of Γ indicates that it is Archimedean, so the inclusion $I_0(K) \subset N(I/K)$ can be proved by showing that $\nabla_\delta(K) \subseteq N(I/K)$, for an arbitrary $\delta \in \Gamma$, $\delta > 0$. Our main step in this direction is contained in the following lemma.

Lemma 3.9. *Assume that (K, v), p and I satisfy the conditions of Lemma 3.5, $v(K)$ is Archimedean, and $L \in I(K(p)/K)$ is a field, such that $[L \colon K] = p$ and $L \neq I$. Suppose further that $v(L) \setminus pv(L)$ contains an element $\gamma > 0$ satisfying the conditions $\gamma = v_L(\lambda) = v_L(\tau(\lambda)\lambda^{-1} - 1) < p^{-1}v(p)$, for some $\lambda \in O_v(L)$, where τ is a generator of $\mathcal{G}(IL/L)$. Then $\nabla_{\gamma'}(L) \subseteq N(IL/L)$ and $\nabla_{\gamma''}(K) \subseteq N(IL/K)$, where $\gamma' = (2p - 2)\gamma$ and $\gamma'' = [(p^2 - 1)(4p - 2)]\gamma$.*

Proof. Fix an element $\theta' \in (IL)^*$ so that $N_L^{IL}(\theta') = \tau(\lambda)\lambda^{-1}$, and put $\tilde{\gamma} = v_{IL}(\tau'(\theta') - \theta')$, for some $\tau' \in \mathcal{G}(IL/L)$, $\tau \neq 1$. As in the proof of (3.1), one obtains that if $\tilde{\gamma} > \gamma$, then $v_{IL}(\lambda - 1 - (\theta' - 1)^p) \geq \tilde{\gamma}$, which implies $\gamma = v_L(\lambda - 1) = v_{IL}(\theta' - 1)^p$. Since IL/L is immediate, this contradicts the assumption that $\gamma \notin pv(L)$ and thereby proves that $\tilde{\gamma} \leq \gamma$. Applying next (2.2) to the minimal polynomial of λ' over L, one obtains that $\nabla_{\gamma'}(L) \subseteq N(IL/L)$. Observe that $pv(K)$ is a dense subgroup of $\mathbb{R}$ (I/K is immediate, whence

$v(K)$ is noncyclic, see [43], Proposition 2.2). Therefore, for each $\varepsilon > 0$, one can find an element $\mu_\varepsilon \in K$ such that $(2p - 3)\gamma < v(\mu_\varepsilon) < (2p - 3)\gamma + \varepsilon$ and $v(\mu_\varepsilon) \in pv(K)$. Hence, by the choice of λ, $\gamma' < v_L(\mu_\varepsilon \lambda)) < \gamma' + \varepsilon$, $v_L(\mu_\varepsilon \lambda) \notin pv(L)$, and $(2p - 1)\gamma < v_L(\tau^j(1 + \mu_\varepsilon \lambda) - 1 - \mu_\varepsilon \lambda) < (2p - 1)\gamma + \varepsilon$, $j = 1, \ldots, p - 1$. As $\nabla_{\gamma'} \subseteq N(IL/L)$, these calculations prove the existence of an element $\mu' \in \nabla_0(IL)$ of norm $N_L^{IL}(\mu') = 1 + \mu_\varepsilon \lambda$. Let f be the minimal polynomial of μ' over K. It is easily seen that f is of degree p^2. Using the above calculations, observing that the natural action of $\mathcal{G}(IL/K)$ on L^* induces on $N(IL/L)$ a structure of a $\mathbb{Z}[\mathcal{G}(L/K)]$-module, and arguing as in the proof of the inclusion $\nabla_{\gamma'} \subseteq N(IL/L)$, one obtains that $0 < v_{IL}(\mu' - \varphi(\mu')) \leq (2p-1)\gamma + \varepsilon$ when φ runs across $\mathcal{G}(IL/K) \setminus \{1\}$. Since ε can be taken smaller than any fixed positive number, this enables one to deduce from (2.2) that $\nabla_{\gamma''} \subseteq N(IL/L)$, so Lemma 3.9 is proved. $\qquad\square$

It is now easy to prove Theorem 3.1 in the remaining case where $v(K) \leq \mathbb{R}$. Take elements $\gamma \in v(K) \setminus pv(K)$ and $\tilde\mu \in K$ so that $\gamma > 0$ and $v(\tilde\mu) = \gamma$. We prove that if γ is sufficiently small, then the extension $L_{\tilde\mu} = L$ of K in K_{sep} generated by a root of the polynomial $f_{\tilde\mu}(X) = X^p - X - \tilde\mu^{-1}$ satisfies the following conditions:

(3.2) $L \subseteq K(p)$, $[L\colon K] = p$ and there exists $\theta \in L$, such that $v_L(\theta) = p^{-1}v(\tilde\mu) = p^{-1}\gamma$, $v_L(\zeta(\theta) - \theta) = (2p^{-1})\gamma$ and $v_L(\zeta(\theta)\theta^{-1} - 1) = p^{-1}\gamma$.

We show that one can take as θ the inverse of some root of $f_{\tilde\mu}$. If $\mathrm{char}(K) = p$, this is obtained by direct calculations (as in the proof of (3.1)). Suppose further that $\mathrm{char}(K) = 0$, take a primitive p-th root of unity $\varepsilon \in K_{\mathrm{sep}}$, and put $m = [K(\varepsilon)\colon K]$. It is well-known (cf. [27], Ch. VIII, Sect. 3) that $K(\varepsilon)/K$ is cyclic and $m \mid (p - 1)$. Set $\mu = m\tilde\mu$, fix a generator φ of $\mathcal{G}(K(\varepsilon)/K)$, and let s and l be positive integers, such that $\varphi(\varepsilon) = \varepsilon^s$ and $p \mid (sl - 1)$. Denote by Λ' some extension of $K(\varepsilon)$ in K_{sep} obtained by adjunction of a p-th root of the element $\rho(\mu) = \prod_{u=0}^{m-1} \varphi^i(1 + (\varepsilon - 1)^p\mu^{-1})^{l^i}$. It is easily verified that $\varphi(\rho(\mu))\rho(\mu)^{-s} \in K(\varepsilon)^{*p}$. Observing also that $v'(\rho(\mu) - 1 - m(\varepsilon - 1)^p\mu^{-1}) \geq (2p)v'(\varepsilon - 1) - 2\gamma$, and that the polynomial $g(X) = (X + 1)^p - \rho(\mu)$ is irreducible over $K(\varepsilon))$, one concludes that $\rho(\mu) \notin K(\varepsilon)^{*p}$. Hence, by Albert's theorem (cf. [1], Ch. IX), $\Lambda' = \Lambda(\varepsilon)$, for some $\Lambda \in I(K(p)/K)$ with $[\Lambda\colon K] = p$. Our calculations also show that $\Lambda'/K(\varepsilon)$ is totally ramified. Since $m \mid (p - 1)$, this proves that Λ/K is totally ramified as well. Note further that when γ is sufficiently small, $\Lambda'/K(\varepsilon)$ possesses a primitive element which is a root of the polynomial $X^p - \tilde\mu^{p-1}X - \tilde\mu^{p-1}$. This is obtained by applying (2.2) to the polynomial $\tilde h(X) = g((\varepsilon - 1)^{-1}\tilde\mu X) \in O_{v'}(K(\varepsilon))$. Thus it becomes clear that $\Lambda'/K(\varepsilon)$ has a primitive element ξ satisfying $f_{\tilde\mu}(X)$. This implies that $[K(\xi)\colon K] = p$, so it follows from the cyclicity of Λ'/K and Galois theory that

$K(\xi) = \Lambda = L$. Using again (2.2), one concludes that when γ is sufficiently small, the element $\theta = \xi^{-1}$ satisfies the inequalities required by (3.2). Since $p^{-1}\gamma \notin pv(L)$, the obtained result and Lemma 3.9 prove Theorem 3.1.

Let (K, v) be a Henselian field, such that $\mathrm{char}(\widehat{K}) = p > 0$, and let I/K be a $\mathbb{Z}_p$-extension, such that $\widehat{I} = \widehat{K}$. Denote by I_n the extension of K in I of degree p^n, and put $v_n = v_{I_n}$, for each $n \in \mathbb{Z}$, $n \geq 0$. The uniqueness, up-to an equivalence, of v_n implies the following inclusion, for every index n:

(3.3) $\{\psi_n(u_n)u_n^{-1}\colon \ u_n \in I_n^*, \psi_n \in \mathcal{G}(I_n/K)\} \subseteq \nabla_0(I_n)$.

We say that I is a norm-inertial extension of K, if $\nabla_0(K) \subseteq N(I_n/K)$, for each $n \in \mathbb{N}$. Suppose that $H \neq pH$, for every $H \in \mathrm{Is}'_v(K)$, $H \neq \{0\}$. We conclude this Section with the proof of the equivalence of the following statements in case I/K is immediate:

(3.4) (i) I/K is norm-inertial;
(ii) I/I_n is norm-inertial, for every index n;
(iii) For each $\gamma \in v(K)$, $\gamma > 0$, there exists $\mu_n(\gamma) \in O_v(I_n)$, such that $v_n(\varphi_n(\mu_n(\gamma)) - \mu_n(\gamma)) < \gamma$, for each $\varphi_n \in \mathcal{G}(I_n/K) \setminus \{1\}$.

The implication (3.4)(ii)→(3.4) (i) is obvious and the implication (3.4) (iii) →(3.4) (ii) follows from (2.2), [5], II, (2.6) and (2.7), and the fact that $H_n \neq pH_n$, for every $H_n \in \mathrm{Is}'_{v_n}(I_n)$, and each $n \in \mathbb{N}$. The implication (3.4) (i)→(3.4) (iii) can be deduced from the following result:

(3.5) Let (K, v) be a Henselian field, $L \in I(K(p)/K)$ a cyclic extension of K of degree p^n, ψ a generator of $\mathcal{G}(L/K)$, λ and λ_0 be elements of $M_v(L)$ and $M_v(K)$, respectively, such that $v(\lambda_0) \in v(K) \setminus pv(K)$ and $N_K^L(1+\lambda) = 1+\lambda_0$, where N_K^L is the norm map. Then $v_L(\psi^j(\lambda) - \lambda) \leq v(\lambda_0)$, for $j = 1, \dots, p^n - 1$.

It is easy to see that $v_L(\psi(\alpha) - \alpha) \leq v_L(\psi^j(\alpha) - \alpha)$, for any $\alpha \in L$ and each index j, and equality holds in the case where $p \nmid j$. When $p \nmid j$, i.e. ψ^j generates $\mathcal{G}(L/K)$, this leads to the conclusion of (3.5). Thus our assertion is proved in case $n = 1$, so we assume further that $n \geq 2$. Let k be an integer with $1 \leq k < n$, L_k the fixed field of ψ^{p^k}, and $\lambda_k = -1 + \prod_{u=0}^{p^k-1}(1 + \psi^u(\lambda))$. Clearly, $N_{L_k}^L(1 + \lambda_k) = 1 + \lambda_0$. Note also that $v_L(\psi^{p^k}(\lambda) - \lambda) \leq v(\lambda_0)$, provided $v_L(\psi^{p^k}(\lambda_k) - \lambda_k) \leq v(\lambda_0)$. Since ψ^{p^k} is a generator of $\mathcal{G}(L/L_k)$, these observations enable one to complete the proof of (3.5) by a standard inductive argument.

At the same time, it is easily deduced from (2.2) (without restrictions on $v(K)$) that the fulfillment of (3.4) (iii) ensures that I/K is norm-inertial.

4   Proof of Theorem 1.1

Let first K be an arbitrary p-quasilocal nonreal field containing a primitive p-th root of unity unless $\operatorname{char}(K) = p$. Then $\operatorname{cd}(\mathcal{G}(K(p)/K)) \leq 2$ and equality holds if and only if $\operatorname{char}(K) \neq p$ and $\operatorname{Br}(K)_p \neq \{0\}$ (see [9], Proposition 5.1, and [40], Ch. I, 4.2). When K possesses a Henselian valuation v with $v(K) \neq pv(K)$, this enables one to deduce from [4], (1.2), [8], Lemma 3.6, and [6], Lemma 1.1 (b) (an analogue to a part of the main result of [29]) that $\operatorname{cd}(\mathcal{G}(K(p)/K)) = r(p)_K$. At the same time, the assumptions on K, Lemma 3.3 (i) and [8], I, Lemma 3.5, indicate that K is a nonreal field. These observations, combined with [45], Lemma 7 (or [9], Corollary 5.3), prove (1.4). Using (1.4), Remark 3.4 and Lemma 3.3 (i), one deduces the assertion of Theorem 1.1 in the special case where $\operatorname{char}(\widehat{K}) \neq p$.

In the rest of our proof of Theorem 1.1, we assume that (K, v) is Henselian p-quasilocal with $v(K) \neq pv(K)$ and $\operatorname{char}(\widehat{K}) = p$. Suppose first that $\operatorname{char}(K) = 0$ and $v_{G(K)}(K) \neq pv_{G(K)}(K)$, $G(K)$ being defined as in Section 2, and fix a primitive p-th root of unity $\varepsilon \in K_{\mathrm{sep}}$. Then $\operatorname{char}(K_{G(K)}) = 0$, so it follows from Remark 3.4 that $\mathcal{G}(K_{G(K)}(p)/K_{G(K)}) \cong \mathbb{Z}_p$ unless $p \notin P(K_{G(K)})$. In view of Remark 2.8 and Proposition 2.6, this yields $G(K) = pG(K)$. Hence, by Proposition 2.5, $K_{G(K)}(p)/K_{G(K)}$ is immediate relative to $\hat{v}_{G(K)}$ unless $p \in P(\widehat{K})$, and by (1.4) and Remark 3.4, applied to $(K, v_{G(K)})$, $\mathcal{G}(K(p)/K)$ has the following properties:

(4.1) (i) $\mathcal{G}(K(p)/K) \cong \mathcal{G}(K_{G(K)}(p)/K_{G(K)})$, provided that $\varepsilon \notin K$; when this occurs, $r(p)_K \leq 1$ and $\operatorname{Br}(K)_p$ is isomorphic to $\operatorname{Br}(K_{G(K)})_p$ or to a divisible hull of $_p\operatorname{Br}(K_{G(K)}) \oplus v(K)/pv(K)$, depending on whether or not $r(p)_K = 0$;

(ii) If $\varepsilon \in K$ and either $p \in P(K_{G(K)})$ or $v(K)/pv(K)$ is noncyclic, then $r(p)_K = 2$, $\mathcal{G}(K(p)/K)$ is a Demushkin group and $\operatorname{Br}(K)_p \cong \mathbb{Z}(p^\infty)$;

(iii) $\mathcal{G}(K(p)/K) \cong \mathbb{Z}_p$, if $\varepsilon \in K$, $p \notin P(K_{G(K)})$ and $v(K)/pv(K)$ is of order p; in this case, $\operatorname{Br}(K)_p = \{0\}$ and finite extensions of K in $K(p)$ are totally ramified.

When $p \in P(K_{G(K)})$, it also becomes clear that the compositum $K'_{G(K)}$ of the inertial lifts in K_{sep}, relative to $v_{G(K)}$, of the finite extensions of $K_{G(K)}$ in $K_{G(K)}(p)$, has the following properties:

(4.2) $K'_{G(K)}$ is a $\mathbb{Z}_p$-extension of K with $v(K'_{G(K)}) = v(K)$; more precisely $K'_{G(K)}/K$ is immediate relative to v unless $p \in P(\widehat{K})$.

Statements (4.1), (4.2) and Corollary 2.7 reduce the proof of Theorem 1.1 to the special case where $\operatorname{char}(K) = p$ or $\operatorname{char}(K) = 0$ and the group $v(K)/G(K)$ is p-divisible. Then it follows from Corollary 2.7 and Remark 2.8 that $r(p)_K \geq$

2 and $v(K)/pv(K)$ is of order p. This, combined with (2.3), (2.4) (i) and Lemma 3.3, proves the following assertions:

(4.3) If $d(K)$ contains a noncommutative defectless K-algebra of p-primary index, then the compositum $U_p(K)$ of the inertial extensions of K in $K(p)$ is a $\mathbb{Z}_p$-extension of K. In addition, every $D \in d(K)$ and each finite extension of K in $K(p)$ are defectless over K.

Note also that $\nabla_0(K) \subseteq N(U/K)$, for every inertial extension U/K; this is a well-known consequence of (2.2). When U/K is cyclic and $U \subseteq K(p)$, this enables one to deduce from [31], Sect. 15.1, Proposition b, that $\mathrm{Br}(U/K) = \{b \in \mathrm{Br}(K): [U:K]b = 0\}$ and $\mathrm{Br}(U/K)$ is cyclic of order $[U:K]$. Thereby, it becomes clear that $\mathrm{Br}(U_p(K)/K) = \mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$, which proves the assertion of Theorem 1.1 in the case singled out by (4.3).

Suppose now that K has a finite extension L' in $K(p)$ of nontrivial defect, choose L' to be of minimal possible degree over K, and fix a maximal subfield L of L' including K. Clearly, L'/L is immediate and $[L':L] = p$, and by [8], Theorem 4.1, L is p-quasilocal. In addition, it follows from (1.1) and Lemma 3.3 (i) that $\mathrm{Cor}_{L/K}$ induces an isomorphism of $\mathrm{Br}(L)_p$ on $\mathrm{Br}(K)_p$. Observing also that $v(K)/pv(K) \cong v(L)/pv(L)$ (see, e.g., [4], Remark 2.2), one concludes that L, v_L and p satisfy the conditions of Theorem 3.1, which yields $\mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$, as claimed. For the rest of the proof of Theorem 1.1, we need the following lemma.

Lemma 4.1. *In the setting of Theorem 1.1, suppose that* $\mathrm{char}(\widehat{K}) = p$ *and there exists a finite extension of K in $K(p)$ of nontrivial defect. Then there is a field $I \in I(K(p)/K)$, such that I/K is immediate and $[I:K] = p$.*

Proof. In view of Galois theory and the subnormality of proper subgroups of finite groups, it suffices to consider the special case in which K has an extension M in $K(p)$ of degree p^2 and defect p. We show that there exists a field $I \in I(M/K)$, such that $[I:K] = p$ and I/K is immediate. Let R be an extension of K in M of degree p. It follows from (2.3) that $v(R)/pv(R) \cong v(K)/pv(K)$, so we have $v(R) \neq pv(R)$. If R/K is immediate, there is nothing to prove, so we assume that this is not the case. Our extra hypothesis guarantees that M/R is immediate, and since R is p-quasilocal [8], I, Theorem 4.1 (i), one obtains from Lemma 3.5 (i) and (ii) that $v(R)/pv(R)$ and $v(K)/pv(K)$ are of order p, $\widehat{R}$ is perfect and $p \notin P(\widehat{R})$. Note further that, by Lemma 3.3 (ii), M is the only extension of R in $R(p) = K(p)$ of degree p, which is not totally ramified. Statement (2.3) and these observations indicate that $p \notin P(\widehat{K})$ and R/K is totally ramified. At the same time, by Theorem 3.1, $\mathrm{Br}(R)_p \cong \mathbb{Z}(p^\infty)$, so it follows from (1.1) and Lemma 3.3 (i) that $\mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$. Using the normality of R/K and the equality $[M:K] = p^2$, one proves that M/K is

abelian. The obtained results, combined with (1.2) (ii) and [9], Lemma 2.1, imply that $\mathcal{G}(M/K) \cong K^*/N(M/K)$. We show that M/K is noncyclic. It is clear from Theorem 3.1 and the Henselian property of v that $N(M/R) = \{\rho \in R^*\colon v_R(\rho) \in pv(R)\}$. This, combined with Hilbert's Theorem 90 and the transitivity of norm maps in the field tower $K \subset R \subset M$, implies that $N(M/K)$ is included in the set $\Omega_p(K) = \{\alpha \in K^*\colon v(\alpha) \in pv(K)\}$. Thus it turns out that $K^{*p} \subseteq N(M/K)$, whence $K^*/N(M/K)$ has exponent p. As $\mathcal{G}(M/K) \cong K^*/N(M/K)$, it is now easy to see that $\mathcal{G}(M/K)$ is noncyclic. By Galois theory, this means that $M = RL$, for some $L \in I(M/K)$ with $[L\colon K] = p$ and $L \neq R$. Clearly, one may assume for the rest of the proof that L/K is totally ramified. By (1.2), $K^* = N(R/K)N(L/K)$, $K^*/N(R/K)$ and $K^*/N(L/K)$ are of order p, and $N(M/K) = N(R/K) \cap N(L/K)$. As $v(K)/pv(K)$ is of order p, these observations prove the existence of elements λ and r of K^*, such that $v(\lambda) = v(r) \notin pv(K)$, $\lambda \in N(L/K)$, $r \in N(R/K)$ and the co-sets $\lambda N(M/K)$ and $r N(M/K)$ generate $K^*/N(M/K)$. In addition, it follows from (1.2) that the set $\{N(K'/K)\colon K' \in I(M/K)\}$ equals the set of subgroups of K^* including $N(M/K)$. Therefore, there is $I \in I(M/K)$, such that $[I\colon K] = p$ and $N(I/K)$ is generated by $N(M/K)$ and λr^{-1}. Hence, $N(I/K) = \Omega_p(K)$, which means that I/K is not totally ramified. As M/R is immediate, this leads to the conclusion that I/K is also immediate, which proves Lemma 4.1. $\qquad\square$

The idea of the remaining part of the proof of Theorem 1.1 is to show that if finite extensions of K in $K(p)$ are defectless, then every $\Delta \in d(K)$ is defectless over K. Its implementation relies on the following two lemmas.

Lemma 4.2. *Let K be a p-quasilocal field, for some $p \in \mathbb{P}$. Assume that $\mathrm{Br}(K)_p$ is divisible, M is an extension of K in $K(p)$ of degree p, ψ is a generator of $\mathcal{G}(M/K)$ and $d(K)$ contains the algebra $(M/K, \psi, c)$, for some $c \in K^* \setminus M^{*p}$. Then $[(M/K, \psi, c)] = \mathrm{Cor}_{M/K}([(M_1/M, \psi_1, c)])$, for some $M_1 \in I(K(p)/M)$ with $[M_1\colon M] = p$, and some generator ψ_1 of $\mathcal{G}(M_1/M)$.*

Proof. Suppose first that $\mathrm{char}(K) = p$. By the Artin-Schreier theorem (cf. [27], Ch. VIII, Sect. 6), then $M = K(\xi)$, where ξ is a root of the polynomial $X^p - X - a$, for some $a \in K^*$. Clearly, a can be chosen so that $(M/K, \psi, c)$ is isomorphic to the p-symbol K-algebra $K[a, c]$, in the sense of [43]. Since M/K is separable, there exists $\eta \in M$ of trace $\mathrm{Tr}_K^M(\eta) = c$. This implies that the polynomial $X^p - X - \eta$ has no zero in M, whence, by the Artin-Schreier theorem, its root field over M is a cyclic extension of M of degree p. Since, by a known projection formula (see [28], Proposition 3 (i)), $\mathrm{Cor}_{M/K}([M[\eta, c]]) = [K[a, c]]$, these observations prove Lemma 4.2 in the case of $\mathrm{char}(K) = p$.

In the rest of the proof, we assume that $\text{char}(K) = 0$, ε is a primitive p-th root of unity in K_{sep}, $[K(\varepsilon) \colon K] = m$ and s is an integer satisfying the equality $\varphi(\varepsilon) = \varepsilon^s$, where φ is a generator of $\mathcal{G}(K(p)(\varepsilon)/K(p))$. Then it follows from (1.1), Lemma 3.3 (i) and [8], I, Theorems 3.1 (i) and 4.1 (iii), that $\text{ind}(A) = \text{ind}(A')$ whenever $A \in d(M)$, $A' \in d(K)$, $[A] \in \text{Br}(M)_p$ and $\text{Cor}_{M/K}([A]) = [A']$. When $\varepsilon \in K$, F/K is cyclic of degree p, by Kummer theory, and by the assumption on c, $F \neq M$, which implies MF/M is cyclic and $[MF \colon M] = p$. Since M is p-quasilocal, these observations enable one to deduce the assertion of Lemma 4.2 from Lemma 3.3.

Suppose now that $\varepsilon \notin K$ and, for each $R \in I(K(p)/K)$, let $R_\varepsilon = \{r \in R(\varepsilon)^* \colon \varphi(r)r^{-s} \in R(\varepsilon)^*\}$. It follows from Albert's theorem (see [1], Ch. IX, Theorem 6) that $M(\varepsilon)$ is generated over $K(\varepsilon)$ by a p-th root of some element $\mu \in K_\varepsilon$. In addition, it becomes clear that μ can be chosen so that the symbol $K(\varepsilon)$-algebra $A_\varepsilon(\mu, c; K(\varepsilon))$ is isomorphic to $(M/K, \psi, c) \otimes_K K(\varepsilon)$. As $p > 2$, one also sees that $\mu \in N(M(\varepsilon)/K(\varepsilon))$, whence μ equals the norm $N_{K(\varepsilon)}^{M(\varepsilon)}(\mu_1 \kappa)$, for some $\mu_1 \in M_\varepsilon$, $\kappa \in K(\varepsilon)$ (see [11], Lemma 3.1). Denote by M_1' the extension of $M(\varepsilon)$ obtained by adjunction of the p-th roots of μ_1 in K_{sep}. Since $\mu_1 \notin M(\varepsilon)^{*p}$, Albert's theorem indicates that $M_1' = M_1(\varepsilon)$, for some $M_1 \in I(K(p)/M)$ with $[M_1 \colon M] = p$. Thereby, it becomes clear that the symbol $M(\varepsilon)$-algebra $A_\varepsilon(\mu_1, c; M(\varepsilon))$ is isomorphic to $(M_1/M, \psi_1, c) \otimes_M M(\varepsilon)$, for some generator ψ_1 of $\mathcal{G}(M_1/M)$. Applying the projection formula for symbol algebras (cf., e.g., [42], Theorem 3.2), one concludes that $\text{Cor}_{M(\varepsilon)/K(\varepsilon)}([A_\varepsilon(\mu_1, c; M(\varepsilon))]) = [A_\varepsilon(\mu, c; K(\varepsilon))]$. Using also the $K(\varepsilon)$-isomorphism $A_\varepsilon(\mu, c; K(\varepsilon)) \cong (M/K, \psi, c) \otimes_K K(\varepsilon)$ (and the fact that $m \mid (p-1)$), one obtains from the RC-formula that $\text{Cor}_{M/K}([M_1/M, \psi_1, c)]) = [(M/K, \psi, c)]$, as claimed by Lemma 4.2. $\qquad\square$

Lemma 4.3. *Let (K, v) be a Henselian p-quasilocal field with $\text{char}(K) = 0$, $\text{char}(\widehat{K}) = p$ and $v(K) \neq pv(K)$. Assume that finite extensions of K in $K(p)$ are totally ramified, (M, c) is a pair satisfying the conditions of Lemma 4.2, and F is an extension of K in K_{sep} obtained by adjunction of a p-th root of c. Then the extension MF/M is totally ramified of degree p.*

Proof. We retain notation as in the proof of Lemma 4.2. In view of Kummer theory, there is nothing to prove in case $\varepsilon \in K$, so we assume that $\varepsilon \notin K$. Then it follows from Lemma 3.3 (iii) that $R_\varepsilon \subseteq \nabla_0(R(\varepsilon))R(\varepsilon)^{*p}$, for each $R \in I(K(p)/K)$. Applying Albert's theorem, Lemma 3.3 and [42], Theorems 2.5 and 3.2, as in the proof of Lemma 4.2, one obtains the following result:

(4.4) The $M(\varepsilon)$-algebra $A_\varepsilon(r, c; M(\varepsilon))$ lies in $d(M(\varepsilon))$ if and only if

$A_\varepsilon(N_\varepsilon^{M(\varepsilon)}(r), c; K(\varepsilon)) \in d(K(\varepsilon))$; equivalently, $r \in N(FM(\varepsilon)/M(\varepsilon))$ if and only if $N_{K(\varepsilon)}^{M(\varepsilon)}(r) \in N(F(\varepsilon)/K(\varepsilon))$.

We prove that MF/M is totally ramified by assuming the opposite. In view of (2.3) and Lemma 3.3 (iii), this requires that MF/M is inertial or immediate, and since $m \mid (p-1)$, $MF(\varepsilon)/M(\varepsilon)$ must be subject to the same alternative. It follows from the Henselity of $v_{K(\varepsilon)}$ that if $F(\varepsilon)/K(\varepsilon)$ is inertial, then $I_0(K(\varepsilon)) \subseteq N(F(\varepsilon)/K(\varepsilon))$. Applying now [31], Sect. 15.1, Proposition b, one concludes that $(M_1/M, \psi_1, c) \notin d(K)$. Since $(M/K, \psi, c) \in d(K)$, this contradicts Lemma 4.2, and thereby proves that $MF(\varepsilon)/M(\varepsilon)$ is not inertial. The final step towards the proof of Lemma 4.3 relies on the fact that M_ε is a module over the integral group ring $\mathbb{Z}[\mathcal{G}(M(\varepsilon)/K(\varepsilon))]$. In view of (4.4), this ensures that $\theta(r)r^{-1} \in N(MF(\varepsilon)/M(\varepsilon))$ whenever $\theta \in \mathcal{G}(M(\varepsilon)/K(\varepsilon))$ and $r \in M_\varepsilon$. Assuming now that MF/M is immediate (or equivalently, that F/K is immediate), using the condition on the finite extensions of K in $K(p)$, and arguing as in the proof of (3.1) and Lemma 3.9, one obtains from this result that $\nabla_0(K(\varepsilon)) \subseteq N(F(\varepsilon)/K(\varepsilon))$. This, however, contradicts the fact that $(M/K, \psi, c) \in d(K)$, so Lemma 4.3 is proved. $\square$

Remark 4.4. *Assume that (K, v) is a Henselian field, such that $\mathrm{char}(K) = p$, $v(K)/pv(K)$ is of order p and finite extensions of K in $K(p)$ are totally ramified. Then, by the proofs of [43], Lemmas 2.2, 3.2 and 3.3, the assertion of Lemma 4.3 remains valid without the assumption that K is p-quasilocal (when K_{sep} is replaced by its algebraic closure). Hence, by the proof of [43], Proposition 3.2, $d(K)$ does not contain cyclic K-algebras of index p. This, combined with [2], Ch. VII, Theorem 28, implies $Br(K)_p = \{0\}$.*

Let now (K, v) be a Henselian p-quasilocal with $\mathrm{char}(K) = 0$ and $\mathrm{char}(\widehat{K}) = p$, and suppose that $p \in P(K)$, $v(K)/pv(K)$ is of order p and finite extensions of K in $K(p)$ are defectless. As noted at the beginning of this Section, then $r(p)_K \geq 2$. Assume further that $Br(K)_p \neq \{0\}$ and fix a primitive p-th root of unity $\varepsilon \in K_{\mathrm{sep}}$. Since K is p-quasilocal, this implies the existence of a cyclic algebra $D \in d(K)$ of index p. As in the proofs of Lemmas 4.2 and 4.3, it is seen that $D \otimes_K K(\varepsilon)$ is $K(\varepsilon)$-isomorphic to $A_\varepsilon(a, b; K(\varepsilon))$, for some $a \in K^*$, $b \in K_\varepsilon$. In addition, it turns out that, by the proof of Albert's cyclicity criterion for an algebra $\Delta \in d(K)$ of index p (see [31], Sect. 15.5, or [11], (3.3)) that if $A_\varepsilon(a, b; K(\varepsilon)) \cong A_\varepsilon(a; b_0; K(\varepsilon))$, for some $b_0 \in \nabla_0(K(\varepsilon))$, then there exists $b_0' \in K_\varepsilon$, such that $A_\varepsilon(a, b_0'; K(\varepsilon)) \cong A_\varepsilon(a, b; K(\varepsilon))$ and $v_{K(\varepsilon)}(b_0' - 1) \geq v_{K(\varepsilon)}(b_0 - 1)$. Considering now $A_\varepsilon(a, b; K(\varepsilon))$ as in the proof of [43], Proposition 3.3 (see also [39], Ch. 2, Lemma 19), and using Lemma 4.3 and the inequality $r(p)_K \geq 2$, one obtains the following result:

(4.5) The K-algebra D is defectless.

We are now in a position to complete the proof of Theorem 1.1. Statements (4.1), (1.4) and (4.5), combined with Lemma 3.3 (ii) and Remark 3.4, as well as with Remark 4.4 and the observation preceding the statement of Lemma 4.1, prove Theorem 1.1 (iii). The conclusion of Theorem 1.1 (i) follows from Lemma 3.3, Remark 3.4, the comment on (4.3) and the pointed observation. It remains for us to prove Theorem 1.1 (ii). Suppose that (K, v) satisfies the conditions of Lemma 4.1. Then K has an immediate extension I_1 in $K(p)$ of degree p. At the same time, Lemma 3.3 (ii), combined with (2.4) (i), (4.3), (4.5) and Remark 4.4, indicates that (R, v_R) satisfies the conditions of Lemma 4.1 whenever $R \in I(K(p)/K)$ and $[R: K] \in \mathbb{N}$. Therefore, one proves without difficulty by induction on n the existence of a unique sequence I_n, $n \in \mathbb{N}$, of subfields of K_{sep}, such that $I_1 = I$, $I_n \subset I_{n+1}$, $[I_{n+1}: I_n] = p$, and I_{n+1}/I_n is immediate, for every index n. In view of Galois theory, this implies that I_n/K is cyclic and immediate with $[I_n: K] = p^n$ whenever $n \in \mathbb{N}$, and the union $I_\infty = \cup_{n=1}^\infty I_n$ is the unique immediate $\mathbb{Z}_p$-extension of K in K_{sep}. This, combined with (4.1), (4.3) and Theorem 1.1 (iii), yields the alternative of Theorem 1.1 (ii) in the case of $\mathrm{char}(\widehat{K}) = p$. Thus Theorem 1.1 is proved.

Remark 4.5. *The proof of Theorem 1.1 is considerably easier in the special case where $K_{\mathrm{sep}} = K(p)$, since then K contains a primitive p-th root of unity or $\mathrm{char}(K) = p$, which simplifies the consideration of the structure of $Br(K)_p$ (see (4.1) and Remark 3.4). In addition, when $\mathrm{char}(\widehat{K}) = p$, (4.5) can be directly deduced from [43], Theorem 3.1.*

5   Brauer groups of Henselian PQL-fields with totally indivisible value groups

The purpose of this Section is to describe the isomorphism classes of several major types of valued PQL-fields considered in this paper. Our first result is particularly useful in the case of quasilocal fields:

Proposition 5.1. *Let (K, v) be a Henselian field, such that $v(K) \neq pv(K)$ and $\mathrm{Br}(K)_p \neq \{0\}$, for some $p \in \mathbb{P}$. Suppose further that finite extensions of K are p-quasilocal. Then $p \in P(K)$ and $\mathrm{Br}(K)_p \cong \mathbb{Z}(p^\infty)$. Moreover, every $D \in d(K)$ of p-primary index is a cyclic K-algebra.*

Proof. Fix Sylow pro-p-subgroups G_p and $\widetilde{G}_p$ of $\mathcal{G}_K$ and $\mathcal{G}_{\widehat{K}}$, respectively, and denote by K_p and $\widehat{K}_p$ the corresponding fixed fields. Our choice of K_p ensures that $p \nmid [K': K]$ whenever $K' \in I(K_p/K)$ and $[K': K] \in \mathbb{N}$, so [4], (1.2) and Remark 2.2, and the results of [31], Sect. 13.4, imply the following:

(5.1) $\mathrm{Br}(K_p/K) \cap \mathrm{Br}(K)_p = \{0\}$ and the natural embedding of K into K_p induces a group isomorphism $v(K)/pv(K) \cong v(K_p)/pv(K_p)$.

Since $\mathrm{Br}(K)_p \neq \{0\}$ and, by [8], I, Lemma 8.3, K_p is p-quasilocal, it can be easily deduced from (5.1), Lemma 3.3 (i) and Theorem 1.1 that $\mathrm{Br}(K_p) \cong \mathbb{Z}(p^\infty) \cong \mathrm{Br}(K)$. In view of [8], I, Theorem 3.1 (i), it remains to be seen that $p \in P(K)$. When $p = \mathrm{char}(K)$, this follows from Lemma 2.2, so we assume further that $p \neq \mathrm{char}(K)$. As $\mathrm{Br}(K)_p \neq \{0\}$, Theorem 2 of [30], Sect. 4, requires the existence of an algebra $D \in d(K)$ of index p. Fix an element $\theta \in K^*$ so that $v(\theta) \notin pv(K)$ and denote by L_θ some extension of K in K_{sep} generated by a p-th root of θ. Also, let $\varepsilon \in K_{\mathrm{sep}}$ be a primitive p-th root of unity. Since $[K(\varepsilon) : K] \mid (p-1)$, we have $D \otimes_K K(\varepsilon) \in d(K(\varepsilon))$ and $[L_\theta : K] = [L_\theta(\varepsilon) : K(\varepsilon)] = p$. Observing also that $L_\theta(\varepsilon)$ is cyclic over $K(\varepsilon)$, one obtains from the p-quasilocal property of $K(\varepsilon)$ that $L_\theta(\varepsilon)$ embeds in $D \otimes_K K(\varepsilon)$ as a $K(\varepsilon)$-subalgebra. Thus it becomes clear that L_θ is isomorphic to a K-subalgebra of D, so it follows from Albert's criterion (cf. [31], Sect. 15.5) that D is a cyclic K-algebra. This shows that $p \in P(K)$, so Proposition 5.1 is proved. $\qquad\square$

Remark 5.2. *In the setting of Proposition 5.1, let $v(K)/pv(K)$ be of order p, and for each finite extension L of K in K_{sep}, put $X_p(L) = \{\chi \in C_L : p\chi = 0\}$, where C_L is the continuous character group of $\mathcal{G}_L$. Denote by κ_L the canonical pairing $X_p(L) \times v(K)/pv(K) \to_p \mathrm{Br}(L)$ (see, e.g., the proof of [8], I, Lemma 1.1), and by $cor_{L/K}$ the homomorphism of $X_p(L)$ into $X_p(K)$ induced by the corestriction map $C_L \to C_K$. It is clear from Proposition 5.1 that $\rho_{K/L}$ maps $\mathrm{Br}(K)_p$ surjectively upon $\mathrm{Br}(L)_p$, and by a well-known projection formula (stated in [41], page 205, for a proof, see, e.g., [46], Proposition 4.3.7), the compositions $\kappa_K \circ cor_{L/K}$ and $Cor_{L/K} \circ \kappa_L$ coincide. Choose as we can (by Remark 4.5) L so that $p \nmid [L : K]$ and there is a field $I_L \in I(L(p)/L)$, such that $[I_L : L] = p$ and $v(I_L) = v(L)$. Then it follows from [42], Theorem 3.2, and the noted property of $\rho_{K/L}$ that $Cor_{L/K}$ induces an isomorphism $\mathrm{Br}(L)_p \cong \mathrm{Br}(K)_p$. Fix a generator σ_L of $\mathcal{G}(I_L/L)$ and an element $\pi \in K^*$ so that $v(\pi) \notin pv(K)$. The preceding observations show that $Cor_{L/K}([I_L/L, \sigma_L, \pi]) \in \mathrm{Br}(I_K/K)$, for some $I_K \in I(K(p)/K)$ with $[I_K : K] = p$ and $v(I_K) = v(K)$. Hence, by Lemma 3.3 (ii), $I_L = I_K L$, which yields $p \in P(K)$ independently of the proof of (4.5).*

It is known (and easy to see, e.g., from Scharlau's generalization of Witt's decomposition theorem or from [8], I, Corollary 8.5) that every divisible subgroup T of $\mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathrm{Br}(K_T)$, provided that (K, v) is a Henselian discrete valued field with a quasifinite residue field and K_T is the compositum of the inertial extensions of K in K_{sep} of degrees not divisible by any $p \in \mathbb{P}$,

for which $T_p \neq \{0\}$. This, combined with our next result, describes the isomorphism classes of Brauer groups of Henselian quasilocal fields with totally indivisible value groups.

Corollary 5.3. *Let (K, v) be a Henselian quasilocal field, such that $v(K)$ is totally indivisible. Then K is nonreal, $\widehat{K}$ is perfect, every $D \in d(K)$ is cyclic and $\mathrm{Br}(K)$ is divisible and embeddable in $\mathbb{Q}/\mathbb{Z}$. Moreover, $P(K)$ contains every $p \in \mathbb{P}$, for which $\mathrm{Br}(K)_p \neq \{0\}$.*

Proof. Our concluding assertion and the one concerning $\mathrm{Br}(K)$ follow from Proposition 5.1. The statements that $\widehat{K}$ is perfect and K is nonreal are implied by Lemma 3.3, the assumption on $v(K)$ and [8], I, Lemma 3.6. Note finally that all $D \in d(K)$ are cyclic. Since it suffices to prove this only in the special case where $[D] \in \mathrm{Br}(K)_p$, for some $p \in \mathbb{P}$ (see [31], Sect. 15.3), the assertion can be viewed as a consequence of Proposition 5.1. $\square$

Corollary 5.4. *Under the hypotheses of Corollary 5.3, let L/K be a finite abelian extension and L_0 the maximal extension of K in L, for which $[L_0 : K]$ is not divisible by any $p \in \mathbb{P}$ with $Br(K)_p = \{0\}$. Then $E^*/N(L/K) \cong \mathcal{G}(L_0/K)$.*

Proof. This follows at once from (1.2) (ii), Theorem 1.1 and [9], Lemma 2.1. $\square$

Corollary 5.5. *An abelian torsion group T with $T_2 \neq \{0\}$ is isomorphic to $\mathrm{Br}(K)$, for some Henselian PQL-field (K, v) such that $v(K)$ is totally indivisible, if and only if T is divisible and $T_2 \cong \mathbb{Z}(2^\infty)$. When this holds, $T \cong \mathrm{Br}(F)$, for some Henselian discrete valued PQL-field (F, w).*

Proof. Theorem 3.1 shows that $\mathrm{Br}(K)_2 \cong \mathbb{Z}(2^\infty)$ whenever (K, v) is a Henselian 2-quasilocal field with $v(K) \neq 2v(K)$ and $\mathrm{Br}(K)_2 \neq \{0\}$. This, combined with [7], Theorem 4.2, proves our assertion. $\square$

Let T be an abelian torsion group and $S_0(T) = \{\pi \in \mathbb{P} \colon T_\pi = \{0\}\}$. Assume that $2 \in S_0(T)$ and denote by $S_1(T)$ the set of those $p \in \mathbb{P} \setminus S_0(T)$, for which $S_0(T)$ contains the prime divisors of $p - 1$. Clearly, if $T \neq \{0\}$, then $S_1(T)$ contains the least element of $\mathbb{P} \setminus S_0(T)$. Using Theorem 3.1, Lemma 3.5 (i) and [7], Theorem 4.2, one obtains the following results:

(5.2) (i) If T is divisible with $T_p \cong \mathbb{Z}(p^\infty)$, for every $p \in S_1(T)$, then there exists a Henselian PQL-field (K, v), such that $\mathrm{Br}(K) \cong T$, $v(K)$ is totally indivisible and $\mathrm{char}(\widehat{K}) = 0$;

(ii) If $\operatorname{Br}(F) \cong T$, for some Henselian PQL-field (F, w) with $w(F)$ totally indivisible, then $T_p \cong \mathbb{Z}(p^\infty)$ and F contains a primitive p-th root of unity, for each $p \in S_1(T)$, $p \neq \operatorname{char}(\widehat{F})$;

(iii) The group T satisfies the condition in (i) if and only if $T \cong \operatorname{Br}(L)$, for some Henselian real-valued PQL-field (L, ω).

The conclusion of (5.2) (iii) is not necessarily true without the condition that $\omega(L) \leq \mathbb{R}$. Indeed, let T be an abelian torsion group with $T_2 = \{0\}$, $S_\pi(T) = S_1(T) \setminus \{\pi\}$, for some $\pi \in S_1(T)$, and $S'_\pi(T)$ the set of those $p' \in \mathbb{P} \setminus S_0(T)$, for which the coset of π in $\mathbb{Z}/p'\mathbb{Z} = \mathbb{F}_{p'}$ has order in $\mathbb{F}_{p'}^*$ not divisible by any $p \in \mathbb{P} \setminus S_\pi(T)$. Fix an algebraic closure $\overline{\mathbb{Q}_\pi}$ of the field $\mathbb{Q}_\pi$ of π-adic numbers, and for each subset Π of $\mathbb{P}$, put $\Pi' = \Pi \cup \{\pi\}$, $\overline{\Pi} = \mathbb{P} \setminus \Pi$, and denote by U_Π the compositum of inertial extensions of $\mathbb{Q}_\pi$ in $\overline{\mathbb{Q}_\pi}$ of degrees not divisible by any $\bar{\pi} \in \overline{\Pi}$. It is well-known that $U_\pi/\mathbb{Q}_\pi$ is a Galois extension with $\mathcal{G}(U_\pi/\mathbb{Q}_\pi)$ isomorphic to the topological group product $\prod_{p \in \Pi} \mathbb{Z}_p$. This implies that $U_\pi(p')/\mathbb{Q}_\pi$ is Galois, for each $p' \in \mathbb{P}$. Observing also that $\mathcal{G}(U_\Pi/\mathbb{Q}_\pi)$ is a projective profinite group [20], Theorem 1, one concludes that, for each $\Pi \subseteq \mathbb{P}$, there exists a field $F_\Pi \in I(U_{\Pi'}(\pi)/U_\Pi)$, such that $U_{\Pi'}.F_\Pi = U_{\Pi'}(\pi)$ and $U_{\Pi'} \cap F_\Pi = U_\Pi$. This observation, combined with (4.1), Proposition 2.5 and Theorem 1.1, enables one to obtain the following result (arguing in the spirit of the proof of (5.2) and Corollary 5.5):

(5.3) $T \cong \operatorname{Br}(K)$, for some Henselian PQL-field (K, v), such that $v(K)$ is totally indivisible and $\operatorname{char}(\widehat{K}) = \pi$, if and only if T is divisible with $T_{p'} \cong \mathbb{Z}(p'^\infty)$, for every $p' \in S'_\pi$. When T has the noted properties and $T_\pi \not\cong \mathbb{Z}(\pi^\infty)$, $\operatorname{char}(K) = 0$ and $G(K) = \pi G(K)$. Moreover (K, v) can be chosen so that $K_{G(K)} = F_{S_\pi(T)}$ and $\hat{v}_{G(K)}$ extends the natural valuation of $\mathbb{Q}_\pi$.

Corollary 5.5, statements (5.2) and (5.3), and the classification of divisible abelian groups (cf. [17], Theorem 23.1) describe the isomorphism classes of Brauer groups of Henselian PQL-fields with totally indivisible value groups.

6   Applications

The first result of this Section together with Theorem 2.1 of [5], I, characterizes the quasilocal property in the class of Henselian quasilocal fields with totally indivisible value groups.

Proposition 6.1. *Let (K, v) be a Henselian field, such that $\operatorname{char}(\widehat{K}) = q \neq 0$, and let K_p be the fixed field of some Sylow pro-p-subgroup $G_p \leq \mathcal{G}_K$, for each $p \in \Pi(K)$. Assume that $v(K) \neq pv(K)$, for every $p \in \Pi(K)$, and K possesses at least one finite extension in K_{sep} of nontrivial defect. Then K is quasilocal if and only if it satisfies the following two conditions:*

(i) $\widehat{K}$ *is perfect,* $q \notin \Pi(\widehat{K})$, $v(K)/qv(K)$ *is of order* q, *and* K_{sep} *contains as a subfield a norm-inertial* $\mathbb{Z}_q$-*extension* Y *of* K_q, *such that every finite extension* L_q *of* K_q *in* K_{sep} *with* $L_q \cap Y = K_q$ *is totally ramified;*

(ii) $r(p)_{K_p} \le 2$, *for each* $p \in \Pi(K) \setminus \{q\}$.

Proof. The Henselian property of (K, v) ensures that $K_p^*/K_p^{*p} \cong \widehat{K}_p^*/\widehat{K}_p^{*p} \oplus v(K)/pv(K)$, for each $p \in \mathbb{P} \setminus \{q\}$. Since $v(K)/pv(K) \cong v(K_p)/pv(K_p)$ and $v(K) \ne pv(K)$, this enables one to deduce from (2.3) that $\mathrm{Br}(K_p) \ne \{0\}$ whenever $r(p)_{K_p} \ge 2$. These observations, combined with (1.4), [45], Lemma 7, [8], I, Lemma 3.8, indicate that condition (ii) holds if and only if K_p is p-quasilocal, for each $p \in \Pi(K) \setminus \{q\}$. At the same time, it follows from (2.3) and our assumptions that there exists a finite extension of K_q in K_{sep} of nontrivial defect. Suppose that K_q is quasilocal. Then (4.2), Lemma 4.1 and the preceding observation imply the existence of an immediate $\mathbb{Z}_q$-extension Y of K_q in K_{sep}. In view of Remark 4.5 and Theorem 4.1, $\mathrm{Br}(K_q) \cong \mathbb{Z}(q^\infty)$, so it follows from [31], Sect. 15.1, Proposition b and the Henselian property of v_{K_q} that Y is norm-inertial over K_q. Since q does not divide the degrees of the finite extensions of K in K_q, the fulfillment of the remaining part of condition (i) can be proved by applying Lemmas 3.3 and 3.5 to (K_q, v_{K_q}).

Our objective now is to show that K_q is q-quasilocal, provided that condition (i) holds. Put $w = v_{K_q}$, denote by H the maximal q-divisible group from $\mathrm{Is}_w(K_q)$, and for each $n \in \mathbb{N}$, let Y_n be the extension of K_q in Y of degree q^n. Since $\widehat{K}$ is perfect, it follows from the assumption on Y/K_q and the choice of H that $\mu \in N(Y_n/K_q)$, for every $n \in \mathbb{N}$ whenever $\mu \in K_q^*$ and $v_{K_q}(\mu) \in H$. At the same time, the q-divisibility of H implies that every finite extension L_q of K_q in K_{sep} with $L_q \cap Y = K_q$ is totally ramified relative to w_H. The obtained results enable one to deduce from Lemma 3.2 that the residue field of (K_q, w_H) is perfect (of characteristic q or zero). Observing also that $w_H(Y) = w_H(K_q)$, one concludes that Y satisfies one of the following two conditions:

(6.1) (i) Y equals the compositum of the inertial extensions of K_q in K_{sep} relative to w_H;

(ii) Y/K_q is norm inertial relative to w_H.

The fulfillment of (6.1) guarantees that K_q is quasilocal (see [5], I, Lemma 2.2, and the proof of [4], Theorem 3.1 (b) (ii)), so we assume further that Y/K_q satisfies (6.1) (ii). Considering, if necessary, w_H instead of w, and using the observations preceding the statement of (6.1), one obtains a reduction of the proof of the q-quasilocal property of K_q to the special case in which every nontrivial group from $\mathrm{Is}_w(K_q)$ is q-indivisible. We first show that $_q\mathrm{Br}(K_q) = \mathrm{Br}(Y_1/K_q)$. Let L be a finite extension of Y in K_{sep}. Then it follows from Galois theory and the projectivity of $\mathbb{Z}_q$ [20], Theorem 1, that $L \subseteq Y.L_q'$, for some finite extension L_q' of K_q in K_{sep}, such that $L_q' \cap Y = K_q$. This

implies that L/Y is defectless, so it follows from [43], Theorem 3.1, that every $D \in d(Y)$ is defectless. Observing, however, that $\widehat{Y}$ is perfect, $q \notin P(\widehat{Y})$, and for each $n \in \mathbb{N}$, $v(Y)/q^n v(Y)$ is a cyclic group of order q^n, one concludes that $\mathrm{ind}(D)$ divides the defect of D over Y. Thus it turns out that $\mathrm{Br}(Y) = \{0\}$ and $\mathrm{Br}(K_q) = \mathrm{Br}(Y/K_q) = \cup_{n=1}^{\infty} \mathrm{Br}(Y_n/K_q)$. Since Y/K_q is norm-inertial, this enables one to deduce from (3.4), Hilbert's Theorem 90 and general properties of cyclic algebras (cf. [31], Corollary b) that $_q\mathrm{Br}(K_q) = \mathrm{Br}(Y_1/K_q)$, as claimed. Let now Y' be an extension of K_q in K_{sep}, such that $Y' \neq Y_1$ and $[Y' : K_q] = q$. Then Y'/K_q is totally ramified. Since $\widehat{K}_q$ is perfect and $w(K_q)/qw(K_q)$ is of order q, this ensures that $K_q^* \subseteq Y'^{*p}\nabla_0(Y')$. Observing also that YY'/Y' is norm-inertial (as follows from (3.4) and (2.3)), one concludes that $K_q^* \subseteq N(Y_1Y'/Y')$. In view of [31], Sect. 15.1, Proposition b, the obtained result implies that $\mathrm{Br}(Y_1/K_q) \subseteq \mathrm{Br}(Y'/K_q)$. As $\mathrm{Br}(Y'/K_q) \subseteq_q \mathrm{Br}(K_q)$, this proves that $\mathrm{Br}(Y'/K_q) = \mathrm{Br}(Y_1/K_q) =_q \mathrm{Br}(K)$, which means that K_q is q-quasilocal. It is now easy to deduce from [8], I, Lemma 8.3, that K is quasilocal when conditions (i) and (ii) hold, which completes the proof of Proposition 6.1. $\square$

The following result shows that every nontrivial divisible subgroup of $\mathbb{Q}/\mathbb{Z}$ is realizable as a Brauer group of a quasilocal field of the type characterized by Proposition 6.1. In view of Corollary 5.3, it clearly illustrates the fact that the study of Henselian quasilocal fields with totally indivisible value groups does not reduce to the special case covered by [5], II, Theorem 2.1.

Proposition 6.2. *Let (Φ, ω) be a Henselian discrete valued field, such that $\widehat{\Phi}$ is quasifinite and $\mathrm{char}(\widehat{\Phi}) = q > 0$, and let T be a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$ with $T_q \neq \{0\}$. Then there exists a Henselian quasilocal field (K, v) with the following properties:*

(i) $\mathrm{Br}(K) \cong T$, K/Φ is a field extension of transcendency degree 1, and v is a prolongation of ω;

(ii) $v(K)$ is Archimedean and totally indivisible, $\widehat{K}/\widehat{\Phi}$ is an algebraic extension and K possesses an immediate $\mathbb{Z}_q$-extension.

The proof of Proposition 6.2 is constructive and can be found in [5], II, Sect. 4. Our next result gives a criterion for divisibility of value groups of Henselian quasilocal fields, and for defectlessness of their central division algebras.

Proposition 6.3. *Let E be a quasilocal field satisfying one of the following two conditions:*

(i) Every finite group G is isomorphic to a subquotient of $\mathcal{G}_E$, i.e. to a homomorphic image of some open subgroup H_G of $\mathcal{G}_E$;

(ii) $\mathrm{Br}(E_p) \neq \{0\}$, for each $p \in \Pi(E)$, E_p being the fixed field of some Sylow pro-p-subgroup G_p of $\mathcal{G}_E$; moreover, if $\mathrm{Br}(E_p) \cong \mathbb{Z}(p^\infty)$, then $2 < r(p)_{E_p} < \infty$.

Assume that E has a Henselian valuation v. Then $v(E)$ is divisible and, in case (i)*, every $D \in d(E)$ is inertial over E relative to v.*

Proof. For each $p \in \mathbb{P}$, E_p is p-quasilocal, $v(E_p)/pv(E_p) \cong v(E)/pv(E)$, and E_p contains a primitive p-th root of unity unless $p = \mathrm{char}(E) = p$. These observations, combined with (1.4), imply that if $v(E) \neq pv(E)$ and $p \neq \mathrm{char}(\widehat{E})$, then $r(p)_{E_p} \leq 2$. When condition (ii) holds, this contradicts Theorem 1.1, and thereby proves that $v(E) = pv(E)$ in case $p \neq \mathrm{char}(\widehat{E})$. Suppose now that $\mathrm{char}(\widehat{E}) = p$, $p \in \Pi(E)$ and E satisfies condition (ii). We first show that $r(p)_{E_p} = \infty$. Assuming the opposite, one obtains from [22], Proposition 4.4.8, and [9], Corollary 5.3, that $\mathrm{char}(E) = 0$, $\mathrm{Br}(E_p) \cong \mathbb{Z}(p^\infty)$ and $r(p)_{E_p} \geq 3$. It is therefore clear from (4.1) that $v(E)/G(E)$ is p-divisible, $G(E)$ being defined as in Section 2. At the same time, [14], Proposition 3.4, and the assumptions on $r(p)_{E_p}$ require that $v(E_p) \neq pv(E_p)$. Thus it turns out that $v(E) \neq pv(E)$ and $G(E) \neq pG(E)$, so it follows from Proposition 2.6 that $G(E)$ is cyclic and $v(E)$ is totally indivisible. The obtained contradiction proves that $r(p)_{E_p} = \infty$. Hence, by condition (ii), $_p\mathrm{Br}(E_p)$ is noncyclic, which enables one to deduce from (5.2) (ii) and Theorem 1.1 that $v(E) = pv(E)$. Thus the fulfillment of (ii) guarantees that $v(E)$ is divisible.

We turn to the proof of the divisibility of $v(E)$ in case (i) of Proposition 6.3. Our assumptions indicate that, for each $p \in \mathbb{P}$, there are finite extensions L_p, L_p' and L_p'' of E in E_{sep}, such that L_p' and L_p'' are Galois over L_p with $\mathcal{G}(L_p'/L_p)$ and $\mathcal{G}(L_p''/L_p)$ isomorphic to the alternating groups Alt_{jp} and Alt_{kp}, respectively, for some integers j and k with $5 \leq j < k$. It is clear from Galois theory, the choice of L_p' and L_p'', and the simplicity of the groups Alt_n, $n \geq 5$, that $L_p'L_p''$ is a Galois extension of L_p with $\mathcal{G}(L_p'L_p''/L_p) \cong \mathrm{Alt}_{jp} \times \mathrm{Alt}_{kp}$. Denote by U_p the compositum of the inertial extensions of L_p in E_{sep}, and put $I_p' = U_p \cap L_p'$, $I_p'' = U_p \cap L_p''$. Note that U_p/L_p is Galois and $\mathcal{G}_{U_p}$ is prosolvable (see [21], page 135, and, e.g., [5], I, page 3102), which means that $\mathcal{G}(L_p'/I_p')$ and $\mathcal{G}(L_p''/I_p'')$ are solvable groups. Therefore, the preceding observation also shows that $L_p'L_p'' \subseteq U_p$. Let now M_p be the fixed field of some Sylow p-subgroup of $\mathcal{G}(L_p'L_p''/L_p)$, and let H_p' and H_p'' be Sylow p-subgroups of $\mathcal{G}(L_p'/L_p)$ and $\mathcal{G}(L_p''/L_p)$, respectively. Then $L_p'L_p''/M_p$ is an inertial Galois extension with $\mathcal{G}(L_p'L_p''/M_p) \cong H_p' \times H_p''$. The obtained result indicates that $r(p)_{\widehat{M_p}} \geq 2$. As M_p is quasilocal, this yields $v(M_p) = pv(M_p)$ (see Remark 3.4). Hence, by the isomorphism $v(E)/pv(E) \cong v(M_p)/pv(M_p)$, we have $v(E) = pv(E)$, which proves that $v(E)$ is divisible, as claimed.

It remains to be seen that the fulfillment of condition (i) of Proposition 6.3 ensures that every $D \in d(E)$ is inertial. As $v(E)$ is divisible, it suffices to show that D/E is defectless. In view of (2.3) and (1.1), one may assume, for the proof, that $\mathrm{char}(\widehat{E}) = q > 0$ and $\mathrm{Br}(E_q) \neq \{0\}$. As shown above, $q \in P(\widehat{E}_q)$

and $v(E_q) = qv(E_q)$, so Lemma 3.6 and our assumptions indicate that finite extensions of E_q in E_{sep} are defectless. It is therefore clear from (2.3) that finite extensions of E in E_{sep} have the same property, so the concluding assertion of Proposition 6.3 follows from [43], Theorem 3.1. $\square$

Our next result characterizes p-adically closed fields in the class of Henselian quasilocal fields with residue fields of characteristic p; it essentially gives an answer to a question posed to the author by Serban Basarab.

Proposition 6.4. *For a Henselian field (K, v) with $\text{char}(\widehat{K}) = p > 0$ and $v(K) \neq pv(K)$, the following conditions are equivalent:*

(i) (K, v) *is p-adically closed;*

(ii) K *is quasilocal,* $\text{char}(K) = 0$, $r(p)_K \in \mathbb{N}$ *and either* $r(p)_K \geq 3$ *or* $\mathcal{G}(K(p)/K)$ *is a free pro-p-group with* $r(p)_K = 2$.

Proof. Note that condition (i) of Proposition 2.6 holds when (K, v) satisfies condition (i) or (ii) of Proposition 6.4 (see (4.1), [45], page 725, and [34], Theorem 3.1). Therefore, one may assume for the proof that $\text{char}(K) = 0$, $\widehat{K}$ is finite and the subgroup $G(K) \leq v(K)$ is cyclic. This enables one to deduce from (1.4) and Remark 2.8 that if K is quasilocal, then $v_{G(K)}(K) = v(K)/G(K)$ is divisible. In view of [34], Theorem 3.1, the obtained results prove that (ii)→(i). Assume now that (K, v) is p-adically closed. This ensures that $v(K)/G(K)$ is divisible, which implies finite extensions of K in K_{sep} are inertial relative to $v_{G(K)}$. Therefore, $\mathcal{G}_K \cong \mathcal{G}_{K_{G(K)}}$ (see [21], page 135), so it follows from [8], I, Theorem 8.1, that K is quasilocal if and only if so is $K_{G(K)}$. As $\widehat{K}$ is finite and $G(K)$ is cyclic, the assertion that $K_{G(K)}$ is quasilocal can be deduced from (2.1) (i), Propositions 2.1 and [5], I, Corollary 2.5. Finally, the isomorphism $\mathcal{G}_K \cong \mathcal{G}_{K_{G(K)}}$, combined with (2.6) (ii) and [40], Ch. II, Theorems 3 and 4, indicates that $\mathcal{G}(K(p)/K)$ and $r(p)_K$ satisfy condition (ii), so the implication (i)→(ii) and Proposition 6.4 are proved. $\square$

The concluding result of this Section supplements Theorem 1.1 by showing that Brauer groups of quasilocal fields with Henselian valuations whose residue fields are separably closed and imperfect do not necessarily embed in $\mathbb{Q}/\mathbb{Z}$. In addition to Proposition 6.3, it simultaneously indicates that central division algebras over such fields are not necessarily defectless.

Proposition 6.5. *Assume that (Φ, ω) is a Henselian field, such that $\omega(\Phi) = \mathbb{Z}$, $\widehat{\Phi}$ is quasifinite and $\text{char}(\widehat{\Phi}) = q > 0$. Then there exists a Henselian field (K, v) with the following properties:*

(i) $v(K) = \mathbb{Q}$, $\text{char}(\widehat{K}) = q$ *and* $[\widehat{K} : \widehat{K}^q] = q$;

(ii) $\mathcal{G}_K$ *is a pro-q-group and* $_q\mathrm{Br}(K) \cong \widehat{K}^*/\widehat{K}^{*q}$; *in particular,* $_q\mathrm{Br}(K)$ *is infinite;*

(iii) K *is quasilocal and has a unique immediate* $\mathbb{Z}_q$-*extension* I_∞ *in* K_{sep}; *for each finite extension* L *of* K *in* K_{sep} *of nontrivial defect,* $L \cap I_\infty \neq K$;

(iv) K/Φ *is a field extension of transcendency degree 1.*

Proof. Let $\overline{\Phi}$ be an algebraic closure of Φ. By the proof of [5], II, Theorem 1.2, there exists a field $\widetilde{K} \in I(\overline{\Phi}/\Phi)$, such that $\overline{\Phi}/\widetilde{K}$ is an immediate $\mathbb{Z}_q$-extension relative to $v_{\widetilde{K}}$; in particular, one can apply (3.4) (iii) to the fields from $I(\overline{\Phi}/\widetilde{K})$ the conditions of [5], II, Lemma 3.1. Put $\tilde{v} = v_{\widetilde{K}}$ and let $\tilde{k}$ be the residue field of $(\widetilde{K}, \tilde{v})$. Denote by v_X the Gauss valuation of the rational function field $\widetilde{K}(X)$, which extends $\tilde{v}$ so that $v_X(f(X)) = 0$, for each $f(X) \in O_{\tilde{v}}[X] \setminus M_{\tilde{v}}[X]$, and fix a Henselization (F, v) of $(\widetilde{K}(X), v_X)$. The definition of v_X shows that the residue class of X is transcendental over $\tilde{k}$. Let G_q be a Sylow pro-q-subgroup of $\mathcal{G}_F$, K the fixed field of G_q and $v = w_K$. Arguing in the spirit of the proof of the quasilocal property of the field K_q (considered in [5], II, Sect. 4), one obtains first that $\mathrm{Br}(\widetilde{K}_1K/K) =_q \mathrm{Br}(K)$, where $\widetilde{K}_1$ is the extension of $\widetilde{K}$ in Γ of degree q. This is used for proving that K, v and $I_\infty = \overline{\Phi}K$ have the properties required by Proposition 6.5. $\qquad\square$

Remark 6.6. *Let* F_0 *be a global field,* w_0 *a discrete valuation of* F_0, $q = \mathrm{char}(\widehat{F_0})$, F *a completion of* F_0 *with respect to* ω_0, *and* w *the valuation of* F *continuously extending* w_0. *It is well-known that* $\mathrm{tr}(F/F_0)$ *is (uncountably) infinite. Fix a purely transcendental extension* F_n *of* F_0 *in* F *so that* $\mathrm{tr}(F_n/F_0) = n \geq 0$, $n \leq \infty$, *denote by* Φ_n *the separable closure of* F_n *in* F, *and let* ω_n *be the valuation of* Φ_n *induced by* w. *Then* ω_n *is discrete and Henselian and* $\widehat{\Phi}_n \cong \mathbb{F}_q$. *Therefore, one can find an extension* R_n *of* Φ_n *with the properties required by Proposition 6.5. When* $n \in \mathbb{N}$, *our construction ensures that* $\mathrm{tr}(R_{n-1}/F_0) = n$.

For each $m \in \mathbb{N}$, the quasilocal fields R_n, $n \in \mathbb{N}$, in Remark 6.6 have infinitely many nonisomorphic algebras $D_{n,m} \in d(R_n)$ of index p^m. By [6], Theorem 4.1 and Corollary 8.6, for each admissible pair (n,m), all $D_{n,m}$ share, up-to R_n-isomorpisms, a common set of maximal subfields, and a common class of splitting fields algebraic over R_n. Since R_n is of transcendency degree $n+1$ or $n+2$ over its prime subfield, for each $n < \infty$, this raises interest in the open problem of whether there exist finitely generated fields F which possess infinitely many nonisomorphic $D \in d(F)$ with some of the noted two properties of the algebras $D_{m,n}$ (see [24], and for the case of quaternion algebras [35], [18] and [33], Remark 5.4). The corresponding problem for arbitrary fields

has an affirmative solution (found by Van den Bergh-Schofield [44], Sect. 3, and Saltman, see [19], Sect. 5.5). In view of [8], I, Corollary 8.5, a complete solution to the general problem is obtained by applying the latter assertion of (1.3) (i), to a field E_0 of zero characteristic and to a divisible abelian torsion group T with infinite components T_p, for all $p \in \mathbb{P}$.

References

[1] A.A. Albert, *Modern Higher Algebra*, Univ. of Chicago Press, Chicago, Ill., 1937.

[2] A.A. Albert, *Structure of Algebras*, Amer. Math. Soc. Coll. Publ., vol. XXIV, Amer. Math. Soc., New York, 1939.

[3] N. Bourbaki, *Elements de Mathematique. Algebre Commutative*, Chaps. V, VI, VII, Hermann, Paris, 1964.

[4] I.D. Chipchakov, *Henselian valued stable fields*, J. Algebra **206** (1998), 344-369.

[5] I.D. Chipchakov, *Henselian valued quasi-local fields with totally indivisible value groups*, I, Comm. Algebra **27** (1999), 3093-3108; II, Preprint, arXiv:1011.2661v3 [math.RA].

[6] I.D. Chipchakov, *On the Galois cohomological dimensions of stable fields with Henselian valuations*, Comm. Algebra **30** (2002), 1549-1574.

[7] I.D. Chipchakov, *Henselian discrete valued fields with one-dimensional local class field theory*, in: Proc. of AGAAP-Conference (V. Brinzanescu, V. Drensky, P. Pragatz, Eds.), Borovets, Bulgaria, 22.9-02.10. 2003, Serdica Math. J. **34** (2004), 363-394.

[8] I.D. Chipchakov, *On the residue fields of Henselian valued stable fields*, I, J. Algebra **319** (2008), 16-49; II, C.R. Acad. Bulg. Sci. **60** (2007), 471-478.

[9] I.D. Chipchakov, *Primarily quasilocal fields and 1-dimensional abstract local class field theory*, Preprint, arXiv:math/0506515v7 [math.RA].

[10] I.D. Chipchakov, *On the Brauer groups of quasilocal fields and the norm groups of their finite Galois extensions*, Preprint, arXiv:math/0707.4245v6 [math.RA].

[11] I.D. Chipchakov, *Demushkin groups and inverse Galois theory for pro-p-groups of finite rank and maximal p-extensions*, Preprint, arXiv:1103.2114v3 [math.RA].

[12] P.K. Draxl, *Skew Fields*, London Math. Soc. Lecture Note Series, vol. 81, Cambridge etc., Cambridge Univ. Press, IX, 1983.

[13] P.K. Draxl, *Ostrowski's theorem for Henselian valued skew fields,* J. Reine Angew. Math. **354** (1984), 213-218.

[14] I. Efrat, *Finitely generated pro-p Galois groups of p-Henselian fields*, J. Pure Appl. Algebra **138** (1999), 215-228.

[15] I. Efrat, *A Hasse principle for function fields over PAC-fields*, Isr. J. Math. **122** (2001), 43-60.

[16] I. Efrat, *Valuations, Orderings, and Milnor K-Theory,* Math. Surveys and Monographs 124, Amer. Math. Soc., Providence, RI, 2006.

[17] L. Fuchs, *Infinite Abelian Groups*, v. I, Pure and Applied Math., 36. New York-London, Academic Press, XI, 1970.

[18] S. Garibaldi and D.J. Saltman, *Quaternion algebras with the same subfields*, Preprint, arXiv:0906.5337v1 [math.RA], Jun 28, 2009.

[19] Ph. Gille and T. Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambridge Studies in Advanced Mathematics 101; Cambridge University Press, XI, Cambridge, 2006.

[20] K.W. Gruenberg, *Projective profinite groups*, J. Lond. Math. Soc. **42** (1967), 155-165.

[21] B. Jacob and A.R. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), 126-179.

[22] N. Jacobson, *Finite-Dimensional Division Algebras over Fields*, Springer-Verlag, Berlin, 1996.

[23] A.J. de Jong, *The period-index problem in the theory of algebraic surfaces*, Duke Math. J. **123** (2004), 71-94.

[24] D. Krashen and K. McKinnie, *Distinguishing division algebras by finite splitting fields*, Preprint, arXiv:1001.3685v1 [math.RA].

[25] T.Y. Lam, *Orderings, valuations and quadratic forms*, Reg. Conf. Ser. Math. 52, 1983.

[26] M. Lieblich, *Twisted sheaves and the period-index problem*, Compos. Math. **144** (2008), 1-31.

[27] S. Lang, *Algebra*, Reading, Mass., Addison-Wesley, Inc., XVIII, 1965.

[28] P. Mammone and A. Merkur'ev, *On the corestriction of the p^n-symbol*, Isr. J. Math. **76** (1991), 73-80.

[29] O.V. Mel'nikov and O.I. Tavgen, *The absolute Galois group of a Henselian field*, Dokl. Akad. Nauk BSSR **29** (1985), 581-583 (Russian).

[30] A.S. Merkur'ev, *Brauer groups of fields*, Comm. Algebra **11** (1983), 2611-2624.

[31] R. Pierce, *Associative Algebras*, Graduate Texts in Math., vol. 88, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[32] F. Pop, *Galoissche Kennzeichnung p-adisch abgeschlossener Körper*, J. Reine Angew. Math. **392** (1988), 145-175.

[33] G. Prasad and A.S. Rapinchuk, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*, Publ. Math. IHES **109** (2009), 113-184.

[34] A. Prestel and P. Roquette, *Formally p-adic Fields*, Lecture Notes in Math., 1050, Berlin etc., Springer-Verlag, 1984.

[35] A.S. Rapinchuk and I.A. Rapinchuk, *Division algebras having the same maximal subfields*, Preprint, arXiv:0910.3368v1 [math.RA], Oct. 18, 2009.

[36] M. Reiner, *Maximal Orders*, London Math. Soc. Monographs, vol. 5, London-New York-San Francisco: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers, 1975.

[37] D.J. Saltman, *The Schur index and Moody's theorem*, K-Theory **7** (1993), 309-332.

[38] W. Scharlau, *Über die Brauer-Gruppe eines Hensel-Körpers*, Abh. Math. Semin. Univ. Hamb. **33** (1969), 243-249.

[39] O.F.G. Schilling, *The Theory of Valuations*, Math. Surveys, IV, Amer. Math. Soc., New York, 1950.

[40] J.-P. Serre, *Cohomologie Galoisienne*, Cours au College de France, 1962/1963, 2nd ed., Springer-Verlag, Berlin-Göttingen-Heidelberg-New York, 1964.

[41] J.-P. Serre, *Local Fields*, Translated from the French by M.J. Greenberg. Graduate Texts in Math., vol. 67, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[42] J.-P. Tignol, *On the corestriction of central simple algebras*, Math. Z. **194** (1987), 267-274.

[43] I.L. Tomchin and V.I. Yanchevskij, *On defects of valued division algebras*, Algebra i Analiz **3** (1991), 147-164 (Russian: English transl. in St. Petersburg Math. J. **3** (1992), 631-647).

[44] M. Van den Bergh and A. Schofield, *Division algebra coproducts of index n*, Trans. Amer. Math. Soc. **341** (1994), 505-517.

[45] R. Ware, *Galois groups of maximal p-extensions*, Trans. Amer. Math. Soc. **333** (1992), 721-728.

[46] E. Weiss, *Cohomology of Groups*, Pure and Applied Math., 34, Academic Press, New York-London, 1969.

[47] G. Whaples, *Algebraic extensions of arbitrary fields*, Duke Math. J. **24** (1957), 201-204.

[48] T. Yamazaki, *Reduced norm map of division algebras over complete discrete valuation rings of certain type*, Compos. Math. **112** (1998), 127-145.

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