An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 197–204

ON GENERALIZED M^*- GROUPS

Sebahattin Ikikardes and Recep Sahin

Abstract

Let X be a compact bordered Klein surface of algebraic genus $p \geq 2$ and let $G = \Gamma^*/\Lambda$ be a group of automorphisms of X where Γ^* is a *non-euclidian chrystalographic* group and Λ is a bordered surface group. If the order of G is $\frac{4q}{(q-2)}(p-1)$, for $q \geq 3$ a prime number, then the signature of Γ^* is $(0; +; [-]; \{(2, 2, 2, q)\})$. These groups of automorphisms are called generalized M^*-groups. In this paper, we give some results and examples about generalized M^*-groups. Then, we construct new generalized M^*-groups from a generalized M^*-group G (or not necessarily generalized M^*-group).

1 Introduction

A compact bordered Klein surface X of algebraic genus $p \geq 2$ has at most $12(p-1)$ automorphisms [9]. The groups which are isomorphic to the automorphism group of such a compact bordered Klein surface with this maximal number of automorphisms are called *M^*-groups*. M^*-groups were first studied in [10], and additional results about these groups are in [4, 5, 6, 12]. Also, the article [3] contains a nice survey of known results about M^*-groups.

The first important result about M^*-groups was that they must have a certain partial presentation [10]. This was established by considering an M^*-group as a quotient of an quadrilateral group $\Gamma^*[2, 2, 2, 3]$. In [13, p.223, Proposition 2], this was extended to the quadrilateral groups $\Gamma^*[2, 2, 2, q]$ where $q \geq 3$ is an integer. By using the quadrilateral groups $\Gamma^*[2, 2, 2, q]$ for $q \geq 3$ prime, Sahin et al. in [15] defined *generalized M^*-group* similar to M^*-group

Key Words: M^*-groups, generalized M^*-groups, Klein surfaces.
Mathematics Subject Classification: 30F50;20D45
Received: October, 2009
Accepted: January, 2010

case. In [15], the authors found a relationship between extended Hecke groups and generalized M^*-groups. The relationship says that a finite group of order at least $4q$ is a generalized M^*-group if and only if it is the homomorphic image of the extended Hecke groups $\overline{H}(\lambda_q)$. In fact, by using known results about normal subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$ given in [14], they obtained many results related to generalized M^*-groups. For example, if G is a generalized M^*-group, then $|G : G'|$ divides 4 and $|G' : G''|$ divides q^2. Finally, they proved that if $q \geq 3$ prime number and G is a generalized M^*-group associated to q, then G is supersoluble if and only if $|G| = 4 \cdot q^r$ for some positive integer r.

In this paper, our main goal is to generalize some results related to the M^*-groups to the generalized M^*-groups. First,we give some results and examples about generalized M^*-groups. Then, we construct new generalized M^*-groups from a generalized M^*-group G (or not necessarily generalized M^*-group). To do these, we shall use the same methods in [3], [5] and [11] for M^*-groups.

2 Preliminaries

We shall assume that all Klein surfaces we are working with are compact and of algebraic genus $p \geq 2$. Let $\mathcal{U}$ be the open upper half plane. An Non-Euclidean crystallographic group, *NEC group* in short, is a discrete subgroup Γ of the group PGL(2,$\mathbb{R}$) of all conformal and anti-conformal automorphisms of $\mathcal{U}$ such that the quotient space $\mathcal{U}/\Gamma$ is compact. If Γ lies wholly within the conformal group $PSL(2,\mathbb{R})$, it is more usually known as a *Fuchsian group*. Also, if Γ contains both conformal and anti-conformal automorphisms of $\mathcal{U}$, it is known as a *proper NEC group*.

An NEC group is called a *bordered surface group* if it contains a reflection but does not contain other elements of finite order. Each compact bordered Klein surface X of algebraic genus $p \geq 2$ can be presented as the orbit space $X = \mathcal{U}/\Lambda$ for some bordered surface group Λ. Moreover, given a surface X so represented, a finite group G acts as a group automorphisms of X if and only if there exists an NEC group Γ^* and an epimorphism $\theta : \Gamma^* \to G$ such that $\ker(\theta) = \Lambda$. All groups of automorphisms of bordered Klein surfaces arise in this way. Such an epimorphism, whose kernel is a bordered surface group, is called a *bordered smooth epimorphism*.

In this paper, we shall be mainly concerned with quadrilateral groups $\Gamma^*[2, 2, 2, q]$. A quadrilateral group Γ^* is an NEC group with signature

$$(0; +; [-]; \{(2, 2, 2, q)\}),$$

where $q \geq 3$ prime number [16]. Also Γ^* is isomorphic to the abstract group with the presentation

$$< c_0, c_1, c_2, c_3 \mid c_i^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^q = I > .$$

It is well-known [13] that large groups of automorphisms of bordered surfaces are quotients of the quadrilateral groups $\Gamma^*[2, 2, 2, q]$.

It is clear that Γ^* has exactly three subgroups of index 2 which contain c_1 (namely Γ_1, Γ_2, and Γ_3) and a unique normal subgroup of index 4 which contains c_1 (namely Γ_4). In fact, Γ_1 is generated by c_0, c_1, $c_2c_0c_2$ and c_3; Γ_2 is generated by c_2c_3, c_3c_0 and c_1; Γ_3 is generated by c_1, c_2, c_3c_0 and $c_3c_1c_3$; and Γ_4 is generated by c_0c_3, $c_2c_3c_0c_2$ and c_1. Also the signatures of Γ_1, Γ_2, Γ_3 and Γ_4 are $(0; +; [-]; \{(2, 2, q, q)\})$, $(0; +; [2, q]; \{(-)\})$, $(0; +; [q]; \{(2, 2)\})$ and $(0; +; [q, q]; \{(-)\})$, respectively (see, [2, p. 564]).

If Λ_{c_1} is the normal subgroup of Γ^* generated by c_1, then $\overline{\Gamma}^* = \Gamma^*/\Lambda_{c_1}$. Also, if there exist a normal subgroup Φ in Γ^*containing c_1, then $\Gamma^*/\Phi \cong \overline{\Gamma}^*/\overline{\Phi}$. Since $\Gamma^*/\Gamma_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\overline{\Gamma}^*/\overline{\Gamma}_4$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is clear that the commutator subgroup $(\overline{\Gamma}^*)' \subset \overline{\Gamma}_4$. Notice that the quotient group $\overline{\Gamma}^*/(\overline{\Gamma}^*)'$ is generated by elements of order 2. Also it is easy to see that $c_0(\overline{\Gamma}^*)'$ and $c_3(\overline{\Gamma}^*)'$ commute, as $\overline{\Gamma}^*/(\overline{\Gamma}^*)'$ is abelian. Since c_0c_3 has order q, $c_0c_3 \in (\overline{\Gamma}^*)'$. Therefore $\overline{\Gamma}^*/(\overline{\Gamma}^*)'$ is generated by two elements of order q. Thus $\overline{\Gamma}_4 = (\overline{\Gamma}^*)'$ and then $\overline{\Gamma}_4$ is a free product generated by two elements of order q. This requires that $\overline{\Gamma}_4/\overline{\Gamma}_4' \cong \mathbb{Z}_q \times \mathbb{Z}_q$, which yields that $\overline{\Gamma}^*/(\overline{\Gamma}^*)'' \cong D_q \times D_q$.

From [3, Theorems 2.2.4 and 2.3.3], if $G = \Gamma^*/\Lambda$ satisfies $|G| = \frac{4q}{(q-2)}(p-1)$, for some NEC group Γ^* and for $q \geq 3$ prime number, then the signature of Γ^* is $(0; +; [-]; \{(2, 2, 2, q)\})$ and for each group G, there is a bordered smooth epimorphism $\theta : \Gamma^* \to G$ which maps $c_0 \to r_1$, $c_1 \to I$, $c_2 \to r_2$ and $c_3 \to r_3$. Thus r_1r_2 and r_1r_3 have orders 2 and q respectively and each group G admits the following partial presentation :

$$\langle r_1,\ r_2,\ r_3 \mid r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^q = \cdots = I\ \rangle.$$

Now we need a definition.

Definition 1 ([15]). *Let $q \geq 3$ be a prime number. A finite group G will be called a generalized M^*-group if it is generated by three distinct nontrivial elements r_1, r_2 and r_3 of order 2 such that r_1r_2 and r_1r_3 have orders 2 and q respectively, i.e.,*

$$r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^q = I. \tag{1}$$

The order t of r_2r_3 is called an index of G and G is said a *generalized $M^* -$ group with index t*. A generalized M^*-group can have more than one index. If G is a generalized M^*-group with index t and l is the order of $(r_1r_2r_3)$, then G is also a generalized M^*-group with index l [15].

From [15], if $G = \Gamma^*/\Lambda$ is a generalized M^*-group, then it can have at most three subgroups of index 2 and one normal subgroup of index 4. A generalized M^*-group G possesses either zero, one or three subgroups of index 2, $G_1 =< r_1, r_3, r_2r_3r_2 >$, $G_2 =< r_1r_2, r_1r_3 >$, $G_3 =< r_2, r_1r_3 >$, respectively. A generalized M^*-group G possesses at most one normal subgroup of index 4, $G_4 =< r_1r_3, r_2r_3r_1r_2 >$. Here the subgroups of G corresponding to each of Γ_1, Γ_2, Γ_3 and Γ_4 are G_1, G_2, G_3 and G_4, respectively.

3 Generalized M^*-Groups and Related Results

A finite group G of order $\frac{4q}{(q-2)}(p-1)$ is a generalized M^*-group if and only if G acts on a bordered Klein surface X of genus $p \geq 2$. If we take $p = (q-2)s+1$ where $q \geq 3$ prime number and $s \in \mathbb{Z}^+$, then we find $|G| = 4qs$. Thus for every positive integer p which is of the form $(q-2)s+1$, there are infinitely many generalized M^*-groups and for every positive integer p which is not of the form $(q-2)s+1$, there are no generalized M^*-groups.

Note that if $s = 1$, then we get $p = (q-2)1+1 = q-1$ and $|G| = 4q$. Therefore for every $q \geq 3$ a prime number, there is a generalized M^*-group G. Here this result coincides with the ones given in [1, Theorem 2.1]. Also, using a result of Bujalance [1, Theorem 2.1], it is easy to see that if X a compact bordered Klein surface of algebraic genus $p \geq 2$, $p \neq 5$, 11 and 29, and the group $G = Aut(X)$ is isomorphic to

$$\left\langle r_1,\ r_2,\ r_3 \mid r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^q = I,\ r_2r_3r_2 = r_1(r_3r_1)^t \right\rangle$$

for some t such that $t^2 \equiv 1 \bmod (q)$ and $1 \leq t \leq q-1$ then X is orientable and has $k = \gcd(q, t+1)$ boundary components. Therefore, if $q \geq 3$ prime number then X is orientable and $k = q$ boundary components or $k = 1$ boundary component. Thus G acts on a sphere with q holes and a surface of genus $\frac{q-1}{2}$ with one hole. Conversely if $p \geq 2$ and $|G| = 4q$ then Γ^* has signature $(0; +; [-]; \{(2,2,2,q)\})$ where Γ^* is an NEC group.

Remark 1. *Generalized M^*-groups are exactly the same as the automorphism groups of regular maps (regular tilings) of type $\{q, t\}$ where t is prime. A map is said to be of type $\{q, t\}$ if it is composed of $q-$gons, with exactly t, $q-$gons meeting at each vertex. Suppose a generalized M^*-group G acts on the bordered surface X with index t. Then the surface X corresponds to a regular map $\mathcal{M}$ of type $\{q, t\}$ on the surface X^* obtained from X by attaching a disc to each boundary component. Also G is isomorphic to the automorphism group of the map $\mathcal{M}$, and the number of boundary components of X is equal to the number of vertices of $\mathcal{M}$.*

Example 1. *Let $G^{q,n,r}$ be the group with generators A, B and C and defining relations*

$$A^q = B^n = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = I.$$

If we set $r_1 = BC$, $r_2 = CA$, and $r_3 = BCA$, then we obtain the presentation

$$r_1^2 = r_2^2 = r_3^2 = (r_1r_2)^2 = (r_1r_3)^q = (r_2r_3)^n = (r_1r_2r_3)^r = I.$$

Thus G is a quotient of $\Gamma[2,2,2,q]$ by a bordered surface group if and only if G is a quotient of the group $G^{q,n,r}$ for some n and r. If $q \geq 3$ is a prime and the group is finite, then we obtain a generalized M^*-group with index n. Some values of n and r which make the group to be finite are given in [7] and [8].

Now using of the first and the second commutator subgroups of generalized M^*-groups, we obtain new generalized M^*-groups.

Theorem 1. *Let G be a generalized M^*-group. Then there exist a normal subgroup N of $D_q \times D_q$, $q \geq 3$ prime, such that we have the following.*

(i) $G/G'' \cong (D_q \times D_q)/N$.

(ii) For each $N_1 \lhd D_q \times D_q$ with $N_1 \leq N$, let $K = N/N_1$. Then there exists a generalized M^*-group $\hat{G}$ such that

$$1 \to K \to \hat{G} \to G \to 1$$

is a short exact sequence. Furthermore, $\hat{G}$ contains a subgroup isomorphic to $G'' \times K$.

Proof. We will prove our theorem as in the case of the M^*-groups in [5].

(i) Firstly, since G is a generalized M^*-group, it is known that there is a smooth epimorphism $\theta : \Gamma^* \to G$, such that $c_1 \in \Lambda := \ker(\theta)$. Then, by using Lemma 2.1 in [5, p.342] and $G \cong \overline{\Gamma}^*/\overline{\Lambda}$, we have $G' \cong (\overline{\Gamma}^*)'\overline{\Lambda}/\overline{\Lambda}$ and $G'' \cong (\overline{\Gamma}^*)''\overline{\Lambda}/\overline{\Lambda}$. Therefore, to complete the proof (i), we define $N := (\overline{\Gamma}^*)''\overline{\Lambda}/(\overline{\Gamma}^*)''$. Using $\overline{\Gamma}^*/(\overline{\Gamma}^*)'' \cong D_q \times D_q$, we get $G/G'' \cong \overline{\Gamma}^*/(\overline{\Gamma}^*)''\overline{\Lambda} \cong (D_q \times D_q)/N$. This concludes the proof of (i).

(ii) Let N_1 be a normal subgroup of $D_q \times D_q$ such that $N_1 \leq N$. Let $K = N/N_1$. From (i), we know that $N = (\overline{\Gamma}^*)''\overline{\Lambda}/(\overline{\Gamma}^*)''$. Then there exist an NEC group $(\overline{\Gamma}_4)_1 \leq (\overline{\Gamma}^*)''\overline{\Lambda}$ such that $N_1 \cong (\overline{\Gamma}_4)_1/(\overline{\Gamma}^*)''$. Since $(\overline{\Gamma}^*)'' \leq (\overline{\Gamma}_4)_1 \leq (\overline{\Gamma}^*)''\overline{\Lambda}$ we get $(\overline{\Gamma}^*)''\overline{\Lambda} = (\overline{\Gamma}_4)_1\overline{\Lambda}$ and $N \cong (\overline{\Gamma}^*)''\overline{\Lambda}/(\overline{\Gamma}^*)'' = (\overline{\Gamma}_4)_1\overline{\Lambda}/(\overline{\Gamma}^*)''$. Define $\hat{G} = \overline{\Gamma}^*/(\overline{\Lambda} \cap (\overline{\Gamma}_4)_1)$. Then $\hat{G}$ contains the subgroup

$$\frac{\overline{\Lambda}}{\overline{\Lambda} \cap (\overline{\Gamma}_4)_1} \cong \frac{\overline{\Lambda}(\overline{\Gamma}_4)_1}{(\overline{\Gamma}_4)_1} \cong \frac{(\overline{\Gamma}^*)''\overline{\Lambda}/(\overline{\Gamma}^*)''}{(\overline{\Gamma}_4)_1/(\overline{\Gamma}^*)''} \cong \frac{N}{N_1} \cong K.$$

Finally, the subgroups $G'' \cong (\overline{\Gamma}_4)_1/(\overline{\Lambda} \cap (\overline{\Gamma}_4)_1)$ and $K \cong \overline{\Lambda}/(\overline{\Lambda} \cap (\overline{\Gamma}_4)_1)$ are normal in $\hat{G}$. Since the subgroups G'' and K generate $\overline{\Lambda}(\overline{\Gamma}_4)_1/(\overline{\Lambda} \cap (\overline{\Gamma}_4)_1)$ and have trivial intersection, we obtain $\overline{\Lambda}(\overline{\Gamma}_4)_1/(\overline{\Lambda} \cap (\overline{\Gamma}_4)_1) \cong G'' \times K$. This completes the proof (ii). □

This theorem provides a way for constructing new families of generalized M^*-groups and has several interesting consequences. For example, it can be applied to perfect groups, which are equal to their first commutator subgroup. Let G be a perfect group. Then $G'' = G$. Therefore, the above theorem shows that if K is a factor group of $D_q \times D_q$, then there is a generalized M^*-group $\hat{G}$ of order $|G|\,|K|$ such that $\hat{G}$ contains a subgroup isomorphic to $G \times K$. But the only normal subgroup of $\hat{G}$ of order $|G|\,|K|$ is $\hat{G}$, then $\hat{G}$ is isomorphic to $G \times K$.

Using this, we obtain the following corollary and examples:

Corollary 1. *İf G is a perfect generalized M^*-group, then $G\times\mathbb{Z}_2$, $G\times\mathbb{Z}_2\times\mathbb{Z}_2$, $G \times D_q$, $G \times \mathbb{Z}_2 \times D_q$, and $G \times D_q \times D_q$ are generalized M^*-groups.*

Example 2. *Many finite simple groups H have been shown to be generated by three involutions, two of which commute, are generalized M^*-groups. Also for these finite simple groups, $H \times \mathbb{Z}_2$, $H \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $H \times D_q$, $H \times \mathbb{Z}_2 \times D_q$, and $H \times D_q \times D_q$ are generalized M^*-groups.*

Example 3. *For any prime $q > 6$, all but finitely many alternating groups A_n are quotients of the extended $(2, 3, q)$ triangle group, and are therefore generalized M^*-groups of index 3. For these values we find that $A_n \times \mathbb{Z}_2$, $A_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $A_n \times D_q$, $A_n \times \mathbb{Z}_2 \times D_q$, and $A_n \times D_q \times D_q$ are generalized M^*-groups.*

Now, we give some methods for constructing new generalized M^*-groups from a group (not necessarily generalized M^*-groups) which may arise as a normal subgroup of index two. These constructions were obtained in [3] and [11] for M^*-groups.

Theorem 2. *Let $q \geq 3$ be a prime number. If G is a generalized M^*-group associated to q with odd index t, then $\mathbb{Z}_2 \times G$ is a generalized M^*-group with index $2t$.*

Proof. Let G be a generalized M^*-group generated by r_1, r_2 and r_3 satisfying the relations in (2.1) and let G has odd index t. If a generate $\mathbb{Z}_2$ then we set $r_1^* = (a, r_1)$, $r_2^* = (1, r_2)$, and $r_3^* = (a, r_3)$. Therefore, r_1^*, r_2^*, and r_3^* generate the direct product $\mathbb{Z}_2 \times G$. Also, they satisfy the relations (2.1) with $o(r_2^* r_3^*) = 2t$. □

Notice that if the index t is even, the construction will not work, since $(2,t) \neq 1$.

Theorem 3. *Let $q \geq 3$ be a prime number. Let H be a finite group generated by two elements x and y, of order 2 and q, respectively. If H admits the automorphism*

$$\gamma : x \to x^{-1} = x,\ y \to y^{-1}$$

then the semidirect product group $G = H \rtimes_\gamma \mathbb{Z}_2$ is a generalized M^*-group.

Proof. If a generate $\mathbb{Z}_2$ then it is easy to see that $G = H \rtimes_\phi \mathbb{Z}_2$ with generators with $r_1 = (y,a)$, $r_2 = (x,a)$, and $r_3 = (1,a)$ is a generalized M^*-group. □

Theorem 4. *Let $q \geq 3$ be a prime number and let G be a generalized M^*-group associated to q. If $[G :< r_1r_2, r_1r_3 >] = 2$, and t is not a multiple of 3, then $\mathbb{Z}_3 \rtimes_\theta G$ is a generalized M^*-group with odd index $3t$.*

Proof. Since $[G :< r_1r_2, r_1r_3 >] = 2$, we take the quotient map θ,

$$\theta : G \to G/ < r_1r_2, r_1r_3 > \cong \mathbb{Z}_2 = Aut(\mathbb{Z}_3)$$

and we construct the semi-direct product $\mathbb{Z}_3 \rtimes_\theta G$. If a generate $\mathbb{Z}_3$ then we set $r_1' = (x, r_1)$, $r_2' = (x, r_2)$, and $r_3' = (1, r_3)$. Thus r_1', r_2', and r_3' generate $\mathbb{Z}_3 \rtimes_\theta G$ and they satisfy the relations (2.1) with $o(r_2'r_3') = 3t$ and $o(r_1'r_2'r_3') = l = o(r_1r_2r_3)$. □

References

[1] E. Bujalance, On compact Klein surfaces with a special automorphism group, *Ann. Acad. Sci. Fenn. Math.*, **22** (1) (1997), 15–20.

[2] E. Bujalance, P. Turbek, On Klein surfaces with 2p or 2p+2 automorphisms. *J. Algebra*, 235 (2001), no. 2, 559–588.

[3] E. Bujalance, F. J. Cirre, P. Turbek, Groups acting on bordered Klein surfaces with maximal symmetry, in *Proceedings of Groups St. Andrews* 2001 in Oxford. Vol. I, London Math. Soc. Lecture Note Ser., 304, Cambridge Univ. Press, Cambridge, 2003, 50–58,.

[4] E. Bujalance, F.J. Cirre, P. Turbek, Subgroups of M^*-groups, *Q. J. Math.*, **54** (1), (2003), 49–60.

[5] E. Bujalance, F.J. Cirre, P. Turbek, Automorphism criteria for M^*-groups, *Proc. Edinb. Math. Soc.*, **47** (2) (2004), 339–351.

[6] E. Bujalance, J. J. Etayo, J. M. Gamboa, G. Gromadzki, *Automorphisms groups of compact bordered Klein surfaces. A Combinatorial Approach*, Lecture Notes in Math. vol. 1439, Springer Verlag. 1990.

[7] H.S.M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, Berlin, 1965.

[8] M. Edjvet, A. Juhász, The groups $G^{m,n,p}$, *J. Algebra,* **319** (1), (2008), 248–266.

[9] C. L. May, Automorphisms of compact Klein surfaces with boundary, *Pacific J. Math.*, **59** (1975), 199-210.

[10] C. L. May, Large automorphism groups of compact Klein surfaces with boundary, *Glasgow Math. J.*, **18** (1977), 1-10.

[11] C. L. May, The species of bordered Klein surfaces with maximal symmetry of low genus, *Pacific J.Math.,* 111 (1984), no. 2, 371–394.

[12] C. L. May, Supersolvable M^*-groups, *Glasgow Math. J.*, **30** (1) (1988), 31–40.

[13] C. L. May, The groups of real genus 4, *Michigan Math. J.*, **39** (1992), 219-228.

[14] R. Sahin, S. İkikardes and Ö. Koruoğlu, Some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_p)$, *Rocky Mountain J. Math.*, **36** (3) (2006), 1033-1048.

[15] R. Sahin, S. İkikardes and Ö. Koruoğlu, Generalized M^*-groups, *Int. J. Algebra Comput.*, **16** (6) (2006), 1211–1219.

[16] A. M. Porto Fereira da Silva and A. F. Costa, On the automorphism group of the canonical double covering of bordered Klein surfaces with large automorphism group, *J. Math. Sci.*, **82** (6) (1996), 3773–3779.

Balıkesir Üniversitesi,
Fen-Edebiyat Fakültesi, Matematik Bölümü,
10145 Çağış Kampüsü, Balıkesir, Turkey
e-mail: skardes@balikesir.edu.tr; rsahin@balikesir.edu.tr