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HÖLDER WEIGHT ESTIMATES OF RIESZ-BESSEL SINGULAR INTEGRALS GENERATED BY A GENERALIZED SHIFT OPERATOR

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Abstract

Riesz-Bessel singular integral operators generated by generalized shift operator in weighted Hölder space $H^{\gamma}_{\alpha,\beta}$ is studied. The $H^{\gamma}_{\alpha,\beta}$ boundedness of this operator is established in certain cases.

1 Introduction

Singular integral operators are playing an important role in Harmonic Analysis, theory of functions and partial differential equations. Singular integrals associated with the Δ_{B_n} Laplace-Bessel differential operator

$$\Delta_{B_n} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_n, \quad B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma_n}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma_n > 0,$$

which is known as an important operator in analysis and its applications, have been the research areas of many mathematicians such as B. Muckenhoupt and E.M. Stein [12, 13], I. Kipriyanov and M. Klyuchantsev [1, 2, 11], K. Trimeche [15], K. Stempak [14], I.A. Aliev and A.D. Gadjiev [3], V.S. Guliyev [9], A.D. Gadjiev and Emin V. Guliyev [8] and others.

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The multidimentional singular integral operators generated by generalized shift operator are studied by Klyuchantsev in [1] and Kipriyanov and Klyuchantsev in [2]. Aliev and Gadjiev [3], and Ekincioglu and Serbetci [7] have studied the boundedness of certain singular integrals generated by generalized shift operator in weighted L_p-spaces with radial weights.

Klyuchantsev and Kipriyanov [2] have firstly introduced and investigated the $L_{p,\gamma}$-boundedness of singular integrals generated by the Δ_{B_n}–Laplace-Bessel diffe-rential operator (B_n–singular integrals). Aliev and Gadjiev [3] have studied the boundedness of the B_n–singular integrals in weighted $L_{p,\omega,\gamma}$-spaces with radial weights. Gadjiev and Guliyev [8] and Ekincioglu [18] have investigated the boundedness of the B_k-singular integrals in weighted $L_{p,\omega,\gamma}$-spaces with general weights.

Matrix singular integral operators and singular integral operators are investigated in weighted Hölder spaces by Kapanadze in [16] and Abdullayev [17].

The paper is organized as follows. In the next section, we collect the background materials. In the Section 3, we recall the definition of weighted Hölder spaces and give some inequalities in weighted Hölder spaces. Finally, we give the boundedness of Riesz-Bessel singular integral operators in weighted Hölder spaces will be studied in certain cases.

Throughout this paper we use the convention that c denotes a generic constant, depending on k, γ or other fixed parameters, its value may change from line to line.

In this paper, Riesz-Bessel singular integral operators generated by generalized shift operator associated with the Δ_{B_k} Laplace-Bessel differential operator

$$\Delta_{B_k} = \sum_{i=1}^{n-k} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_{n-k+i}} \frac{\partial}{\partial x_{n-k+i}} + \frac{\partial^2}{\partial x_{n-k+i}^2}, \; k = 1, \dots, n.$$

is studied in weighted Hölder spaces.

2 Preliminaries

Let $\mathbb{R}^n$ be n–dimensional Euclidean space and $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ be vectors in $\mathbb{R}^n_{k,+}$, then $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$, $x = (x', x'')$, $x' = (x_1, \ldots, x_{n-k}) \in \mathbb{R}^{n-k}_{k,+}$, $x'' = (x_{n-k+1}, \ldots, x_n) \in \mathbb{R}^k_{k,+}$, $x''' = (x_{n+1}, \ldots, x_{n+k}) \in \mathbb{R}^k_{k,+}$ and $\tilde{x} = (x', x'', x''') \in \mathbb{R}^{n+k}$. Denote , $S^n_{k,+} = \{x \in \mathbb{R}^n_{k,+} : \; |x| = 1\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_{n-k}^{\gamma_{n-k}}$.

Let $\mathbb{R}^n_{k,+} = \{x : \ x = (x_1, x_2, \ldots, x_n), \ x_{n-k+i} > 0, \ldots, x_n > 0, \ 1 \le i \le k\}$. We now introduce the generalized shift operator is defined by

$$T^y f(x) = c_\gamma \int_0^\pi \ldots \int_0^\pi f\left(x' - y', (x_{n-k+1}, y_{n-k+1})_{\alpha_1}, \ldots, (x_n, y_n)_{\alpha_k}\right) \, d\gamma(\alpha),$$

where $c_\gamma = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma(\gamma_i + \frac{1}{2})}{\Gamma(\gamma_i)}$, $d\gamma(\alpha) = \prod_{i=1}^{k} \sin^{\gamma_i - 1} \alpha_i \quad d\alpha_i$ and $x', y' \in \mathbb{R}^{n-k}$ and $(x_{n-k+i}, y_{n-k+i})_{\alpha_i} = [x^2_{n-k+i} - 2x_{n-k+i} y_{n-k+i} \cos\alpha_i + y^2_{n-k+i}]^{\frac{1}{2}}$, $1 \le i \le k$, [18].

Note that the generalized shift operator is closely connected with Laplace-Bessel differential operator (see [4],[1]). The shift T^y generates the corresponding convolution

$$(f * g)(x) = \int_{\mathbb{R}_n^{k,+}} f(y) \left[T^y g(x)\right] (y'')^\gamma dy$$

which satisfies the property $(f * g) = (g * f)$.

The Riesz-Bessel singular integral operator generated by a generalized shift operator $(R_{B_k}) f \quad (1 \le k \le n)$ is defined by

$$\begin{aligned} (R_{B_k}) f(x) \quad &\equiv \ p.v. \ c_k \int_{\mathbb{R}^n_{k,+}} \frac{P_k\ (\theta)}{|y|^{Q+k}} \left[T^y f(x)\right] (y'')^\gamma dy \\ &\equiv c_k \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^n_{k,+} : \ |y| > \varepsilon\}} \frac{P_k\ (y)}{|y|^{Q+k}} \left[T^y f(x)\right] (y'')^\gamma dy \\ &\equiv \lim_{\varepsilon \to 0} (R_{B_k})_\varepsilon f(x), \end{aligned} \tag{2.1}$$

where $Q = n + |\gamma|$, $c_k = 2^{Q/2} \Gamma\Big(\frac{Q+k}{2}\Big) \big(\Gamma\big(\frac{k}{2}\big)\big)^{-1}$, $k = 1, 2, \ldots, n$, $\theta = y/|y|$, and the characteristic $P_k(\theta)$ belongs to some function space on the hemisphere $S^n_{k,+}$ such that $P_k(x) = P_k(x_1, \ldots, x_{n-k}, x^2_{n-k+1}, \ldots, x^2_n)$ is a homogeneous polynomial with order k which holds $\Delta_{B_k} P_k = 0$ and satisfies the cancellation condition

$$\int_{S^n_{k,+}} P_k(\theta) (\theta'')^\gamma d\theta = 0. \tag{2.2}$$

Unless stated otherwise we will assume throughout that $P_k(\theta)$ satisfies a uniform Hölder condition on S^n_{k+}, i.e., if θ_1 and θ_2 are unit vectors, then for some positive constants A and $0 < \delta \le 1$,

$$|P_k(\theta_1) - P_k(\theta_2)| \le cA |\theta_1 - \theta_2|^\delta \tag{2.3}$$

where $|\theta_1-\theta_2|$ denotes the geodesic distance on S^n_{k+} from θ_1 to θ_2, c is a constant and $\sup|P_k(\theta)|=A\le\infty$.

Lemma 2.1. *Let α and β be are arbitrary numbers such that $0<\alpha<\beta\le+\infty$ then for any point $x\in\mathbb{R}^n_{k,+}$ the following equality holds*

$$\int\limits_{\alpha<|y|<\beta}\frac{P_k\left(\frac{y}{|y|}\right)}{|y|^{Q+k}}\left[T^y f(x)\right](y'')^\gamma dy=c_\gamma\int\limits_{\alpha<|\tilde{x}-\tilde{s}|<\beta}\frac{P_k(\tilde{\theta})}{|\tilde{x}-\tilde{s}|^{Q+k}}\tilde{f}(s)\prod_{i=1}^k|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s},$$

where $\tilde{f}(s)=f\left(s',[s^2_{n-k+1}+s^2_{n-k+2}]^{\frac12}\dots,[s^2_{n+k-1}+s^2_{n+k}]^{\frac12}\right)$, $d\tilde{s}=ds'ds''ds'''$,
$\tilde{\theta}=\left(\frac{x'-s'}{|\tilde{x}-\tilde{s}|},\frac{[(x_{n-k+1}-s_{n-k+1})^2+s^2_{n-k+2}]^{\frac12}}{|\tilde{x}-\tilde{s}|},\dots,\frac{[(x_{n+k-1}-s_{n+k-1})^2+s^2_{n+k}]^{\frac12}}{|\tilde{x}-\tilde{s}|}\right)$,
and $\tilde{x}=(x',x'',0,\dots,0)\in\mathbb{R}^{n+k}_{k+}$. *(see [1])*

Proof. We denote the first part by I

$$\begin{aligned}I&=c_\gamma\int\limits_{|y|>\epsilon}\frac{P_k(\frac{y}{|y|})}{|y|^{Q+k}}\int\limits_0^\pi\dots\int\limits_0^\pi f\left(x'-y',(x_{n-k+1},y_{n-k+1})_{\alpha_1},\dots,(x_n,y_n)_{\alpha_k}\right)(y'')^\gamma d\gamma(\alpha)dy\\&=c_\gamma\int\limits_{|y|>\epsilon}\frac{P_k(\frac{y}{|y|})}{|y|^{Q+k}}\int\limits_0^\pi\dots\int\limits_0^\pi f\Big(x'-y',[x^2_{n-k+1}-2x_{n-k+1}y_{n-k+1}\cos\alpha_1+(y_{n-k+1}\cos\alpha_1)^2\\&+(y_{n-k+1}\sin\alpha_1)^2]^{\frac12},\dots,[x^2_n-2x_ny_n\cos\alpha_k+(y_n\cos\alpha_k)^2+(y_n\sin\alpha_k)^2]^{\frac12}\Big)\\&\times(y'')^\gamma\sin^{\gamma_i-1}\alpha_id\alpha_idy\\&=c_\gamma\int\limits_{|y|>\epsilon}\frac{P_k(\frac{y}{|y|})}{|y|^{Q+k}}\int\limits_0^\pi\dots\int\limits_0^\pi f\Big(x'-y',[(x_{n-k+1}-y_{n-k+1}\cos\alpha_1)^2+(y_{n-k+1}\sin\alpha_1)^2]^{\frac12},\\&\dots,[(x_n-y_n\cos\alpha_k)^2+(y_n\sin\alpha_k)^2]^{\frac12}\Big)(y'')^\gamma\sin^{\gamma_i-1}\alpha_id\alpha_idy.\end{aligned}$$

Performing a change of variables in the last integral to $x'-y'=s'$, $s_{n-k+(2i-1)}=x_{n-k+i}-y_{n-k+i}\cos\alpha_i$, $s_{n-k+2i}=y_{n-k+i}\sin\alpha_i$, $0\le\alpha_i<\pi$ and $y_{n-k+i}>0$, $i=1,2,\dots,k$. Since the jacobian of the transformation is equal to $(y'')^{-1}$, we have

$$\int\limits_{\alpha<|y|<\beta}\frac{P_k\left(\frac{y}{|y|}\right)}{|y|^{Q+k}}\left[T^y f(x)\right](y'')^\gamma dy=c_\gamma\int\limits_{|\tilde{x}-\tilde{s}|>\epsilon}\frac{P_k(\tilde{\theta})}{|\tilde{x}-\tilde{s}|^{Q+k}}\tilde{f}(s)\prod_{i=1}^k|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}.$$

This completes the proof. □

Lemma 2.2.

$$\int\limits_{\mathbb{R}^n_{k+}} f(y)(y'')^\gamma dy \quad = c_\gamma \int\limits_{\mathbb{R}^{n+k}_{k+}} \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}.$$

Proof. Changing variables $y' = s'$, $s_{n-k+(2i-1)} = y_{n-k+i}\cos\alpha_i$, $s_{n-k+2i} = y_{n-k+i}\sin\alpha_i$,
$0 \le \alpha_i < \pi$ and $y_{n-k+i} > 0$, $i = 1, 2, \ldots, k$ show that

$$\int\limits_{\mathbb{R}^n_{k,+}} f(y)(y'')^\gamma dy \quad = c_\gamma \int\limits_{\mathbb{R}^{n+k}_{k,+}} \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s},$$

which completes the proof. □

Setting $f(x) = 1$ in Lemma 2.1 ($T^y f(x) = 1$) and using the polar coordinates show that

$$\int\limits_{S^n_{k,+}} P_k(\theta) \prod_{i=1}^{k} \theta^{\gamma_i}_{n-k+i} d\theta = c_\gamma \int\limits_{S^{n+k}_{k,+}} P_k(\tilde{\theta}) \prod_{i=1}^{k} |\theta_{n-k+2i}|^{\gamma_{i-1}} d\tilde{\theta}. \tag{2.4}$$

3 Holder Space with weight $H^\nu_{\alpha,\beta}(\mathbb{R}^n_{k,+})$

Let $\nu > 0$, $\alpha > 0$, β be a real number,

$$\rho(x) = \sum_{i=1}^{k} x^\alpha_{n-k+i}(1 + |x|)^{\beta-\alpha}, \quad x \in \mathbb{R}^n_{k,+}.$$

If $\lim\limits_{x\to\infty} f(x)\rho(x) = 0$, $\lim\limits_{x_{n-k+i}\to 0} f(x)\rho(x) = 0$, $1 \le i \le k$, then $f \in H^\nu_{\alpha,\beta}(\mathbb{R}^n_{k,+})$ and the norm in the space $H^\nu_{\alpha,\beta}$is defined by the equality

$$||f||_{H^\nu_{\alpha,\beta}} = \sup_{x,y\in\mathbb{R}^n_{k,+}} \big(|f(x)\rho(x) - f(y)\rho(y)|d^{-\nu}(x,y)\big),$$

where $d(x,y) = \dfrac{|x-y|}{(1+|x|)(1+|y|)}$ (see [6]).

If the contrary is not stipulated, then later on we will assume that $0 < \nu < 1$, $0 < \alpha - \nu < 1$, $0 < \beta + \nu < n$. Let $x \in \mathbb{R}^n_{k,+}$. We introduce the following notation,

$$\Psi_\nu(x) = \sum_{i=1}^{k} x^{\nu-\alpha}_{n-k+i}(1+|x|)^{-\ell} \equiv \rho^{-1}(x) \sum_{i=1}^{k} x^\nu_{n-k+i}(1+|x|)^{-2\nu}$$

where $\ell = 2\nu + \beta - \alpha$. Let E' and E be defined as above

$$E' = \left\{ \tilde{s} \in \mathbb{R}^{n+k}_{k,+} : |\tilde{x} - \tilde{s}| < \sum_{i=1}^{k} \frac{x_{n-k+i}}{2} \right\}, \quad E = \left\{ y \in \mathbb{R}^{n}_{k,+} : |y - x| < \sum_{i=1}^{k} \frac{x_{n-k+i}}{2} \right\}.$$

Remark 3.1. Note that if $\tilde{s} \in E'$, then $|\tilde{x}| < 2|\tilde{s}| < 3|\tilde{x}|$; $E' \subset E \times \big\{(s_{n-k+2i}) : |s_{n-k+2i}| < \sum_{i=1}^{k} \frac{x_{n-k+i}}{2},\ 1 \le k \le n\big\}$ and at $\tilde{s} \in \mathbb{R}^{n+k}_{k,+} \setminus E'$, $|\tilde{x} - \tilde{s}| \approx |x' - s'| + \sum_{i=1}^{k} x_{n-k+i} + \sum_{i=1}^{k} |s_{n-k+i}| + \sum_{i=1}^{k} |s_{n-k+2i}|$.

The following lemma gives an explicit expression for $H^{\nu}_{\alpha,\beta}$.

Lemma 3.2. *Let $0 < \gamma < \alpha$, $\beta + \gamma > 0$. $f \in H^{\nu}_{\alpha,\beta}(\mathbb{R}^n_{k,+})$ if and only if there exits C_1 and C_2 depend on f such that*

(i) $|f(x)| \le C_1(f)\Psi_\gamma(x)$, $\forall x \in \mathbb{R}^n_{k,+}$

(ii) $|f(\tilde{x}) - f(\tilde{s})| \le C_2(f)\rho^{-1}(x)d^{\gamma}(\tilde{x}, \tilde{s})$ $\ \forall x \in \mathbb{R}^n_{k,+}$ *and* $\forall \tilde{s} \in E'$.

Moreover $||f||_{H^{\nu}_{\alpha,\beta}} \approx (\max C_1(f) + \max C_2(f))$ *(see [5]).*

We have the following result.

Corollary 3.3. *If $f \in H^{\nu}_{\alpha,\beta}$, then*

(i) $|\tilde{f}(s)| \le ||f||_{H^{\nu}_{\alpha,\beta}} \sum_{i=1}^{k} \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} (|1 + |s|)^{-\ell}$
$\le ||f||_{H^{\nu}_{\alpha,\beta}} \sum_{i=1}^{k} \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} \big(1 + |s'| + \sum_{i=1}^{k} \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)\big)^{-\ell}$

(ii) $\big|\tilde{f}(s) - f(x)\big| \le ||f||_{H^{\nu}_{\alpha,\beta}} \rho^{-1}(x) d^{\nu}(\tilde{x}, \tilde{s}) \le ||f||_{H^{\nu}_{\alpha,\beta}} \rho^{-1}(x) |\tilde{x} - \tilde{s}|^{\nu} (1 + |x|)^{-2\nu}$ *for* $\forall x \in \mathbb{R}^n_{k,+}$ *and* $\forall \tilde{s} \in E'$.

4 Riesz-Bessel Singular Integral Operators

It is well known that the Riesz-Bessel singular integral operators R_{B_k} $(1 \le k \le n)$ are bounded in $L_p(\mathbb{R}^n)$. The basic goal of this paper is to establish an estimate in the weighted Hölder spaces for singular integral operators generated by a generalized shift operator such as Riesz-Bessel singular integral operators R_{B_k}. We obtain sufficient conditions for α, β, ν so that R_{B_k} operators are bounded from the weighted $H^{\nu}_{\alpha,\beta}(\mathbb{R}^n_{k,+})$ spaces into the weighted $H^{\nu}_{\alpha,\beta}(\mathbb{R}^n_{k,+})$ spaces.

Theorem 4.1. *Let $\alpha > 0$, $\nu > 0$ and β be a real number such that $0 < \nu < 1$, $0 < \alpha - \nu < 1$, $0 < \beta + \nu < n$. Suppose that the characteristic $P_k(\theta)$ of the singular integral (2.1) is bounded and satisfies the condition (2.2). Then there exists Riesz-Bessel singular integrals generated by generalized shift operator such that*

$$\begin{aligned} R_{B_k} f(x) &= p.v.\ c_k \int\limits_{\mathbb{R}^n_{k,+}} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y\ f(x)\big](y'')^{\gamma} dy \\ &= c_{k,\gamma} \int\limits_{|\tilde{x}-\tilde{s}|\geq\eta} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\tilde{f}(s)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &+ c_{k,\gamma} \int\limits_{\epsilon<|\tilde{x}-\tilde{s}|<\eta} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\big[\tilde{f}(s)-f(x)\big]\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \end{aligned} \tag{4.1}$$

for any $f \in H^{\nu}_{\alpha\beta}$ and for a.e. $x \in \mathbb{R}^n_{k,+}$.

Proof. From Lemma 2.1

$$\begin{aligned} R_{B_k} f(x) &= p.v.\ c_k \int\limits_{\mathbb{R}_n^{k,+}} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y\ f(x)\big](y'')^{\gamma} dy \\ &= c_k \int\limits_{|y|\geq\eta} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y\ f(x)\big](y'')^{\gamma} dy \\ &+ c_k \int\limits_{\epsilon<|y|<\eta} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y f(x)-f(x)\big](y'')^{\gamma} dy \\ &= c_{k,\gamma} \int\limits_{|\tilde{x}-\tilde{s}|\geq\eta} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\tilde{f}(s)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &+ c_{k,\gamma} \int\limits_{\epsilon<|\tilde{x}-\tilde{s}|<\eta} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\big[\tilde{f}(s)-f(x)\big]\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &= c_{k,\gamma} \int\limits_{\mathbb{R}^{n+k}_{k,+}\setminus E'} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\tilde{f}(s)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &+ c_{k,\gamma} \int\limits_{E'} P_k(\tilde{\theta})|\tilde{x}-\tilde{s}|^{-Q-k}\big[\tilde{f}(s)-f(x)\big]\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \end{aligned}$$

where $E'(\epsilon) = \left\{ \tilde{s} \in \mathbb{R}^{n+k}_{k,+} : \epsilon < |\tilde{x} - \tilde{s}| < \eta, \ \eta = \sum_{i=1}^{k} \dfrac{x_{n-k+i}}{2} \right\}$. By definition $E'(\epsilon)$

$$\begin{aligned} R_{B_k} f(x) &= p.v.\ c_k \int\limits_{\mathbb{R}^{k,+}_n} P_k\big(\tfrac{y}{|y|}\big) |y|^{-Q-k} \big[T^y\ f(x) \big] (y'')^{\gamma} dy \\ &= c_{k,\gamma} \int\limits_{\mathbb{R}^{n+k}_{k,+} \setminus E'(\epsilon)} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \\ &+ c_{k,\gamma} \int\limits_{E'(\epsilon)} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \big[\tilde{f}(s) - f(x) \big] \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}. \end{aligned} \tag{4.2}$$

If we calculate the second integral in (4.2), we get

$$\begin{aligned} &\int\limits_{E'(\epsilon)} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \big[\tilde{f}(s) - f(x) \big] \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \\ &\quad = \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \\ &\quad - f(x) \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}. \end{aligned} \tag{4.3}$$

Passing to the limit of (4.3) as $\epsilon \to 0$ and polar coordinates, it can be easily see that the second integral in the right-hand side is vanish.

$$\begin{aligned} &\lim_{\epsilon \to 0} \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \big[\tilde{f}(s) - f(x) \big] \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \\ &\quad = \lim_{\epsilon \to 0} \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} P_k(\tilde{\theta}) |\tilde{x} - \tilde{s}|^{-Q-k} \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}. \end{aligned}$$

Hence, we get

$$\lim_{\epsilon \to 0} \big(R_{B_k} \big)_{\epsilon} f(x) = c_{k,\gamma} \big(i_1(\tilde{x}, E') + i_2(\tilde{x}, G) \big)$$

where $E \subseteq E'$ and $G \subseteq \mathbb{R}^{n+k}_{k,+} \setminus E'$. This completes the proof of the theorem. □

We just need to the estimates for $i_1(\tilde{x};\tilde{E})$ and $i_2(\tilde{x};G)$ when a.e. $\tilde{x}\in\mathbb{R}^{n+k}_{k,+}$ and $E\subseteq E'$ and $G\subseteq\mathbb{R}^{n+k}_{k,+}\setminus E'$. Firstly, let $f\in H^{\nu}_{\alpha,\beta}$ and $P_k(\theta)$ be bounded for $\theta\in S^n_{k,+}$. For $0<\nu<1$, $0<\alpha-\nu<1$ and $0<\beta+\nu<n$, then we prove the absolute convergence of the following integrals

$$i_1(\tilde{x};E') \quad =\int\limits_{E'}\frac{P_k(\tilde{\theta})}{|\tilde{x}-\tilde{s}|^{Q+k}}\left[\tilde{f}(s)-f(x)\right]\prod_{i=1}^{k}\left|s_{n-k+2i}\right|^{\gamma_i-1}d\tilde{s}$$

$$i_2(\tilde{x},G) \quad =\int\limits_{G}\frac{P_k(\tilde{\theta})}{|\tilde{x}-\tilde{s}|^{Q+k}}\tilde{f}(s)\prod_{i=1}^{k}\left|s_{n-k+2i}\right|^{\gamma_i-1}d\tilde{s}$$

To estimate $i_1(\tilde{x};E')$ we use Corollary 3.3 which states that

$$\begin{aligned}\left|i_1(\tilde{x};E')\right| &=\left|\int\limits_{E'}\frac{P_k(\tilde{\theta})}{|\tilde{x}-\tilde{s}|^{Q+k}}\left[\tilde{f}(s)-f(x)\right]\prod_{i=1}^{k}\left|s_{n-k+2i}\right|^{\gamma_i-1}d\tilde{s}\right|\\ &\le\int\limits_{E'}\frac{\left|P_k(\tilde{\theta})\right|}{|\tilde{x}-\tilde{s}|^{Q+k}}\left|\tilde{f}(s)-f(x)\right|\prod_{i=1}^{k}\left|s_{n-k+2i}\right|^{\gamma_i-1}d\tilde{s}\\ &\le c||f||_{H^{\nu}_{\alpha,\beta}}\rho^{-1}(x)(1+|x|)^{-2\nu}A\int\limits_{E'}\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}|\tilde{x}-\tilde{s}|^{-Q-k}d\tilde{s}\end{aligned}$$

where $||P_k(\tilde{\theta})||=\sup\limits_{\theta\in S^n_{k,+}}|P_k(\theta)|=A<\infty$. Since

$$\int\limits_{E'}\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}|\tilde{x}-\tilde{s}|^{-Q-k}d\tilde{s}\le\sum_{i=1}^{k}\left(\frac{x_{n-k+i}}{2}\right)^{\nu}=\eta,$$

we conclude that

$$\left|i_1(\tilde{x};E')\right|\le cA||f||\Psi_{\nu}(x). \tag{4.4}$$

We use the Corollary 3.3 to get the $i_2(\tilde{x},G)$ estimate. Then we introduce the following notation

$$\begin{aligned} L(\tilde{x},\tilde{s}) &= \frac{\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}}{|\tilde{x}-\tilde{s}|^{Q+k}}\Big(\sum_{i=1}^{k}\big(|s_{n-k+i}|+|s_{n-k+2i}|\big)\Big)^{\nu-\alpha}(1+|\tilde{s}|)^{-\ell} \\ &= \frac{\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}\Big(\sum_{i=1}^{k}\big(|s_{n-k+i}|+|s_{n-k+2i}|\big)\Big)^{\nu-\alpha}}{\Big(|x'-s'|+\sum_{i=1}^{k}x_{n-k+i}+\sum_{i=1}^{k}|s_{n-k+i}|+\sum_{i=1}^{k}|s_{n-k+2i}|\Big)^{Q+k}} \\ &\times\frac{\Big(1+|s'|+\sum_{i=1}^{k}\big(|s_{n-k+i}|+|s_{n-k+2i}|\big)\Big)^{-\ell}}{\Big(|x'-s'|+\sum_{i=1}^{k}x_{n-k+i}+\sum_{i=1}^{k}|s_{n-k+i}|+\sum_{i=1}^{k}|s_{n-k+2i}|\Big)^{Q+k}}. \end{aligned} \tag{4.5}$$

Now we shall obtain the estimate $i_2(\tilde{x};G)$. In the case, let us introduce the spaces

$$D_1 = \Big\{\tilde{s}\in\mathbb{R}^{n+k} : |\tilde{x}-\tilde{s}|<\frac{|\tilde{s}|}{2}\Big\},\quad D_2 = \Big\{\tilde{s}\in\mathbb{R}^{n+k} : \frac{|\tilde{s}|}{2}<|\tilde{x}-\tilde{s}|\le 3|\tilde{s}|\Big\},$$

$$D_3 = \Big\{\tilde{s}\in\mathbb{R}^{n+k} : 3|\tilde{s}|<|\tilde{x}-\tilde{s}|\Big\}.$$

Thus $i_2(\tilde{x},G)$ can be written as

$$\begin{aligned} |i_2(\tilde{x},G)| &\le ||f||\,||P_k(\theta)||i_2\big(\tilde{x},\mathbb{R}^{n+k}_{k,+}\setminus E'\big) \\ &= ||f||\,||P_k(\theta)||\Big(i_2\big(\tilde{x},D_1\big)+i_2\big(\tilde{x},D_2\big)+i_2\big(\tilde{x},D_3\big)\Big). \end{aligned} \tag{4.6}$$

Now we get the estimates $i_2\big(\tilde{x},D_1\big)$, $i_2\big(\tilde{x},D_2\big)$ and $i_2\big(\tilde{x},D_3\big)$ in (4.6). Let us take $\tilde{s}\in D_1$. We know that $(1+|\tilde{x}|)\approx(1+|\tilde{s}|)$. Hence we obtain

$$\begin{aligned} i_2(\tilde{x},D_1) &\le c.(1+|x|)^{-\ell}\int_0^\infty\dots\int_0^\infty\sum_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}ds''' \\ &\times\int_0^\infty\dots\int_0^\infty\sum_{i=1}^{k}\big(|s_{n-k+i}|+|s_{n+i}|\big)^{\nu-\alpha}ds'' \\ &\times\int_{\mathbb{R}^{n-k}_{k,+}}\Big(|y'|+\sum_{i=1}^{k}\big(|s_{n-k+i}|+|s_{n-k+2i}|+x_{n-k+i}\big)\Big)^{-Q-k}dy' \\ i_2(\tilde{x},D_1) &\le (1+|x|)^{-\ell}\Big(\sum_{i=1}^{k}x_{n-k+i}\Big)^{\nu-\alpha}=\Psi_\nu(x). \end{aligned}$$

To get estimate $i_2(\tilde{x};D_2)$, we define

$$i_2\big(\tilde{x},D_2\big)=\int_{G\cap D_2}L(\tilde{x},\tilde{s})d\tilde{s}.$$

Let us take first $|x| \geq 1$ and $\tilde{s} \in D_1$. We see that $|\tilde{x}-\tilde{s}| \approx |\tilde{s}|$ and $|\tilde{s}| \geq \dfrac{|x|}{4}$.
Hence, we get

$$|\tilde{x}-\tilde{s}| \approx |\tilde{s}| + |x| \approx |\tilde{s}| + 1.$$

We obtain

$$i_2\big(\tilde{x}, D_2\big) = \int_{D_2} \frac{\sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} \prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i - 1}}{\big(|\tilde{s}| + |x|\big)^{Q+k+\ell}} d\tilde{s}.$$

Suppose $\mu = (\nu+\beta) + (1+\nu-\alpha)$. By virtue of $0 < \nu < 1,\ 0 < \alpha-\nu < 1$, and $0 < \beta+\nu < n)$, $\mu > 0$. Taking into account that $n+2\nu+\ell = (n-1)+\mu+2\nu$, we obtain

$$i_2(\tilde{x}, D_2) < \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1} ds''' \int_0^\infty \ldots \int_0^\infty \sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} ds''$$
$$\times \int_{\mathbb{R}^{n-k}_{k,+}} \big(|y'| + \textstyle\sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}| + x_{n-k+i}\big)\big)^{(n-1)+\mu+2\nu} dy' < \Psi_\nu(x), \quad (|x| \geq 1).$$

Let use take second $|x| < 1$ and $\tilde{s} \in D_2$. Then it follows from $|\tilde{s}| \geq 1$, $|\tilde{s}| \approx |\tilde{s}| + 1 \approx |\tilde{s}| + 1 + |x|$ and $|\tilde{s}| < 1,\ 1 + |\tilde{s}| \approx 1$ that $|\tilde{x}-\tilde{s}| \approx |\tilde{s}| + \sum_{i=1}^k |x_{n-k+i}| \approx |s'| + \sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}|\big) + \sum_{i=1}^k x_{n-k+i}$.

Therefore, we deduce that

$$\begin{aligned} i_2(\tilde{x}, D_2) \quad &\leq \int_{\{\tilde{s} \in \mathbb{R}^{n+k}_{k,+} :\, |\tilde{s}|<1\}} \frac{\sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} \prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}}{\big(|s'| + \sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}| + x_{n-k+i}\big)\big)^{Q+k+\ell}} d\tilde{s} \\ &+ \int_{\{\tilde{s} \in \mathbb{R}^{n+k}_{k,+} :\, |\tilde{s}|\geq 1\}} \frac{\sum_{i=1}^k \big(|s_{n-k+i}| + |s_{n-k+2i}|\big)^{\nu-\alpha} \prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}}{\big(|\tilde{s}| + 1 + |x|\big)^{Q+k+\ell}} d\tilde{s} \\ &\leq \Big(\sum_{i=1}^k x_{n-k+i}^{\nu-\alpha} + \frac{1}{(1+|x|)^{\beta+\nu}} \Big) < \Psi_\nu(x) \quad (|x| < 1). \end{aligned}$$

Thus, we prove that

$$i_2(\tilde{x}, D_2) \leq \Psi_\nu(x). \tag{4.7}$$

Finally, $i_2(\tilde{x}, D_3)$ estimate can be proved by similar way. Thus we prove that

$$i_2\big(\tilde{x}, G\big) < \Psi_\nu(x) \quad \text{and} \quad |i_2(\tilde{x}, G)| < ||f||\, ||P_k(\theta)|| \Psi_\nu(x). \tag{4.8}$$

Consequently, the four estimates show that the absolute convergence of integrals $i_1(\tilde{x}, E')$ and $i_2(\tilde{x}, G)$.

Theorem 4.2. *Under the assumption (2.3), above if $f \in L_{p,\nu}$, and f satisfies the Hölder condition*

$$|f(x) - f(s)| \leq M|x - s|^{\delta}, \tag{4.9}$$

$\nu < \delta$, then for the integral operator

$$\big(R_{B_k}\big)_{\epsilon} f(x) = c_k \int\limits_{\{y \in \mathbb{R}^n_{k,+}:\ |y| > \epsilon\}} \frac{P_k(\frac{y}{|y|})}{|y|^{Q+k}} T^y f(x)(y'')^{\gamma} dy \tag{4.10}$$

the limit $R_{B_k} f(x) = \lim\limits_{\epsilon \to 0} \big(R_{B_k}\big)_{\epsilon} f(x)$ exists, and this limit satisfies the inequality

$$|\big(R_{B_k}\big) f(x) - \big(R_{B_k}\big) f(s)| \leq c_k M A |x - s|^{\delta}. \tag{4.11}$$

Proof. We show first of all that the limit of $\big(R_{B_k}\big)_{\epsilon} f(x)$ exists as $\epsilon \to 0$ and equal to $R_{B_k} f(x)$. The explicit formula of (4.10) shows that

$$\begin{aligned} \big(R_{B_k}\big)_{\epsilon} f(x) &= I_1 + I_2 = c_k \int\limits_{|y| > \eta} P_k\Big(\frac{y}{|y|}\Big)|y|^{-Q-k} T^y f(x)(y'')^{\gamma} dy \\ &+ c_k \int\limits_{\epsilon < |y| < \eta} P_k\Big(\frac{y}{|y|}\Big)|y|^{-Q-k}\big[T^y f(x) - T^0 f(x)\big](y'')^{\gamma} dy \end{aligned}$$

so that the Riesz-Bessel singular integral operator can be written as a sum of two terms. We consider the term I_2. By Lemma 2.1, we have

$$I_2 = c_k \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} \frac{P_k(\tilde{\theta})}{|\tilde{x} - \tilde{s}|^{Q+k}} \big[\tilde{f}(s) - f(x)\big] \prod_{i=1}^{k} s_{n-k+2i}^{\gamma_i - 1} d\tilde{s}.$$

From the condition (4.9) and the triangle inequality we obtain for I_2 the estimate

$$|I_2| \leq c_k M A \int\limits_{\epsilon < |\tilde{x} - \tilde{s}| < \eta} \frac{1}{|\tilde{x} - \tilde{s}|^{Q+k-\delta}} \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \leq \frac{c_k M A}{\delta}(\eta^{\delta} - \epsilon^{\delta}),$$

whence there exits the limit of the function $\big(R_{B_k}\big)_{\epsilon} f(x)$ as $\epsilon \to 0$. □

We now state the main theorem.

Theorem 4.3. *Let $P_k(\theta)$ satisfy condition (2.2) and (2.3). If $0 < \nu < \delta \leq 1$ and $0 < \alpha - \nu < 1$, $\beta + \nu < n$, then Riesz-Bessel Singular integral operator generated by generalized shift operator is bounded for each $f \in H^{\nu}_{\alpha,\beta}(\mathbb{R}^n_{k,+})$ in the weighted space $H^{\nu}_{\alpha\beta}$.*

Proof. By virtue of Theorem 4.1, from (4.4) and (4.8) we obtain

$$|\big(R_{B_k}\big)f(x)| \leq c_{k,\gamma}\big(|i_1(\tilde{x},E')| + |i_2(\tilde{x},G)|\big).$$

By virtue of Lemma 3.2, to prove the inequality it suffices to show that $\forall x \in R^n_{k+}$ and $|h| \leq \sum_{i=1}^k \dfrac{x_{n-k+i}}{8}$

$$|(R_{B_k})f(x) - (R_{B_k})f(x+h)| \leq c_k.M.A|h|^{\nu}$$

where c_k is independent of x and h.

$$\begin{aligned}(R_{B_k})f(x) - (R_{B_k})f(x+h) &= c_k \int\limits_{\mathbb{R}^n_{k,+}} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y\ f(x)\big](y'')^{\gamma}dy \\ &-c_k \int\limits_{\mathbb{R}^n_{k+}} P_k\big(\tfrac{y}{|y|}\big)|y|^{-Q-k}\big[T^y\ f(x+h)\big](y'')^{\gamma}dy.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}(R_{B_k})f(x) - (R_{B_k})f(x+h) &= c_{k,\gamma} \int\limits_{\mathbb{R}^n_{k+}} \frac{P_k\big(\tilde{\theta}\big)}{|\tilde{x}-\tilde{s}|^{Q+k}}\tilde{f}(s)\prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &-c_{k,\gamma} \int\limits_{\mathbb{R}^n_{k+}} \frac{P_k\big(\tilde{\theta}_1\big)}{|\widetilde{x+h}-\tilde{s}|^{Q+k}}\tilde{f}(s)\prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &= c_{k,\gamma}\int\limits_{E'} \big[K(\tilde{s},\tilde{x}) - K(\tilde{s},\widetilde{x+h})\big]\big[\tilde{f}(s)-f(x)\big]\prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}d\tilde{s} \\ &+c_{k,\gamma}\int\limits_{\mathbb{R}^{n+k}_{k+}\setminus E'} \big[K(\tilde{s},\tilde{x}) - K(\tilde{s},\widetilde{x+h})\big]\tilde{f}(s)\prod_{i=1}^k |s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}\end{aligned}$$

where

$$K(\tilde{s},\tilde{x}) = \frac{P_k\big(\tilde{\theta}\big)}{|\tilde{x}-\tilde{s}|^{Q+k}}.$$

For $\tilde{x} \in \mathbb{R}^{n+k}_{k,+}$, $\tilde{s} \in \mathbb{R}^{n+k}_{k,+} \setminus E'$ and $|h| < \sum_{i=1}^k \dfrac{x_{n-k+i}}{8}$ it can be easily seen that

$$|\tilde{x}-\tilde{s}| \approx |\widetilde{x+h}-\tilde{s}|. \tag{4.12}$$

It is well known that if $|x-(x+h)| \leq \max\{|x|,|x+h|\}$ then

$$\big|K(\tilde{s},\tilde{x}) - K(\tilde{s},\widetilde{x+h})\big| \leq \frac{c_k.A.|h|^{\delta}}{|\tilde{x}-\tilde{s}|^{Q+k+\delta}}. \tag{4.13}$$

Suppose $E_1(x) = E(\tilde{x}, 2|h|)$, $E_2(x) = E(\widetilde{x+h}, 3|h|)$ and $E_3(x) = E(\widetilde{x+h}, \sum_{i=1}^{k} \frac{x_{n-k+i}}{2} - |h|)$. Obviously $E_1(x) \subset E_2(x) \subset E_3(x) \subset E'$. Subject to (2.4) and (2.2), it can easily prove that

$$(R_{B_k})f(x) - (R_{B_k})f(x+h) = \mathrm{I}_1(x;h) + \mathrm{I}_2(x;h) + \mathrm{I}_3(x;h) + \mathrm{I}_4(x;h),$$

where

$$\begin{aligned}
\mathrm{I}_1(x;h) &= \Big(\int\limits_{E_2} + \int\limits_{E' \setminus E_2} \Big) K(\tilde{s}, \tilde{x}) \big(\tilde{f}(s) - f(x) \big) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}, \\
\mathrm{I}_2(x;h) &= -\int\limits_{E_2} K(\tilde{s}, \widetilde{x+h}) \big(\tilde{f}(s) - f(x) \big) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}, \\
\mathrm{I}_3(x;h) &= -\int\limits_{E' \setminus E_3} K(\tilde{s}, \widetilde{x+h}) \big(\tilde{f}(s) - f(x) \big) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}, \\
\mathrm{I}_4(x;h) &= \int\limits_{\mathbb{R}^{n+k}_{k,+} \setminus E'} \big[K(\tilde{s}, \tilde{x}) - K(\tilde{s}, \widetilde{x+h}) \big] \tilde{f}(s) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}.
\end{aligned}$$

Let us $|\mathrm{I}_i(x;h)|$ for $i = 1, 2, 3, 4$. Taking into account ii) of Corollary 3.3 and also (4.13), we get

$$\begin{aligned}
\mathrm{I}_1(x;h) &= \Big(\int\limits_{E_2} + \int\limits_{E' \setminus E_2} \Big) K(\tilde{s}, \tilde{x}) \big(\tilde{f}(s) - f(x) \big) \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s}, \\
|\mathrm{I}_1(x;h)| &\leq \Big(\int\limits_{E_2} + \int\limits_{E' \setminus E_2} \Big) \frac{|P_k(\tilde{\theta})|}{|\tilde{s} - \tilde{x}|^{Q+k}} \rho^{-1}(x) \Big(\frac{|s - \tilde{x}|}{(1+|x|)^2} \Big)^{\nu} \prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1} d\tilde{s} \\
&\leq A \rho^{-1}(x) (1+|x|)^{-2\nu} \Big(\int\limits_{E_2} + \int\limits_{E' \setminus E_2} \Big) \frac{\prod_{i=1}^{k} |s_{n-k+2i}|^{\gamma_i - 1}}{|\widetilde{x+h} - \tilde{s}|^{Q+k-\nu}} d\tilde{s} \\
&< A \rho^{-1}(x) (1+|x|)^{-2\nu} |h|^{\nu}
\end{aligned} \tag{4.14}$$

The following expressions are proved similarly:

$$\begin{aligned}
\mathrm{I}_2(x;h) &= -\int\limits_{E_2} K(\tilde{s},\widetilde{x+h})\big(\tilde{f}(s)-f(x)\big)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s},\\
|\mathrm{I}_2(x;h)| &\leq \int\limits_{E_2} \frac{|P_k(\tilde{\theta})|}{\widetilde{|x+h}-\tilde{s}|^{Q+k-\nu}}\rho^{-1}(x)(1+|x|)^{-2\nu}\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}\\
&< A\rho^{-1}(x)(1+|x|)^{-2\nu}\int\limits_{E_2}\frac{\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}}{\widetilde{|x+h}-\tilde{s}|^{Q+k-\nu}}d\tilde{s}\\
&< A\rho^{-1}(x)(1+|x|)^{-2\nu}|h|^{\nu}.
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\mathrm{I}_3(x;h) &= -\int\limits_{E'\setminus E_3} K(\tilde{s},\widetilde{x+h})\big(\tilde{f}(s)-f(x)\big)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s},\\
|\mathrm{I}_3(x;h)| &\leq \int\limits_{E'\setminus E_3} \frac{|P_k(\tilde{\theta})|}{\widetilde{|x+h}-\tilde{s}|^{Q+k-\nu}}\rho^{-1}(x)(1+|x|)^{-2\nu}\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}\\
&< A\rho^{-1}(x)(1+|x|)^{-2\nu}\int\limits_{E'\setminus E_3}\frac{\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}}{\widetilde{|x+h}-\tilde{s}|^{Q+k-\nu}}d\tilde{s}\\
&< A\rho^{-1}(x)(1+|x|)^{-2\nu}|h|^{\nu}.
\end{aligned} \tag{4.16}$$

According to (4.12) and (4.13), we obtain

$$
\begin{aligned}
\mathrm{I}_4(x;h) &= \int\limits_{\mathbb{R}^{n+k}_{k,+}\setminus E'} \left[K(\tilde{s},\tilde{x}) - K(\tilde{s},\widetilde{x+h})\right]\tilde{f}(s)\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}\\
|\mathrm{I}_4(x;h)| &\leq \int\limits_{\mathbb{R}^{n+k}_{k,+}\setminus E'} |K(\tilde{s},\tilde{x}) - K(\tilde{s},\widetilde{x+h})||\tilde{f}(s)|\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}d\tilde{s}\\
&< A||\tilde{f}|| \int\limits_{\mathbb{R}^{n+k}_{k,+}\setminus E'} \frac{|h|^{\delta}}{|\tilde{x}-\tilde{s}|^{Q+k+\delta}}\prod_{i=1}^{k}|s_{n-k+2i}|^{\gamma_i-1}\\
&\times\frac{\left(\sum_{i=1}^{k}|s_{n-k+i}|+|s_{n-k+2i}|\right)^{\nu-\alpha}}{\left(1+|s'|+\left(\sum_{i=1}^{k}|s_{n-k+i}|+|s_{n-k+2i}|\right)\right)^{\ell}}d\tilde{s}\\
&\leq A|h|^{\delta}||\tilde{f}||\sum_{i=1}^{k}(x_{n-k+i})^{-\delta} i_2(x,\mathbb{R}^{n+k}_{k,+}\setminus E')\\
&\leq A||\tilde{f}||\left(\frac{|h|}{\sum_{i=1}^{k}x_{n-k+i}}\right)^{\delta}\Psi_{\nu}(x) \leq A||\tilde{f}||\left(\frac{|h|}{\sum_{i=1}^{k}x_{n-k+i}}\right)^{\nu}\Psi_{\nu}(x)\\
&\leq A||\tilde{f}||\rho^{-1}(x)(1+|x|)^{-2\nu}|h|^{\nu}
\end{aligned}
\tag{4.17}
$$

Thus, the estimates (4.14)-(4.17) show that the theorem is proved. □

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