An. Şt. Univ. Ovidius Constanţa Vol. 18(1), 2010, 287–294

MATLAB EVALUATION OF THE $\Omega_{j,k}^{m,n}(x)$ COEFFICIENTS FOR PDE SOLVING BY WAVELET - GALERKIN APPROXIMATION

Constantin I. Popovici

Abstract

After approaching the problem of the numerical solution of Stefan's Problem by Finite Element Methods (FEM) or by Finite Difference Methods in his earlier papers [5]-[11], the author goes on with using the wavelet - Galerkin method for the solution of the earlier mentioned problem.

The evaluation of the coefficients $\Omega_{j,k}^{m,n}(x)$ is a first step in the numerical approximation of the Navier - Stokes equation, of the problem with free boundary of Stefan type, etc.

This paper is one in a set of articles dealing with solutions to PDEs or ODEs using the wavelet - Galerkin method. In order to approximate the solution, a couple of families of coefficients are needed; they occur in wavelet series and they are involved in discretizing differential equations that characterize mathematical-mechanical models. Following some earlier ideas (of references [1],[2],[3],[4],[12],[13],[14]), we have worked out several algorithms and MATLAB - based programs that differ from the algorithms in [2], allowing to obtain high precision results for the necessary functionals. The accuracy of the results follows from the performances of the MATLAB programming environment. In this paper it is described the MATLAB evaluation of the integral

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \Phi(y)\, \Phi^{(m)}(y-j)\, \Phi^{(n)}(y-k)\, dy.$$

The orthonormal wavelet class with compact support, developed by Daubechies (in 1988) follows to be adapted as a Galerkin basis in the space domain.

Key Words: wavelets, Galerkin method.
Mathematics Subject Classification: 65T60
Received: March, 2009
Accepted: January, 2010

1. INTRODUCTION

Using some results due to Prof. Ingrid Daubechies (Princeton University, USA) regarding the determination of an orthonormal basis of functions with compact support on $L^2(\mathbb{R})$ [4], the team led by Prof. Chen (National Cheng Kung University of Taiwan) has proposed in [2] some algorithms for calculating seven functionals that occur in wavelet - Galerkin discretization of differential equations. In our paper we present the algorithms and programs needed for the calculation of one of these functionals, namely

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \Phi(y)\,\Phi^{(m)}(y-j)\,\Phi^{(n)}(y-k)\,dy. \tag{1}$$

using the programming environment MATLAB. In expression (1), $j,k \in \mathbb{Z}$, $m,n \in \mathbb{N}^*$ and $\Phi^{(n)}(u)$ denotes the n-order derivative of function Φ. We will calculate the coefficients $\Omega_{j,k}^{m,n}$ (5). Having the values of $\Omega_{j,k}^{m,n}$ (5) for $j,k \in \mathbb{Z}$ calculated, we can determine the values of $\Omega_{j,k}^{m,n}(x)$ for $x = 1,2,3,4$ and $j,k \in \mathbb{Z}$, taking into account the properties of the coefficients $\Omega_{j,k}^{m,n}(x)$ of the formulae (2)-(4).

2. CALCULATION OF COEFFICIENTS $\Omega_{j,k}^{m,n}$

Each member of the family of wavelets built by Daubechies is governed by a set of L (an integer number) coefficients $\{p_k : k = 0,1,\ldots,L-1\}$ and two functions $\Phi(x)$ and $\Psi(x)$.

The function $\Phi(x)$, called the scalar or waveleted function is defined on $[0, L-1]$ and it has the expression

$$\Phi(x) = \sum_{j=0}^{L-1} p_j \Phi(2x - j).$$

The function $\Psi(x)$, called wavelet-mother, is defined on $[1 - L/2, L/2]$ and its expression is

$$\Psi(x) = \sum_{j=2-L}^{1} (-1)^j p_{1-j} \Phi(2x - j).$$

The Daubechies filtration coefficients p_k, $k = \overline{0, L-1}$ for $L = 6$ are the following:

$$p_0 = \frac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}, \qquad p_1 = \frac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16},$$
$$p_2 = \frac{10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}}{16}, \qquad p_3 = \frac{10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}}{16},$$
$$p_4 = \frac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16}, \qquad p_5 = \frac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}.$$

It can be seen that the equation $\sum_{k=0}^{5} p_k = 2$ is satisfied. We are going to calculate $\Omega_{j,k}^{m,n}(x)$ for $x = 5$, $L = 6$, $m = 0$, $n = 1$ and $-4 \le j, k \le 4$. $\Omega_{j,k}^{m,n}(x)$ plays an important role in the numerical solution of nonlinear differential equations by the wavelet - Galerkin method, according to the assertion and the example given by A. Latto and E. Tenenbaum, "Les ondelettes a support compact et la solution numerique de l'equation de Burgers", C. R. Acad. Sci. Paris, 311, 903-909 (1990). In the paper "The evaluation of connection coefficients of compactly supported wavelets" authored by A. Latto, H. L. Resnikoff and E. Tenenbaum and published in Proc. French - USA Workshop on wavelets and Turbulence, Y. Maday (ed.), the coefficient $\Omega_{j,k}^{m,n}(x)$ is called the third coefficient of wavelets connection.

The coefficients $\Omega_{j,k}^{m,n}(x)$ have the following properties:

$$\Omega_{j,k}^{m,n}(x) = 0 \quad \text{for} \quad |j|,\ |k|, \text{ or } |j-k| \ge L-1, \tag{2}$$
$$\Omega_{j,k}^{m,n}(x) = 0 \quad \text{for} \quad x-j,\ x-k, \text{ or } x \le 0, \tag{3}$$
$$\Omega_{j,k}^{m,n}(x) = \Omega_{j,k}^{m,n}(L-1) \quad \text{for} \quad x-j,\ x-k, \text{ or } x \ge L-1. \tag{4}$$

In equation (113) of [2] we consider $-4 \le j, k \le 4$ and, taking into account formulas (2) $-$ (4), we will obtain a homogeneous system in the unknowns $\Omega_{j,k}^{m,n}(x)$ with $m = 0$ and $n = 1$. Equation (113) of [2] has the form

$$\Omega_{j,k}^{m,n}(x) = 2^{m+n-1} \sum_{i_a=0}^{L-1} \sum_{i_b=0}^{L-1} \sum_{i_c=0}^{L-1} p_{i_a} p_{i_b} p_{i_c} \Omega_{2j+i_b-i_a,2k+i_c-i_a}^{m,n}(2x-i_a). \tag{5}$$

We have a system with $3L^2 - 9L + 7$ unknowns $\Omega_{j,k}^{m,n}(L-1)$. We obtain, from equation (5), a homogeneous system of the form

$$v = 2^{1-m-n} S v. \tag{6}$$

where

$$v = \left[v_{2-L},\ v_{3-L},\ \ldots,\ v_{L-2}\right]^T, \tag{7}$$
$$v_j = \left[\Omega_{j,\alpha}^{m,n}(L-1),\ \Omega_{j,\alpha+1}^{m,n}(L-1),\ \ldots,\ \Omega_{j,\beta}^{m,n}(L-1)\right], \tag{8}$$

$\alpha = \max(j+2-L, 2-L)$, $\beta = \min(j+L-2, L-2)$, and the entries of matrix S are sums of products of the form $p_{i_a}p_{i_b}p_{i_c}$. Since $x=5$, $m=0$ and $n=1$, we denote the unknown by $\Omega_{j,k}$.

The unknowns of system are: $\Omega_{-4,-4}$; $\Omega_{-4,-3}$; $\Omega_{-4,-2}$; $\Omega_{-4,-1}$; $\Omega_{-4,0}$; $\Omega_{-3,-4}$; $\Omega_{-3,-3}$; $\Omega_{-3,-2}$; $\Omega_{-3,-1}$; $\Omega_{-3,0}$; $\Omega_{-3,1}$; $\Omega_{-2,-4}$; $\Omega_{-2,-3}$; $\Omega_{-2,-2}$; $\Omega_{-2,-1}$; $\Omega_{-2,0}$; $\Omega_{-2,1}$; $\Omega_{-2,2}$; $\Omega_{-1,-4}$; $\Omega_{-1,-3}$; $\Omega_{-1,-2}$; $\Omega_{-1,-1}$; $\Omega_{-1,0}$; $\Omega_{-1,1}$; $\Omega_{-1,2}$; $\Omega_{-1,3}$; $\Omega_{0,-4}$; $\Omega_{0,-3}$; $\Omega_{0,-2}$; $\Omega_{0,-1}$; $\Omega_{0,0}$; $\Omega_{0,1}$; $\Omega_{0,2}$; $\Omega_{0,3}$; $\Omega_{0,4}$; $\Omega_{1,-3}$; $\Omega_{1,-2}$; $\Omega_{1,-1}$; $\Omega_{1,0}$; $\Omega_{1,1}$; $\Omega_{1,2}$; $\Omega_{1,3}$; $\Omega_{1,4}$; $\Omega_{2,-2}$; $\Omega_{2,-1}$; $\Omega_{2,-0}$; $\Omega_{2,1}$; $\Omega_{2,2}$; $\Omega_{2,3}$; $\Omega_{2,4}$; $\Omega_{3,-1}$; $\Omega_{3,0}$; $\Omega_{3,1}$; $\Omega_{3,2}$; $\Omega_{3,3}$; $\Omega_{3,4}$; $\Omega_{4,0}$; $\Omega_{4,1}$; $\Omega_{4,2}$; $\Omega_{4,3}$; $\Omega_{4,4}$.

Taking $j=-4$ and $k=-4$ in equation (5), and also taking into account $(2)-(4)$, we obtain

$$\Omega_{-4,-4} = (p_0p_4p_4 + p_1p_5p_5)\,\Omega_{-4,-4} + p_0p_4p_5\Omega_{-4,-3} + p_0p_5p_4\Omega_{-3,-4} + p_0p_5p_5\Omega_{-3,-3}.$$

It follows that

$$s_{11} = p_0p_4p_4 + p_1p_5p_5, \quad s_{12} = p_0p_4p_5, \quad s_{13} = p_0p_5p_4, \quad s_{14} = p_0p_5p_5,$$

the remaining entries on the first row being equal to zero.

Similarly, if we consider $j=-4$ and $k=-3$ in formula (5) we have

$$\begin{aligned}\Omega_{-4,-3} = {}&(p_0p_4p_2 + p_1p_5p_3)\,\Omega_{-4,-4} + (p_1p_5p_4 + p_0p_4p_3)\,\Omega_{-4,-3}+\\ &+(p_0p_4p_4 + p_1p_5p_5)\,\Omega_{-4,-2} + p_0p_4p_5\Omega_{-4,-1} + p_0p_5p_2\Omega_{-3,-4}+\\ &+p_0p_5p_3\Omega_{-3,-3} + p_0p_5p_4\Omega_{-3,-2} + p_0p_5p_5\Omega_{-3,-1}.\end{aligned}$$

It follows that

$$\begin{gathered} s_{21} = p_0p_4p_2 + p_1p_5p_3, \quad s_{22} = p_1p_5p_4 + p_0p_4p_3,\\ s_{23} = p_0p_4p_4 + p_1p_5p_5, \quad s_{24} = p_0p_4p_5, \quad s_{25} = 0,\\ s_{26} = p_0p_5p_2, \quad s_{27} = p_0p_5p_3, \quad s_{28} = p_0p_5p_4, \quad s_{29} = p_0p_5p_5; \end{gathered}$$

the remaining entries in the second row are $=0$. Following this procedure, the matrix S is generated as follows:

```
% The generate matrix omega
clc
p1=0.47046720778416;
p2=1.14111691583144;
p3=0.65036500052623;
p4=-0.19093441556833;
p5=-0.12083220831040;
p6=0.04981749973688;
```

```
a=[-4 -4 -4 -4 -4 -3 -3 -3 -3 -3 -3 -2 -2 -2 -2 -2 -2 -2 -1 ...
-1 -1 -1 -1 -1 -1 -1 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 2 2 2 ...
2 2 2 2 3 3 3 3 3 3 4 4 4 4 4];
b=[-4 -3 -2 -1 0 -4 -3 -2 -1 0 1 -4 -3 -2 -1 0 1 2 -4 -3 -2 ...
-1 0 1 2 3 -4 -3 -2 -1 0 1 2 3 4 -3 -2 -1 0 1 2 3 4 -2 -1 0 ...
1 2 3 4 -1 0 1 2 3 4 0 1 2 3 4];
L=6;
s=zeros(61);
for r=1:61
    j=a(r);
    k=b(r);
    for ia=1:L
        for ib=1:L
            for ic=1:L
                jj=2*j+ib-ia;
                kk=2*k+ic-ia;
                for t=1:61
                    if ((jj==a(t))&(kk==b(t))
                        q=t;
                        s(r,q)=s(r,q)+p(ia)*p(ib)*p(ic);
                        break
                    end
                end
            end
        end
    end
end
[vp,dp]=eig(s)
```

The matrix S has the eigenvalues 2^{1-k}, $k = 0, 1, \ldots, L-2$ with their multiplicity order $k+1$.

It can be seen from (6) that v is an eigenvector corresponding to the eigenvalue 2^{m+n-1}. In our case v is the solution corresponding to the eigenvalue 1 with its multiplicity order two. It follows that we cannot determine a unique solution from (6).

In order to determine a solution to system (6) we attach the equations resulting from the equation of moments (formulas (117) and (118) of [2]), namely

$$
\begin{aligned}
&-4\Omega_{-4,-4}-3\Omega_{-4,-3}-2\Omega_{-4,-2}-1\cdot\Omega_{-4,-1}+0\cdot\Omega_{-4,0} &&=\Gamma^0{}_{-4}\\
&-4\Omega_{-3,-4}-3\Omega_{-3,-3}-2\Omega_{-3,-2}-1\cdot\Omega_{-3,-1}+0\cdot\Omega_{-3,0}+1\cdot\Omega_{-3,1} &&=\Gamma^0{}_{-3}\\
&-4\Omega_{-2,-4}-3\Omega_{-2,-3}-2\Omega_{-2,-2}-1\cdot\Omega_{-2,-1}+0\cdot\Omega_{-2,0}+1\cdot\Omega_{-2,1}+\\
&+2\Omega_{-2,2} &&=\Gamma^0{}_{-2}\\
&-4\Omega_{-1,-4}-3\Omega_{-1,-3}-2\Omega_{-1,-2}-1\cdot\Omega_{-1,-1}+0\cdot\Omega_{-1,0}+1\cdot\Omega_{-1,1}+\\
&+2\Omega_{-1,2}+3\Omega_{-1,3} &&=\Gamma^0{}_{-1}\\
&-4\Omega_{0,-4}-3\Omega_{0,-3}-2\Omega_{0,-2}-1\cdot\Omega_{0,-1}+0\cdot\Omega_{0,0}+1\cdot\Omega_{0,1}+2\Omega_{0,2}+\\
&+3\Omega_{0,3}+4\Omega_{0,4} &&=\Gamma^0_0\\
&-3\Omega_{1,-3}-2\Omega_{1,-2}-1\cdot\Omega_{1,-1}+0\cdot\Omega_{1,0}+1\cdot\Omega_{1,1}+2\Omega_{1,2}+3\Omega_{1,3}+\\
&+4\Omega_{1,4} &&=\Gamma^0_1\\
&-2\Omega_{2,-2}-1\cdot\Omega_{2,-1}+0\cdot\Omega_{2,0}+1\cdot\Omega_{2,1}+2\Omega_{2,2}+3\Omega_{2,3}+4\Omega_{2,4} &&=\Gamma^0_2\\
&-1\cdot\Omega_{3,-1}+0\cdot\Omega_{3,0}+1\cdot\Omega_{3,1}+2\Omega_{3,2}+3\Omega_{3,3}+4\Omega_{3,4} &&=\Gamma^0_3\\
&0\cdot\Omega_{4,0}+1\cdot\Omega_{4,1}+2\Omega_{4,2}+3\Omega_{4,3}+4\Omega_{4,4} &&=\Gamma^0_4\\
&\Omega_{-4,-4}+\Omega_{-3,-4}+\Omega_{-2,-4}+\Omega_{-1,-4}+\Omega_{0,-4} &&=\Gamma^1{}_{-4}\\
&\Omega_{-4,-3}+\Omega_{-3,-3}+\Omega_{-2,-3}+\Omega_{-1,-3}+\Omega_{0,-3}+\Omega_{1,-3} &&=\Gamma^1{}_{-3}\\
&\Omega_{-4,-2}+\Omega_{-3,-2}+\Omega_{-2,-2}+\Omega_{-1,-2}+\Omega_{0,-2}+\Omega_{1,-2}+\Omega_{2,-2} &&=\Gamma^1{}_{-2}\\
&\Omega_{-4,-1}+\Omega_{-3,-1}+\Omega_{-2,-1}+\Omega_{-1,-1}+\Omega_{0,-1}+\Omega_{1,-1}+\Omega_{2,-1}+\Omega_{3,-1} &&=\Gamma^1{}_{-1}\\
&\Omega_{-4,0}+\Omega_{-3,0}+\Omega_{-2,0}+\Omega_{-1,0}+\Omega_{0,0}+\Omega_{1,0}+\Omega_{2,0}+\Omega_{3,0}+\Omega_{4,0} &&=\Gamma^1_0
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_{-3,1}+\Omega_{-2,1}+\Omega_{-1,1}+\Omega_{0,1}+\Omega_{1,1}+\Omega_{2,1}+\Omega_{3,1}+\Omega_{4,1} &&=\Gamma^1_1\\
&\Omega_{-2,2}+\Omega_{-1,2}+\Omega_{0,2}+\Omega_{1,2}+\Omega_{2,2}+\Omega_{3,2}+\Omega_{4,2} &&=\Gamma^1_2\\
&\Omega_{-1,3}+\Omega_{0,3}+\Omega_{1,3}+\Omega_{2,3}+\Omega_{3,3}+\Omega_{4,3} &&=\Gamma^1_3\\
&\Omega_{0,4}+\Omega_{1,4}+\Omega_{2,4}+\Omega_{3,4}+\Omega_{4,4} &&=\Gamma^1_4
\end{aligned}
$$

The numbers $\Gamma^0{}_{-4},\Gamma^0{}_{-3},\Gamma^0{}_{-2},\ldots,\Gamma^1_4$ are known.

The attachment of these equations to system (6) is accomplished after the elimination of the rows corresponding to the unknowns $\Omega_{-4,0}$, $\Omega_{-3,0}$ and $\Omega_{0,0}$. The replacement is rather difficult; the obtained solution must satisfy the conditions (117) and (118) of [2].

```
% Program for determinate solution
for i=1:61
    for j=1:61
        if i==j
            s(i,j)=-1+s(i,j)
        end
```

```
    end
end
s(5,1:4)=[-4 -3 -2 -1]; s(5,5:61)=0;
s(10,1:5)=0; s(10,6:11)=[-4 -3 -2 -1 0 1]; s(10,12:61)=0;
% s(31,1:11)=0; s(31,12:18)=[-4 -3 -2 0 1 2]; s(31,20:61)=0;
rang=rank(s)
dets=det(s);
d=zeros(61,1);
d(5,1)=-0.34246575e-3;
d(10,1)=-0.14611872e-1;
% d(31,1)=0.14520548;
format long
sol=s\d
```

The solution thus obtained has a higher accuracy.

3. CONCLUSIONS

The calculation of the cefficients $\Omega_{j,k}^{m,n}(x)$, for $x = 1, 2, 3, 4, 5$ and $j, k \in \mathbb{Z}$, is a first step in the numerical solution of the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

– the Burgers equation. The approach to the solution of Burgers equation will be developed in a subsequent paper.

References

[1] A. Bultheel, *Learning to swim in a sea of wavelets.* Bull. Belg. Math. Soc., **2** (1995), 1-44.

[2] M. Q. Chen, C. Hwang and Y. P. Shih, *The computation of wavelet - Galerkin approximation on a bounded interval.* International Journal for Numerical Methods in Engineering, **39** (1996), 2921-2944.

[3] I. Daubechies, *Ten Lectures on Wavelets.* Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1992.

[4] I. Daubechies, *Orthonormal bases of compactly supported wavelets.* Commun. Pure Appl. Math., **41** (1988), 909-996.

[5] C. I. Popovici, *A finite element for the one-dimensional problem with two phases*. Studii şi cercetări matematice, 48 (1996), nr. 5-6, 383-391.

[6] C. I. Popovici, *The finite element method for solving the two-dimensional Stefan problem with two phases (I)*. Math. Reports, **3** (53) (2001), nr. 1, 67-74.

[7] C. I. Popovici, *The finite element method for solving the two-dimensional Stefan problem with two phases (II)*. Math. Reports, **3** (53) (2001), nr. 3, 273-286.

[8] C. I. Popovici, *A difference scheme for solving two dimensional Stefan's problem with two phases heat equation*. Studii şi cercetări ştiinţifice, Bacau, Matematica 6, (1996), 179-184.

[9] C. I. Popovici, *The numerical solving of a free boundary problem, on a fixed network*. Studii şi cercetări ştiinţifice, Bacau, Matematica 11, (2001), 165-170.

[10] C. I. Popovici and E. Popovici, *A finite difference scheme and the computer program for solving Stefan's problem two phases of the heat equation*. Bul. I.P. Iaşi, 1985, suplimentul Cercetări Matematice, 183-186.

[11] C. I. Popovici, *Numerical solving of Stefan problem with two phases*. Bul. I.P. Iaşi, 2000, tomul XLVI (L), suplimentul Mecanica Fluidelor, 109-114.

[12] C. I. Popovici and C. Filipescu, *Matlab evaluation of the $\Gamma_k^n(x)$ coefficients for PDE solving by wavelet - Galerkin approximation*. Libertas Mathematica, XXVII (2007), 85-93.

[13] C. I. Popovici, *Matlab evaluation of multiple integrals for the scalar wavelet function $\Phi(x)$*. Bul. I.P. Iaşi, Tomul LIII(LVII), Fasc. **5**, 2007, Section Mathematics, Theoretical Mechanics, Physics.

[14] M. Q. Chen, C. Hwang and Y. P. Shin, *A wavelet - Galerkin method for solving Stefan problems*. Journal of the Chin. I. Ch. E., 26, No. **2** (1995), 103-117.

Technical University "Gh. Asachi" Iassy,
Department of Mathematics,
11, Carol Bvd., Ro-700506, Iassy, Romania