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NEW SPHERICAL INDICATRICES AND THEIR CHARACTERIZATIONS

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Abstract

In this work, we introduce new spherical images by translating Bishop frame vectors of a regular curve to the center of the unit sphere of the three dimensional Euclidean space. Such curves are called as Bishop Spherical Indicatrices. Then, the Frenet-Serret apparatus of these new curves is obtained in terms of base curve's Bishop invariants. Additionally, illustrations of two examples are presented.

1 Introduction

In the existing literature, it can be seen that, most of classical differential geometry topics have been extended to Lorentzian manifolds. In this process, generally, researchers used standard moving Frenet-Serret frame. Using transformation matrix among derivative vectors and frame vectors, some of kinematical models were adapted to this special moving frame. Researchers aimed to have an alternative frame for curves and other applications. Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields. Recently, many research papers have been treated in the Euclidean space, see [3], [4]; in the Minkowski space, see [1], [2], [8], [13]; and in the dual space, see [9].

Key Words: Bishop Frame, Spherical Images, Regular Curves, General Helix, Slant Helix, Euclidean Space

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Spherical images of a regular curve in the Euclidean space are obtained by means of Frenet-Serret frame vector fields, so this classical topic is a well-known concept in differential geometry of the curves, see [6]. In the light of the existing literature, this paper aims to determine new spherical images of regular curves using Bishop frame vector fields. We shall call such curves, respecvitely, *Tangent,* M_1 and M_2 *Bishop spherical images* of regular curves. Considering classical methods, we investigated relations among Frenet-Serret invariants of spherical images in terms of Bishop invariants. Additionally, two examples of Bishop spherical indicatrices are presented.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^3 are briefly presented; a more complete elementary treatment can be found in [6].

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $a \in \mathrm{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. φ is called a *unit speed curve* if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in \mathrm{E}^3$ it is said to be *orthogonal* if and only if $\langle v, w\rangle = 0$. Let $\vartheta = \vartheta(s)$ be a regular curve in E^3. If the tangent vector of this curve forms a constant angle with a fixed constant vector U, then this curve is called a *general helix* or an *inclined curve.* The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \left\{p = (p_1, p_2, p_3) \in E^3 : \langle p, p\rangle = r^2\right\}.$$

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve φ in the space E^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E^3, the following Frenet-Serret formulae are given in [6] written under matrix form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1,$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0.$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -\langle N, B' \rangle$.

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in E^3 and e_1, e_2, e_3 be positive oriented natural basis of E^3. Cross product of u and v is defined by

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Mixed product of u, v and w is defined by the determinant

$$[u, v, w] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Torsion of the curve φ is given by the aid of the mixed product

$$\tau = \frac{[\varphi', \varphi'', \varphi''']}{\kappa^2}.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [4]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [5]). The Bishop frame is expressed as [4, 5]

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}. \tag{1}$$

Here, we shall call the set $\{T, M_1, M_2\}$ as *Bishop trihedra* and k_1 and k_2 as *Bishop curvatures.* The relation matrix may be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$\begin{cases} k_1 = \kappa \cos\theta(s) \\ k_2 = \kappa \sin\theta(s) \end{cases}.$$

Izumiya and Takeuchi [7] have introduced the concept of slant helix in the Euclidean 3-space E^3 saying that the normal lines makes a constant angle with a fixed direction [7]. They characterized a slant helix by the condition that the function

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is constant. In further researches, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves are presented (see [10] and [11]). In the same space, in [4], the authors defined and gave some characterizations of slant helices according to Bishop frame with the following definition and theorem:

Definition 2.1. A regular curve $\gamma : I \to E^3$ is called a slant helix according to Bishop frame provided the unit vector $M_1(s)$ of γ has constant angle θ with some fixed unit vector u; that is,

$$\langle M_1, u\rangle = \cos\theta$$

for all $s \in I$.

Theorem 2.2. *Let $\gamma : I \to E^3$ be a unit speed curve with nonzero natural curvatures. Then γ is a* slant helix *if and only if*

$$\frac{k_1}{k_2} = constant.$$

(See [4]).

To separate a slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as "B-slant helix".

It is well-known that for a unit speed curve with non vanishing curvatures the following propositions hold [6], [7]:

Proposition 2.3. *Let $\varphi = \varphi(s)$ be a regular curve with curvatures κ and τ. The curve φ lies on the surface of a sphere if and only if*

$$\frac{\tau}{\kappa} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' = 0.$$

Proposition 2.4. *Let $\varphi = \varphi(s)$ be a regular curve with curvatures κ and τ. φ is a general helix if and only if*

$$\frac{\kappa}{\tau} = constant.$$

Proposition 2.5. *Let $\varphi = \varphi(s)$ be a regular curve with curvatures κ and τ. φ is a slant helix if and only if*

$$\sigma(s) = \left[\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} (\frac{\tau}{\kappa})' \right] = constant.$$

3 Main Results

3.1 Tangent Bishop Spherical Images of a Regular Curve

Definition 3.1. Let $\gamma = \gamma(s)$ be a regular curve in E^3. If we translate of the first (tangent) vector field of Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\xi = \xi(s_\xi)$. This curve is called *tangent Bishop spherical image or indicatrix of the curve $\gamma = \gamma(s)$.*

Let $\xi = \xi(s_\xi)$ be tangent Bishop spherical image of a regular curve $\gamma = \gamma(s)$. One can differentiate of ξ with respect to s :

$$\xi' = \frac{d\xi}{ds_\xi} \frac{ds_\xi}{ds} = k_1 M_1 + k_2 M_2.$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_ξ by a dot. In terms of Bishop frame vector fields (1), we have the tangent vector of the spherical image as follows:

$$T_\xi = \frac{k_1 M_1 + k_2 M_2}{\sqrt{k_1^2 + k_2^2}},$$

where

$$\frac{ds_\xi}{ds} = \sqrt{k_1^2 + k_2^2} = \kappa(s).$$

In order to determine the first curvature of ξ, we write

$$\dot{T}_\xi = -T + \frac{k_2^3}{(k_1^2 + k_2^2)^2} \left(\frac{k_1}{k_2} \right)' M_1 + \frac{k_1^3}{(k_1^2 + k_2^2)^2} \left(\frac{k_2}{k_1} \right)' M_2.$$

Since, we immediately arrive at

$$\kappa_\xi = \left\| \dot{T}_\xi \right\| = \sqrt{1 + \left[\frac{k_2^3}{(k_1^2 + k_2^2)^2} \left(\frac{k_1}{k_2} \right)' \right]^2 + \left[\frac{k_1^3}{(k_1^2 + k_2^2)^2} \left(\frac{k_2}{k_1} \right)' \right]^2}. \tag{2}$$

Therefore, we have the principal normal

$$N_\xi = \frac{1}{\kappa_\xi}\left\{-T + \frac{k_2^3}{(k_1^2+k_2^2)^2}\left(\frac{k_1}{k_2}\right)' M_1 + \frac{k_1^3}{(k_1^2+k_2^2)^2}\left(\frac{k_2}{k_1}\right)' M_2\right\}.$$

By the cross product of $T_\xi \times N_\xi$, we obtain the binormal vector field

$$B_\xi = \frac{1}{\kappa_\xi}\left\{\begin{array}{c}\left[\frac{k_1^4}{(k_1^2+k_2^2)^{\frac{5}{2}}}\left(\frac{k_2}{k_1}\right)' - \frac{k_2^4}{(k_1^2+k_2^2)^{\frac{5}{2}}}\left(\frac{k_1}{k_2}\right)'\right] T \\ -\left[\frac{k_2}{\sqrt{k_1^2+k_2^2}}\right] M_1 + \left[\frac{k_1}{\sqrt{k_1^2+k_2^2}}\right] M_2\end{array}\right\}.$$

By means of obtained equations, we express the torsion of the tangent Bishop spherical image

$$\tau_\xi = \frac{\left(\begin{array}{c} -k_1\left\{3k_2'(k_1k_1'+k_2k_2') - (k_1^2+k_2^2)\left[k_2'' - k_2(k_1^2+k_2^2)\right]\right\} \\ +k_2\left\{3k_1'(k_1k_1'+k_2k_2') - (k_1^2+k_2^2)\left[k_1'' - k_1(k_1^2+k_2^2)\right]\right\}\end{array}\right)}{\left[k_1^2\left(\frac{k_2}{k_1}\right)'\right]^2 + (k_1^2+k_2^2)^3}. \qquad (3)$$

Consequently, we determined Frenet-Serret invariants of the tangent Bishop spherical indicatrix according to Bishop invariants.

Corollary 3.2. *Let $\xi = \xi(s_\xi)$ be the tangent Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If $\gamma = \gamma(s)$ is a B-slant helix, then the tangent spherical indicatrix ξ is a circle in the osculating plane.*

Proof. Let $\xi = \xi(s_\xi)$ be the tangent Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If $\gamma = \gamma(s)$ is a B-slant helix, then Theorem 2.2 holds. So, $\frac{k_1}{k_2} = constant$. Substituting this to equations (2) and (3), we have $\kappa_\xi = constant$ and $\tau_\xi = 0$, respectively. Therefore, ξ is a circle in the osculating plane. □

Remark 3.3. Considering $\theta_\xi = \int_0^{s_\xi} \tau_\xi ds_\xi$ and using the transformation matrix, one can obtain the Bishop trihedra $\{T_\xi, M_{1\xi}, M_{2\xi}\}$ of the curve $\xi = \xi(s_\xi)$.

Here, one question may come to mind about the obtained tangent spherical image, since, Frenet-Serret and Bishop frames have a common tangent vector field. Images of such tangent images are the same as we shall demonstrate in section 4. But, here we are concerned with the tangent Bishop spherical image's Frenet-Serret apparatus according to Bishop invariants.

3.2 M_1 Bishop Spherical Images of a Regular Curve

Definition 3.4. Let $\gamma = \gamma(s)$ be a regular curve in E^3. If we translate of the second vector field of Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\delta = \delta(s_\delta)$. This curve is called M_1 *Bishop spherical image or indicatrix of the curve* $\gamma = \gamma(s)$.

Let $\delta = \delta(s_\delta)$ be M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. We follow the same procedure to investigate the relations among Bishop and Frenet-Serret invariants. Thus, we differentiate

$$\delta' = \frac{d\delta}{ds_\delta}\frac{ds_\delta}{ds} = -k_1 T.$$

First, we have

$$T_\delta = T \text{ and } \frac{ds_\delta}{ds} = -k_1. \tag{4}$$

So, one can calculate

$$T'_\delta = \dot{T}_\delta \frac{ds_\delta}{ds} = k_1 M_1 + k_2 M_2$$

or

$$\dot{T}_\delta = -M_1 - \frac{k_2}{k_1} M_2.$$

Since, we express

$$\kappa_\delta = \left\| \dot{T}_\delta \right\| = \sqrt{1 + \left(\frac{k_2}{k_1}\right)^2} \tag{5}$$

and

$$N_\delta = -\frac{M_1}{\kappa_\delta} - \frac{k_2}{k_1 \kappa_\delta} M_2.$$

Cross product of $T_\delta \times N_\delta$ gives us the binormal vector field of M_1 spherical image of $\gamma = \gamma(s)$

$$B_\delta = \frac{k_2}{k_1 \kappa_\delta} M_1 - \frac{1}{\kappa_\delta} M_2.$$

Using the formula of the torsion, we write

$$\tau_\delta = -\frac{k_1 \left(\frac{k_2}{k_1}\right)'}{k_1^2 + k_2^2}. \tag{6}$$

Considering equations (5) and (6) by the Theorem 2.2, we get:

Corollary 3.5. *Let $\delta = \delta(s_\delta)$ be the M_1 Bishop spherical image of the curve $\gamma = \gamma(s)$. If $\gamma = \gamma(s)$ is a B-slant helix, then, the M_1 Bishop spherical indicatrix $\delta(s_\delta)$ is a circle in the osculating plane.*

Theorem 3.6. *Let $\delta = \delta(s_\delta)$ be the M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. There exists a relation among Frenet-Serret invariants $\delta(s_\delta)$ and Bishop curvatures of $\gamma = \gamma(s)$ as follows:*

$$\frac{k_2}{k_1} = \int_0^{s_\delta} \kappa_\delta^2 \tau_\delta ds_\delta. \tag{7}$$

Proof. Let $\delta = \delta(s_\delta)$ be M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. Then, the equations (4) and (6) hold. Using (4) in (6), we have

$$\tau_\delta = -\frac{k_1 \frac{d}{ds_\delta}\left(\frac{k_2}{k_1}\right)\frac{ds_\delta}{ds}}{k_1^2 + k_2^2}. \tag{8}$$

Substituting (5) to (8) and integrating both sides, we have (7) as desired. □

In the light of the Propositions 2.4 and 2.5, we state the following theorems without proofs:

Theorem 3.7. *Let $\delta = \delta(s_\delta)$ be M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If δ is a **general helix**, then, Bishop curvatures of γ satisfy*

$$\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} = constant.$$

Theorem 3.8. *Let $\delta = \delta(s_\delta)$ be the M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If δ is a **slant helix**, then, the Bishop curvatures of γ satisfy*

$$\left[\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}}\right]' \frac{(k_1^2 + k_2^2)^4}{k_1^3\left[\left(\frac{k_2}{k_1}\right)'^2 + (k_1^2 + k_2^2)^3\right]^{\frac{3}{2}}} = constant.$$

We know that δ is a spherical curve, so, by the Proposition 2.3 one can prove:

Theorem 3.9. *Let δ be the M_1 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. The Bishop curvatures of the regular curve $\gamma = \gamma(s)$ satisfy the following differential equation*

$$\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} - \left[\frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}}\right]' = constant.$$

Remark 3.10. Considering $\theta_\delta = \int\limits_0^{s_\delta} \tau_\delta ds_\delta$ and using the transformation matrix, one can obtain the Bishop trihedra $\{T_\delta, M_{1\delta}, M_{2\delta}\}$ of the curve $\delta = \delta(s_\delta)$.

3.3 $\mathbf{M_2}$ Bishop Spherical Images of a Regular Curve

Definition 3.11. Let $\gamma = \gamma(s)$ be a regular curve in E^3. If we translate of the third vector field of Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image of $\psi = \psi(s_\psi)$. This curve is called the *M_2 Bishop spherical image or the indicatrix of the curve $\gamma = \gamma(s)$.*

Let $\psi = \psi(s_\psi)$ be M_2 spherical image of the regular curve $\gamma = \gamma(s)$. We can write

$$\psi' = \frac{d\psi}{ds_\psi}\frac{ds_\psi}{ds} = -k_2 T.$$

Similar to the M_1 Bishop spherical image, one can have

$$T_\psi = T \quad \text{and} \quad \frac{ds_\psi}{ds} = -k_2. \tag{9}$$

So, by differentiating of the formula (9), we get

$$T'_\psi = \dot{T}_\psi \frac{ds_\psi}{ds} = k_1 M_1 + k_2 M_2$$

or, in another words,

$$\dot{T}_\psi = -\frac{k_1}{k_2} M_1 - M_2,$$

since, we express

$$\kappa_\psi = \left\| \dot{T}_\psi \right\| = \sqrt{1 + \left(\frac{k_1}{k_2}\right)^2} \tag{10}$$

and

$$N_\psi = -\frac{k_1}{k_2 \kappa_\psi} M_1 - \frac{M_2}{\kappa_\psi}.$$

The cross product $T_\psi \times N_\psi$ gives us

$$B_\psi = \frac{1}{\kappa_\psi} M_1 - \frac{k_1}{k_2 \kappa_\psi} M_2.$$

By the formula of the torsion, we have

$$\tau_\psi = \frac{k_2 \left(\frac{k_1}{k_2}\right)'}{k_1^2 + k_2^2}. \tag{11}$$

In terms of equations (10) and (11) and by the Theorem 2.2, we may obtain:

Corollary 3.12. *Let $\psi = \psi(s_\psi)$ be the M_2 spherical image of a regular curve $\gamma = \gamma(s)$. If $\gamma = \gamma(s)$ is a B-slant helix, then the M_2 Bishop spherical image $\psi(s_\psi)$ is a circle in the osculating plane.*

Theorem 3.13. *Let $\psi = \psi(s_\psi)$ be the M_2 spherical image of a regular curve $\gamma = \gamma(s)$. Then, there exists a relation among Frenet-Serret invariants of $\psi(s_\psi)$ and the Bishop curvatures of $\gamma = \gamma(s)$ as follows:*

$$\frac{k_1}{k_2} + \int_0^{s_\psi} \kappa_\psi^2 \tau_\psi ds_\psi = 0.$$

Proof. Similar to proof of the theorem 3.6, above equation can be obtained by the equations (9), (10) and (11). □

In the light of the propositions 2.4 and 2.5, we also give the following theorems for the curve $\psi = \psi(s_\psi)$:

Theorem 3.14. *Let $\psi = \psi(s_\psi)$ be the M_2 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If ψ is a* ***general helix****, then, Bishop curvatures of γ satisfy*

$$\frac{k_2^2 \left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} = constant.$$

Theorem 3.15. *Let $\psi = \psi(s_\psi)$ be the M_2 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. If ψ is a* ***slant helix****, then, the Bishop curvatures of γ satisfy*

$$\left[\frac{k_2^2 \left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}}\right]' \frac{(k_1^2 + k_2^2)^4}{k_2^3 \left[\left(\frac{k_1}{k_2}\right)'^2 + (k_1^2 + k_2^2)^3\right]^{\frac{3}{2}}} = constant.$$

We also know that ψ is a spherical curve. By the Proposition 2.3, it is safe to report the following theorem:

Theorem 3.16. *Let $\psi = \psi(s_\psi)$ be the M_2 Bishop spherical image of a regular curve $\gamma = \gamma(s)$. The Bishop curvatures of the regular curve $\gamma = \gamma(s)$ satisfy the following differential equation*

$$\frac{k_2^2 \left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} + \left[\frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}}\right]' = constant.$$

Remark 3.17. Considering $\theta_\psi = \int_0^{s_\psi} \tau_\psi ds_\psi$ and using the transformation matrix, one can obtain the Bishop trihedra $\{T_\psi, M_{1\psi}, M_{2\psi}\}$ of the curve $\psi = \psi(s_\psi)$.

4 Examples

In this section, we give two examples of Bishop spherical images.

Example 4.1

First, let us consider a unit speed circular helix by

$$\beta = \beta(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}), \tag{12}$$

where $c = \sqrt{a^2+b^2} \in R$. One can calculate its Frenet-Serret apparatus as the following:

$$\begin{cases} \kappa = \frac{a}{c^2} \\ \tau = \frac{b}{c^2} \\ T = \frac{1}{c}(-a\sin\frac{s}{c}, a\cos\frac{s}{c}, b) \\ N = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0) \\ B = \frac{1}{c}(b\sin\frac{s}{c}, -b\cos\frac{s}{c}, a) \end{cases}$$

In order to determine the Bishop frame of the curve $\beta = \beta(s)$, let us form

$$\theta(s) = \int_0^s \frac{b}{c^2} ds = \frac{bs}{c^2}.$$

Since, we can write the transformation matrix

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{bs}{c^2} & \sin\frac{bs}{c^2} \\ 0 & -\sin\frac{bs}{c^2} & \cos\frac{bs}{c^2} \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

by the method of Cramer, one can obtain the Bishop trihedra as follows:

The tangent:

$$T = \frac{1}{c}(-a\sin\frac{s}{c}, a\cos\frac{s}{c}, b) \tag{13}$$

The M_1:

$$M_1 = (-\cos\frac{s}{c}\cos\frac{bs}{c^2} - \frac{b}{c}\sin\frac{s}{c}\sin\frac{bs}{c^2}, \frac{b}{c}\cos\frac{s}{c}\sin\frac{bs}{c^2} - \sin\frac{s}{c}\cos\frac{bs}{c^2}, -\frac{a}{c}\sin\frac{bs}{c^2}) \tag{14}$$

The M_2:

$$M_2 = (\frac{b}{c}\sin\frac{s}{c}\cos\frac{bs}{c^2} - \cos\frac{s}{c}\sin\frac{bs}{c^2}, -\frac{b}{c}\cos\frac{s}{c}\cos\frac{bs}{c^2} - \sin\frac{s}{c}\sin\frac{bs}{c^2}, \frac{a}{c}\cos\frac{bs}{c^2}) \tag{15}$$

We may choose $a = 12$, $b = 5$ and $c = 13$ in the equations (12–15). Then, one can see the curve at the Figure 1. So, we can illustrate spherical images see Figure 2.

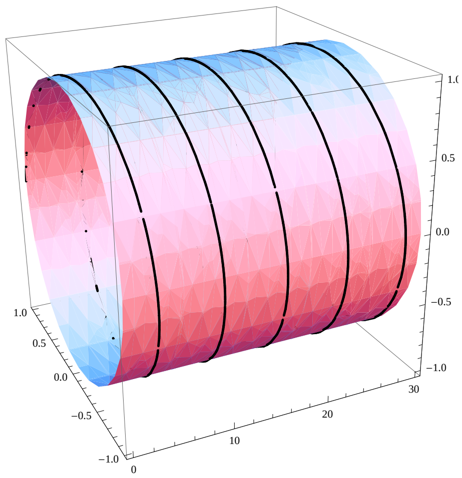

Figure 1: Circular Helix $\beta = \beta(s)$ for $a = 12, b = 5$ and $c = 13$.

Example 4.2

Next, let us consider the following unit speed curve $\gamma(s) = (\gamma_1, \gamma_2, \gamma_3)$:

$$\begin{cases} \gamma_1 = \frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s \\ \gamma_2 = -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s \\ \gamma_3 = \frac{6}{65}\sin 10s \end{cases}.$$

It is rendered in Figure 3. And, this curve's curvature functions are expressed as in [12]:

$$\begin{cases} \kappa(s) = -24\sin 10s \\ \tau(s) = 24\cos 10s \end{cases}$$

It is an easy problem to calculate Frenet-Serret apparatus of the unit speed curve $\gamma = \gamma(s)$. We also need

$$\theta(s) = \int_0^s 24\cos(10s)ds = \frac{24}{10}\sin(10s).$$

The transformation matrix for the curve $\gamma = \gamma(s)$ has the form

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{24}{10}\sin 10s) & \sin(\frac{24}{10}\sin 10s) \\ 0 & -\sin(\frac{24}{10}\sin 10s) & \cos(\frac{24}{10}\sin 10s) \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}$$

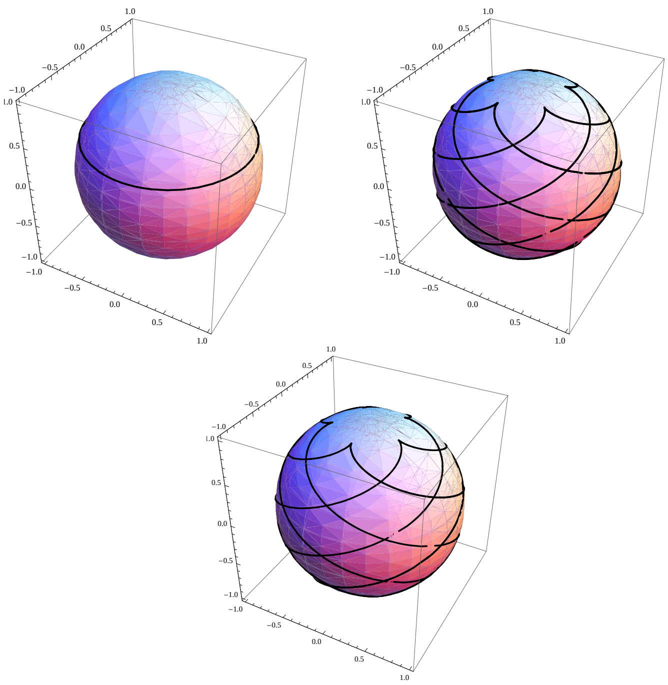

Figure 2: Tangent, M_1 and M_2 Bishop Spherical Images of $\beta = \beta(s)$ for $a = 12, b = 5$ and $c = 13$.

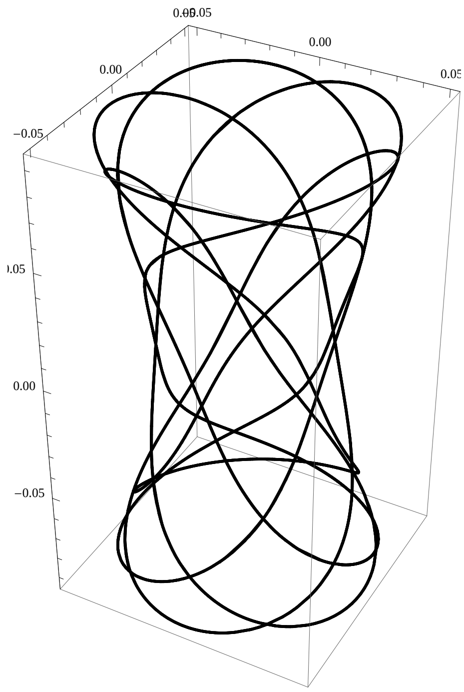

Figure 3: The curve $\gamma = \gamma(s)$

By the solution of the system above, we have Bishop spherical images of the unit speed curve $\gamma = \gamma(s)$, see figures 4, 5 and 6.

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References

[1] Bükcü, B. and Karacan, M.K. *The Bishop Darboux Rotation Axis of the Spacelike Curve in Minkowski 3-Space*, Ege University, Journal of the Faculty of Science, **3**, (1), 1-5, 2007.

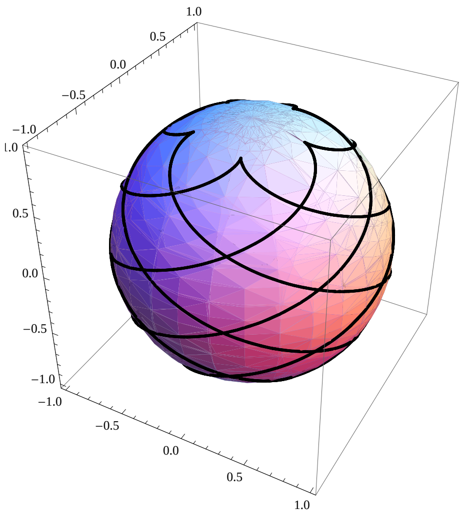

Figure 4: Tangent Spherical Image of $\gamma = \gamma(s)$.

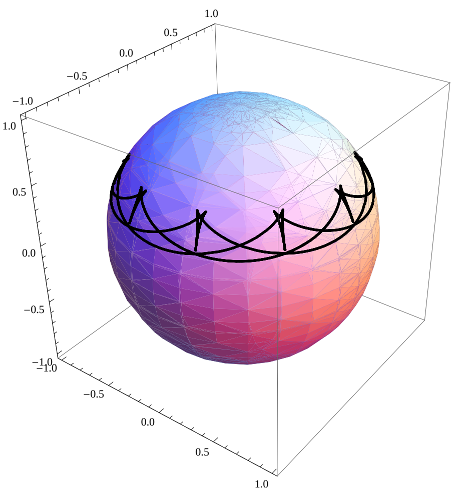

Figure 5: M_1 Spherical Image of $\gamma = \gamma(s)$.

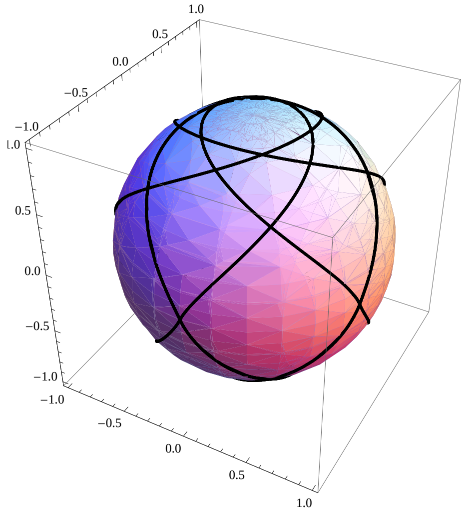

Figure 6: M_2 Spherical Image of $\gamma = \gamma(s)$.

[2] Bükcü, B. and Karacan, M.K. *On the slant helices according to Bishop frame of the timelike curve in Lorentzian space*, Tamkang J. Math., **39**, (3), 255-262, 2008.

[3] Bükcü, B. and Karacan, M.K. *Special Bishop motion and Bishop Darboux rotation axis of the space curve*, J. Dyn. Syst. Geom. Theor., **6**, (1), 27–34, 2008.

[4] Bükcü, B. and Karacan, M.K. *The Slant Helices According to Bishop Frame*, Int. J. Comput. Math. Sci., **3**, (2), 67–70, 2009.

[5] Bishop, L. R. *There is More Than one way to Frame a Curve*, Amer. Math. Monthly, **82** (3), 246–251, 1975.

[6] Do Carmo, M. P., *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.

[7] Izumiya, S. and Takeuchi, N. *New special curves and developable surfaces*, Turk. J. Math., **28** (2), 531–537, 2004.

[8] Karacan, M.K. and Bükcü, B. *Bishop frame of the timelike curve in Minkowski 3-space*, Fen Derg., **3**, (1), 80–90, 2008.

[9] Karacan, M.K., Bükcü, B. and Yuksel, N. *On the dual Bishop Darboux rotation axis of the dual space curve*, Appl. Sci., **10**, 115–120, 2008.

[10] Kula, L. and Yayli, Y. *On slant helix and its spherical indicatrix*, Appl. Math. Comput., **169** (1), 600-607, 2005.

[11] Kula, L., Ekmekçi, N., Yaylı, Y. and İlarslan, K. *Characterizations of slant helices in Euclidean 3-space*, Turk. J. Math., **34**, 261-274, 2010.

[12] Scofield, P.D., *Curves of Constant Precession*, Amer. Math. Monthly, **102**, 531–537, 1995.

[13] Yılmaz, S. *Position Vectors of Some Special Space-like Curves according to Bishop frame in Minkowski Space* E_1^3, Sci. Magna, **5** (1), 48–50, 2009.

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