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A UNIQUENESS RESULT RELATED TO MEROMORPHIC FUNCTIONS SHARING TWO SETS

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Abstract

With the help of a new unique range set we investigate the well known question of Gross and prove a uniqueness theorem on meromorphic functions sharing two sets. The result in this paper will improve and supplement some earlier results.

1 Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [5]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let f and g be two non-constant meromorphic functions and let a be a complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

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Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) = a\}$, where each point is counted according to its multiplicity. Denote by $\overline{E}_f(S)$ the reduced form of $E_f(S)$. If $E_f(S) = E_g(S)$, we say that f and g share the set S CM. If $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In 1970s F. Gross and C.C. Yang started to study the set sharing problem of entire function instead of the value sharing problem, and prove that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$. In 1976 F. Gross proposed the following question in [8]:

Question A *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

In [8] Gross wrote *If the answer of* Question A *is affirmative it would be interesting to know how large both sets would have to be ?*

Yi [21] and independently Fang and Xu [7] gave the same positive answer in this direction.

Now it is natural to ask the following question [19].

Question B *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

Gradually the research on *Question B* gained pace and today it has become one of the most prominent branches of the uniqueness theory. For the last two decades several attempts have been made by different authors to consider the shared value problems relative to a meromorphic function sharing two sets and at the same time give affirmative answers to *Question B* under weaker hypothesis {see [1]-[7], [10], [14], [16]-[17], [19]-[21], [23]-[28]}.

Dealing with the question of Gross in [6] Fang and Lahiri obtained a unique range set S with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g.

Theorem A. *[6] Let $S = \{z : z^n + az^{n-1} + b = 0\}$ where $n(\geq 7)$ is an integer and a and b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g are two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ then $f \equiv g$.*

Let $S = \{z : z^7 - z^6 - 1 = 0\}$ and

$$f = \frac{e^z + e^{2z} + \ldots + e^{6z}}{1 + e^z + \ldots + e^{6z}}, \quad g = \frac{1 + e^z + \ldots + e^{5z}}{1 + e^z + \ldots + e^{6z}}$$

Obviously $f = e^z g$, $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$ but $f \not\equiv g$. So for the validity of *Theorem B*, f and g must not have any simple pole.

In 2001 an idea of gradation of sharing known as weighted sharing has been introduced in {[12], [13]} which measure how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1. *[12, 13] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. *[12] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.*
Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

With the notion of weighted sharing of sets improving *Theorem A*, Lahiri [14] proved the following theorem.

Theorem B. *[14] Let S be defined as in* Theorem A *and $n (\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty; f) + \Theta(\infty; g) > 1$, $E_f(S, 2) = E_g(S, 2)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ then $f \equiv g$.*

In the paper we consider a new range set different from that mentioned earlier and with the help of that set we will improve *Theorem B*.

The following theorem is the main results of the paper.

Theorem 1.1. *Let*

$$S = \left\{ z : \frac{(n-1)(n-2)}{4} z^n - \frac{n(n-2)}{2} z^{n-1} + \frac{n(n-1)}{4} z^{n-2} - 1 = 0 \right\},$$

where n (≥ 6) is an integer. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$, If

(i) $m \geq 2$ and $\Theta_f + \Theta_g + \min\{\Theta(1; f), \Theta(1; g) > 8 - n$

(ii) or if $m = 1$ and $\Theta_f + \Theta_g + \min\{\Theta(1; f), \Theta(1; g)\} + \frac{1}{2} \min\{\Theta(0; f) + \Theta(\infty; f), \Theta(0; g) + \Theta(\infty; g)\} > 9 - n$

(iii) or if $m = 0$ and $\Theta_f + \Theta_g + \Theta(0; f) + \Theta(\infty; f) + \Theta(0; g) + \Theta(\infty; g) + \min\{\Theta(0; f) + \Theta(1; f) + \Theta(\infty; f), \Theta(0; g) + \Theta(1; g) + \Theta(\infty; g) > 14 - n$

then $f \equiv g$*, where* $\Theta_f = 2\Theta(0; f) + \Theta(\infty; f) + \Theta(1; f)$ *and* Θ_g *can be similarly defined.*

Corollary 1.1. *Let* S *be given as in* Theorem 1.1 *where* n (≥ 7) *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S, 2) = E_g(S, 2)$ *and* $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ *and* $\Theta_f + \Theta_g + \min\{\Theta(1; f), \Theta(1; g)\} > 1$ *then* $f \equiv g$*, where* Θ_f *and* Θ_g *have the same meaning as in* Theorem 1.1.

It is assumed that the readers are familiar with the standard definitions and notations of the value distribution theory as those are available in [9]. We are still going to explain some notations as these are used in the paper.

Definition 1.3. *[11] For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting function of simple* a *points of* f*. For a positive integer* m *we denote by* $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ *the counting function of those* a *points of* f *whose multiplicities are not greater(less) than* m*, where each* a *point is counted according to its multiplicity. We denote by* $N(r, a; f \mid< m)$, $(N(r, a; f \mid> m))$ *the counting function of those* a*-points of* f *whose multiplicities are less (greater) than* m*, where each point is counted according to its multiplicity. We denote by* $\overline{N}(r, a; f \mid\leq m)$, $\overline{N}(r, a; f \mid\geq m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ *the reduced forms of* $N(r, a; f \mid\leq m)$, $N(r, a; f \mid\geq m)$, $N(r, a; f \mid< m)$ *and* $N(r, a; f \mid> m)$ *respectively.*

Definition 1.4. *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(1, 0)$*. Let* z_0 *be a* 1*-point of* f *with multiplicity* p*, a* 1*-point of* g *with multiplicity* q*. We denote by* $\overline{N}_L(r, 1; f)$ *the reduced counting function of those* 1*-points of* f *and* g *where* $p > q$*, by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q = 1$*, by* $\overline{N}_E^{(2}(r, 1; f)$ *the reduced counting function of those* 1*-points of* f *and* g *where* $p = q \geq 2$*. In the same way we can define* $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$*. In a similar manner we can define* $\overline{N}_L(r, a; f)$ *and* $\overline{N}_L(r, a; g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$*. When* f *and* g *share* $(1, m)$*,* $m \geq 1$ *then* $N_E^{1)}(r, 1; f) = N(r, 1; f \mid= 1)$*.*

Definition 1.5. *[13] We denote by* $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$*.*

Definition 1.6. *[8, 9] Let* f*,* g *share* $(a, 0)$*. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g*.*

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f and g be two non-constant meromorphic function and for an integer $n \geq 3$

$$F = \frac{(n-1)(n-2)}{4} f^n - \frac{n(n-2)}{2} f^{n-1} + \frac{n(n-1)}{4} f^{n-2}, \tag{2.1}$$

$$G = \frac{(n-1)(n-2)}{4} g^n - \frac{n(n-2)}{2} g^{n-1} + \frac{n(n-1)}{4} g^{n-2}. \tag{2.2}$$

Henceforth we shall denote by H the following functions

$$H = \left(\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1} \right) - \left(\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1} \right).$$

Lemma 2.1. *[23] If F, G be two non-constant meromorphic functions such that they share (1,0) and $H \not\equiv 0$ then*

$$N_E^{1)}(r, 1; F \mid= 1) = N_E^{1)}(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. *Let F, G be given by (2.1) and (2.2). If $H \not\equiv 0$, F, G share $(1, 2)$ and f, g share $(\infty; k)$ then*

$$\begin{aligned} N(r, H) \quad \leq \quad & \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, 1; f) + \overline{N}(r, 1; g) + \overline{N}_*(r, 1; F, G) \\ & + \overline{N}_*(r, \infty; f, g) + \overline{N}_0(r, 0; f^{'}) + \overline{N}_0(r, 0; g^{'}), \end{aligned}$$

where $\overline{N}_0(r, 0; f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$ and $F-1$, $\overline{N}_0(r, 0; g^{'})$ is similarly defined.

Proof. First we note that $F^{'} = n(n-1)(n-2)f^{n-3}(f-1)^2 f^{'}/4$ and $G^{'} = n(n-1)(n-2)g^{n-3}(g-1)^2 g^{'}/4$. We can easily verify that possible poles of H occur at (i) zeros (1-points) of f and g, (ii) poles of f and g with different multiplicities, (iii) those 1-points of F and G whose multiplicities are distinct from the multiplicities of the corresponding 1-points of G and F respectively, (iv) zeros of $f^{'}$ which are not the zeros of $f(f-1)$ and $F-1$, (v) zeros of g' which are not the zeros of $g(g-1)$ and $G-1$.

Since H has only simple poles, clearly the lemma follows from above explanations. □

Lemma 2.3. *[15] If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f).$$

Lemma 2.4. *[20] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 2.5. *Let f, g be two non-constant meromorphic functions and suppose α_1 and α_2 are the roots of the equation $\frac{(n-1)(n-2)}{4} z^2 - \frac{n(n-2)}{2} z + \frac{n(n-1)}{4} = 0$. Then*

$$(n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2) g^{n-2}(g-\alpha_1)(g-\alpha_2) \not\equiv 16$$

and n (≥ 5) is an integer.

Proof. If possible, let us suppose

$$(n-1)^2(n-2)^2 f^{n-2}(f-\alpha_1)(f-\alpha_2) g^{n-2}(g-\alpha_1)(g-\alpha_2) \equiv 16. \tag{2.3}$$

Let z_0 be a zero of f with multiplicity p. Then z_0 is a pole of g with multiplicity q such that

$$(n-2)p = (n-2)q + 2q = nq. \tag{2.4}$$

From (2.4) we see that $2q = (n-2)(p-q) \geq n-2$ and so $p = \frac{n}{n-2} q \geq \frac{n}{2}$.
Let z_0 be a zero of $f - \alpha_i$ $i = 1, 2$ with multiplicity p. Then z_0 is a pole of g with multiplicity q such that $p = (n-2)q + 2q = nq \geq n$.
Since the poles of f are the zeros of g and $g - \alpha_i$ $i = 1, 2$, we get

$$\begin{aligned}
\overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, \alpha_1; g) + \overline{N}(r, \alpha_2; g) \\
&\leq \frac{2}{n} N(r, 0; g) + \frac{1}{n}\, N(r, \alpha_1; g) + \frac{1}{n}\, N(r, \alpha_2; g) \\
&\leq \frac{4}{n} T(r, g).
\end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}
2T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \alpha_1; f) + \overline{N}(r, \alpha_2; f) + \overline{N}(r, \infty; f) + S(r, f) \\
&\leq \frac{2}{n} N(r, 0; f) + \frac{1}{n}\, N(r, \alpha_1; f) + \frac{1}{n}\, N(r, \alpha_2; f) + \frac{4}{n} T(r, g) + S(r, f) \\
&\leq \frac{4}{n} T(r, f) + \frac{4}{n} T(r, g) + S(r, f).
\end{aligned}$$

i.e.,

$$(2 - \frac{4}{n})\, T(r, f) \leq \frac{4}{n}\, T(r, g) + S(r, f). \tag{2.5}$$

Similarly

$$(2-\frac{4}{n})\,T(r,g) \le \frac{4}{n}\,T(r,f)+S(r,g) \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$(2-\frac{8}{n})\,\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

a contradiction for $n \ge 5$. This proves the lemma. □

Lemma 2.6. *[5] Let f, g be two non-constant meromorphic functions and suppose n (≥ 6) is an integer. If*

$$\begin{aligned} & \frac{(n-1)(n-2)}{2}f^n - n(n-2)f^{n-1} + \frac{n(n-1)}{2}f^{n-2} \\ \equiv\ & \frac{(n-1)(n-2)}{2}g^n - n(n-2)g^{n-1} + \frac{n(n-1)}{2}g^{n-2}, \end{aligned}$$

then $f \equiv g$.

Lemma 2.7. *Let F, G be given by (2.1), where $n \ge 7$ is an integer. Also let S be given as in* Theorem 1.1. *If $E_f(S,0)=E_g(S,0)$ then $S(r,f)=S(r,g)$.*

Proof. Since $E_f(S,0)=E_g(S,0)$, it follows that F and G share $(1,0)$. We first note that the polynomial

$$p(z)=\frac{(n-1)(n-2)}{4}z^n - \frac{n(n-2)}{2}z^{n-1} + \frac{n(n-1)}{4}z^{n-2} - 1$$

has only simple zeros. In fact

$$p^{'}(z)=\frac{n(n-1)(n-2)}{4}z^{n-3}(z-1)^2.$$

Also we note that $p(0), p(1) \ne 0$. Thus all the zeros of $p(z)$ are simple and we denote them by w_j, $j=1,2,\ldots n$. Since F, G share $(1,0)$ from the second fundamental theorem we have

$$\begin{aligned} (n-2)T(r,g) &\le \sum_{j=1}^{n} \overline{N}\,(r,w_j;g)+S(r,g) \\ &= \sum_{j=1}^{n} \overline{N}\,(r,w_j;f)+S(r,g) \\ &\le nT(r,f)+S(r,g). \end{aligned}$$

Similarly we can deduce

$$(n-2)T(r,f) \leq nT(r,g) + S(r,f).$$

The last inequalities imply $T(r,f) = O\left(T(r,g)\right)$ and $T(r,g) = O\left(T(r,f)\right)$ and so we have $S(r,f) = S(r,g)$. □

3 Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1) and (2.2). Since $E_f(S,m) = E_g(S,m)$ it follows that F, G share $(1,m)$. By a simple computation it can be easily seen that 1 is a root with multiplicity 3 of $F - \frac{1}{2}$ and hence $F - \frac{1}{2} = (f-1)^3\ \underline{Q_{n-3}(f)}$, where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$ and thus $N_2\left(r, \frac{1}{2}; F\right) \leq 2\overline{N}(r,1;f) + N\left(r,0;Q_{n-3}(f)\right) \leq 2\overline{N}(r,1;f) + (n-3)T(r,f) + S(r,f)$.

Case 1. If possible let us suppose that $H \not\equiv 0$.

Subcase 1.1. $m \geq 1$. While $m \geq 2$, using *Lemma 2.3* we note that

$$\begin{aligned}
& \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) && (3.1)\\
\leq\ & \overline{N}_0(r,0;g^{'}) + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3)\\
\leq\ & \overline{N}_0(r,0;g^{'}) + \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g \mid= 2) + 2\overline{N}(r,\omega_j;g \mid\geq 3)\}\\
\leq\ & N(r,0;g^{'} \mid g \neq 0) + S(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g).
\end{aligned}$$

According to the condition (i) of the theorem there exist a $\delta > 0$ such that

$$2\Theta(0;f) + 2\Theta(0;g) + \Theta(\infty;f) + \Theta(\infty;g) + \Theta(1;f) + \Theta(1;g) +$$
$$\min\{\Theta(1;f), \Theta(1;g)\} = 8 - n + \delta.$$

Hence using (3.1), *Lemmas 2.1* and *2.2* we get from second fundamental theorem for $0 < \varepsilon < \delta$ that

$$\begin{aligned}
& (n+1)\ T(r,f) && (3.2)\\
\leq\ & \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2)\\
& -N_0(r,0;f^{'}) + S(r,f)\\
\leq\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,1;f)\right\} + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,1;g)\\
& +\overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g)\\
\leq\ & 2\left\{\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,1;f)\right\} + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\\
& +\overline{N}(r,1;g) + S(r,f) + S(r,g)\\
\leq\ & (9 - 2\Theta(0;f) - 2\Theta(0;g) - \Theta(\infty;f) - \Theta(\infty;g) - 2\Theta(1;f) -\\
-\ & \Theta(1;g) + \varepsilon)T(r) + S(r),
\end{aligned}$$

where $T(r) = \max\{T(r,f), T(r,g)\}$. In a similar way we can obtain

$$\begin{aligned} & (n+1)\, T(r,g) && (3.3)\\ \leq\ & (9 - 2\Theta(0;f) - 2\Theta(0;g) - \Theta(\infty;f) - \Theta(\infty;g) - \Theta(1;f) \\ -\ & 2\Theta(1;g) + \varepsilon)T(r) + S(r). \end{aligned}$$

Combining (3.2) and (3.3) we see that

$$\begin{aligned} & (n+1)\, T(r) \\ \leq\ & (9 - 2\Theta(0;f) - 2\Theta(0;g) - \Theta(\infty;f) - \Theta(\infty;g) - \Theta(1;f) - \Theta(1;g) \\ & - \min\{\Theta(1;f), \Theta(1;g)\} + \varepsilon)\, T(r) + S(r). \end{aligned}$$

That is

$$\begin{aligned} & (n - 8 + 2\Theta(0;f) + 2\Theta(0;g) + \Theta(\infty;f) + \Theta(\infty;g) + \Theta(1;f) + \quad (3.4)\\ & \Theta(1;g) + \min\{\Theta(1;f), \Theta(1;g)\} - \varepsilon)\, T(r) \leq S(r). \end{aligned}$$

Since $\delta > \varepsilon > 0$, (3.4) leads to a contradiction.
While $m = 1$, using *Lemma 2.3*, (3.1) changes to

$$\begin{aligned} & \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) && (3.5)\\ \leq\ & \overline{N}_0(r,0;g') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_L(r,1;G) + \overline{N}(r,1;F \mid\geq 3) \\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\sum_{j=1}^{n}\{N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\} \\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + S(r,f) + S(r,g) \end{aligned}$$

So using (3.5), *Lemmas 2.1* and *2.2* and proceeding as in (3.2) we get from second fundamental theorem for $\varepsilon > 0$ that

$$\begin{aligned} & (n+1)\, T(r,f) && (3.6)\\ \leq\ & \left\{\frac{5}{2}\overline{N}(r,0;f) + 2\overline{N}(r,1;f) + 2\overline{N}(r,0;g)\right\} + \frac{3}{2}\overline{N}(r,\infty;f) + \overline{N}(r,1;g) + \overline{N}(r,\infty;g) \\ & + S(r,f) + S(r,g) \\ \leq\ & \left(10 - \frac{5}{2}\Theta(0;f) - 2\Theta(0;g) - \frac{3}{2}\Theta(\infty;f) - \Theta(\infty;g) - 2\Theta(1;f) - \Theta(1;g) + \varepsilon\right) T(r) \\ & + S(r). \end{aligned}$$

Similarly we can obtain

$$\begin{aligned} & (n+1)\, T(r,g) && (3.7)\\ \leq\ & \left(10 - 2\Theta(0;f) - \frac{5}{2}\Theta(0;g) - \Theta(\infty;f) - \frac{3}{2}\Theta(\infty;g) - \Theta(1;f) - 2\Theta(1;g) + \varepsilon\right) T(r) \\ & + S(r). \end{aligned}$$

Combining (3.6) and (3.7) we see that

$$(n-9+2\Theta(0;f)+2\Theta(0;g)+\Theta(\infty;f)+\Theta(\infty;g)+\Theta(1;f)+\Theta(1;g) \tag{3.8}$$
$$+\min\{\Theta(1;f),\Theta(1;g)\}+\frac{1}{2}\min\{\Theta(0;f)+\Theta(\infty;f),\Theta(0;g)+\Theta(\infty;g)\}-\varepsilon\,)\;T(r)$$
$$\leq S(r).$$

Since $\varepsilon>0$ is arbitrary, using condition (ii) of the theorem and resorting to the same argument as used in the case when $m=2$ we see that (3.8) leads to a contradiction.

Subcase 1.2. $m=0$. Using *Lemma 2.3* we note that

$$\begin{aligned}
& \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F) && (3.9)\\
\leq\; & \overline{N}_0(r,0;g^{'})+\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\leq\; & \overline{N}_0(r,0;g^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\leq\; & N(r,0;g^{'}\mid g\neq 0)+\overline{N}(r,1;G\mid\geq 2)+2\overline{N}(r,1;F\mid\geq 2)\\
\leq\; & 2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g)
\end{aligned}$$

Hence using (3.9), *Lemmas 2.1* and *2.2* we get from second fundamental theorem for $\varepsilon>0$ that

$$\begin{aligned}
& (n+1)\;T(r,f) && (3.10)\\
\leq\; & \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
& +\overline{N}_E^{(2}(r,1;F)-N_0(r,0;f^{'})+S(r,f)\\
\leq\; & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,1;f)\right\}+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,1;g)\\
& +\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g)\\
\leq\; & 4\overline{N}(r,0;f)+3\overline{N}(r,\infty;f)+3\overline{N}(r,0;g)+2\overline{N}(r,\infty;g)+2\overline{N}(r,1;f)+\overline{N}(r,1;g)\\
& +S(r,f)+S(r,g)\\
\leq\; & (15-4\Theta(0;f)-3\Theta(\infty;f)-3\Theta(0;g)-2\Theta(\infty;g)-2\Theta(1;f)-\Theta(1;g)\\
+\; & \varepsilon)T(r)+S(r).
\end{aligned}$$

In a similar manner we can obtain

$$\begin{aligned}
& (n+1)\;T(r,g) && (3.11)\\
\leq\; & (15-3\Theta(0;f)-2\Theta(\infty;f)-4\Theta(0;g)-3\Theta(\infty;g)-\Theta(1;f)-2\Theta(1;g)+\\
+\; & \varepsilon)T(r)+S(r).
\end{aligned}$$

Combining (3.10) and (3.11) we see that

$$(n-14+3\Theta(0;f)+3\Theta(0;g)+2\Theta(\infty;f)+2\Theta(\infty;g)+\Theta(1;f)+\Theta(1;g) \tag{3.12}$$
$$+\min\{\Theta(0;f)+\Theta(1;f)+\Theta(\infty;f),\Theta(0;g)+\Theta(1;g)+\Theta(\infty;g)\}-\varepsilon)\;T(r)\leq S(r).$$

Since $\varepsilon > 0$ is arbitrary, using condition (iii) of the theorem and applying the same argument as used in the case when $m = 2$ it is clear that (3.12) leads to a contradiction.
Case 2. $H \equiv 0$. Then

$$F \equiv \frac{aG + b}{cG + d}, \tag{3.13}$$

where a, b, c, d are constants such that $ad - bc \neq 0$. Also

$$T(r, F) = T(r, G) + O(1). \tag{3.14}$$

We now consider the following cases.
Case I. Let $ac \neq 0$. From (3.13) we get

$$\overline{N}(r, \infty; G) = \overline{N}\left(r, \frac{a}{c}; F\right). \tag{3.15}$$

So in view of (3.14), by the second fundamental theorem we get

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}\left(r, \frac{a}{c}; F\right) + S(r, F) \\ &= \overline{N}(r, 0; f) + 2T(r, f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) \\ &\leq 5T(r, f) + S(r, f), \end{aligned}$$

which in view of by *Lemma 2.4* gives a contradiction for $n \geq 6$.
Case II. Let $a \neq 0$ and $c = 0$. Then $F = \alpha G + \beta$, where $\alpha = \frac{a}{d}$ and $\beta = \frac{b}{d}$.
If F has no 1-point, by the second fundamental theorem we get

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq 3T(r, f) + \overline{N}(r, \infty; f) + S(r, f), \end{aligned}$$

which implies a contradiction in view of *Lemma 2.4*.
If F and G have some 1-points then $\alpha + \beta = 1$ and so

$$F \equiv \alpha G + 1 - \alpha. \tag{3.16}$$

Suppose $\alpha \neq 1$. If $1 - \alpha \neq \frac{1}{2}$ then in view of (3.14) and the second fundamental theorem we get

$$\begin{aligned} 2T(r, F) &\leq \overline{N}(r, 0; F) + \overline{N}(r, 1 - \alpha; F) + \overline{N}\left(r, \frac{1}{2}; F\right) + \overline{N}(r, \infty; F) + S(r, F) \\ &\leq 3T(r, f) + \overline{N}(r, 0; G) + (n - 2)T(r, f) + \overline{N}(r, \infty; f) + S(r, f) \\ &\leq (n + 5)T(r, f) + S(r, f), \end{aligned}$$

which implies a contradiction in view of *Lemma 2.4* and $n \geq 6$. If $\alpha = \frac{1}{2}$, then we have from (3.16)

$$F \equiv \frac{1}{2}(G+1).$$

So by the second fundamental theorem we can obtain using (3.14) that

$$\begin{aligned} 2T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{2};G\right) + \overline{N}(r,-1;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq 3T(r,g) + (n-2)T(r,g) + \overline{N}(r,0;F) + \overline{N}(r,\infty;g) + S(r,g) \\ &\leq (n+5)T(r,g) + S(r,g), \end{aligned}$$

which implies a contradiction in view of *Lemma 2.4* and $n \geq 6$.

So $\alpha = 1$ and hence $F \equiv G$. So by *Lemma 2.6* we get $f \equiv g$.

Case III. Let $a = 0$ and $c \neq 0$. Then $F \equiv \frac{1}{\gamma G + \delta}$, where $\gamma = \frac{c}{b}$ and $\delta = \frac{d}{b}$. If F has no 1-point then as in *Case II* we can deduce a contradiction.
If F and G have some 1-points then $\gamma + \delta = 1$ and so

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}. \tag{3.17}$$

Suppose $\gamma \neq 1$ If $\gamma \neq -1$, then by the second fundamental theorem we get

$$\begin{aligned} 2\,T(r,F) &\leq \overline{N}(r,0;F) + \overline{N}(r,\frac{1}{1-\gamma};F) + \overline{N}\left(r,\frac{1}{2};F\right) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq 3T(r,f) + \overline{N}(r,0;G) + (n-2)T(r,f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq (n+5)T(r,f) + S(r,f), \end{aligned}$$

which gives a contradiction in view of *Lemma 2.4* and $n \geq 6$. If $\gamma = -1$ from (3.17) we have

$$F \equiv \frac{1}{-G+2}.$$

Now the second fundamental theorem with the help of (3.14) yields

$$\begin{aligned} 2T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{2};G\right) + \overline{N}(r,2;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq 3T(r,g) + (n-2)T(r,g) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,g) \\ &\leq (n+3)T(r,g) + S(r,g), \end{aligned}$$

which implies a contradiction in view of *Lemma 2.4* and $n \geq 6$.

So we must have $\gamma = 1$ then $FG \equiv 1$, which is impossible by *Lemma 2.5*. This completes the proof of the theorem. □

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