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SEVERAL ASPECTS ON THE HYPERGROUPS ASSOCIATED WITH n-ARY RELATIONS

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Abstract

This paper deals with hypergroupoids obtained from n-ary relations. We give conditions for an n-ary relation such that the hypergroupoid associated with it is a hypergroup. First we analyze this construction using a ternary relation and then we generalize it for an n-ary relation.

1 Introduction and Preliminaries

The applications of mathematics in other disciplines, for example in informatics, play a key role and they represent, in the last decades, one of the purpose of the study of the experts of Hyperstructures Theory all over the world. This theory was born during the 8th Congress of Scandinavian Mathematicians since 1934, when F. Marty [11] defined hypergroups, a natural generalization of the concept of group, and presented some of their applications to non-commutative groups, algebraic functions, rational fractions. Since then various connections between hypergroups and other subjects of theoretical and applied mathematics have been established. The most important applications to geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy sets and rough sets, automata theory can be found in [6].

The first association between binary relations and hyperstructures appeared in J. Nieminem [13], who studied hypergroups related to connected

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simple graphs. In the same direction P. Corsini [3] worked, considering different hyperoperations associated with graphs. J. Chvalina [1] used ordered structures for the construction of semihypergroups and hypergroups. Later on, I. Rosenberg [17] extended Chvalina's definition, introducing a hyperoperation obtained by a binary relation; the new hypergroupoid has been investigated by P. Corsini [4], P. Corsini and V. Leoreanu [7] and recently by I. Cristea and M. Ştefănescu [8, 9]. Lately, S. Spartalis and C. Mamaloukas [18], and Ch.G. Massouros and Ch. Tsitouras [12] have created several algorithms in order to calculate the number of non isomorphic hypergroupoids determined by a binary relation on a finite set. The study of the relations between two sets, in particular on a set, points out important properties of the discrete structures. But also relationships among elements of more than two sets often arise and may be expressed in terms of n-ary relations.

This paper deals with hypergroupoids obtained from n-ary relations. The n-ary relations were studied for their applications in the theory of dependence space. Moreover, they are used in Database Theory, providing a convenient tool for database modeling. First we associate a hypergroupoid with a ternary relation defined on a nonempty set H and investigate when the new hypergroupoid is a hypergroup. We use some fundamental operations on databases to generalize the results to the case of n-ary relations. Finally we discuss some aspects connected with the reduced hypergroupoids.

For a nonempty set H, we denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H. A nonempty set H, endowed with a mapping, called *hyperoperation*, $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$ is called a *hypergroupoid*. A hypergroupoid which verifies the following conditions:

1. $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$
2. $x \circ H = H = H \circ x$, for all $x \in H$ (reproduction axiom)

is called a *hypergroup*.
If, for any $x, y \in H$, $x \circ y = H$, then $\langle H, \circ \rangle$ is called the *total hypergroup*.
If A and B are nonempty subsets of H, then we denote $A \circ B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a \circ b$.
We say that two partial hypergroupoids $\langle H, \circ_1 \rangle$ and $\langle H, \circ_2 \rangle$ are *mutually associative* if, for any $x, y, z \in H$ we have

$$(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z); (x \circ_2 y) \circ_1 z = x \circ_2 (y \circ_1 z).$$

For each pair $(a, b) \in H^2$, set $a/b = \{x \mid a \in x \circ b\}$ and $b \backslash a = \{y \mid a \in b \circ y\}$.
If A and B are nonempty subsets of H, then we denote $A/B = \bigcup\limits_{\substack{a \in A \\ b \in B}} a/b$.

A commutative hypergroupoid $\langle H, \circ \rangle$ is called a *join space* if the following implication holds: for any $(a, b, c, d) \in H^4$,

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset \quad \text{("transposition axiom")}.$$

For more details on hypergroup theory, see [2].

2 Properties of the n-ary relations

In this section we present some basic notions about the n-ary relations defined on a set H. We suppose that $H \neq \emptyset$ is a set, $n \in \mathbb{N}$ a natural number such that $n \geq 2$, and $\rho \subseteq H^n$ is an n-ary relation on H.

Definition 1. (see[15]) The relation ρ is said to be

1. *reflexive* if, for any $x \in H$, the n-tuple $(x, \ldots, x) \in \rho$;
2. *n-transitive* if it has the following property: if $(x_1, \ldots, x_n) \in \rho$, $(y_1, \ldots, y_n) \in \rho$ hold and if there exist natural numbers $i_0 > j_0$ such that $1 < i_0 \leq n$, $1 \leq j_0 < n$, $x_{i_0} = y_{j_0}$, then the n-tuple $(x_{i_1}, \ldots, x_{i_k}, y_{j_{k+1}}, \ldots, y_{j_n}) \in \rho$ for any natural number $1 \leq k < n$ and $i_1, \ldots, i_k, j_{k+1}, \ldots, j_n$ such that $1 \leq i_1 < \ldots < i_k < i_0$, $j_0 < j_{k+1} < \ldots < j_n \leq n$;
3. *symmetric* if $(x_1, x_2, \ldots, x_n) \in \rho$ implies $(x_n, x_{n-1}, \ldots, x_1) \in \rho$;
4. *strongly symmetric* if $(x_1, \ldots, x_n) \in \rho$ implies $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in \rho$ for any permutation σ of the set $\{1, \ldots, n\}$;
5. *n-ary preordering* on H if it is reflexive and n-transitive;
6. *n-equivalence* on H if it is reflexive, strongly symmetric and n-transitive.

Example 2. A ternary relation ρ is 3-transitive if and only if it satisfies the following conditions:

(i) If $(x, y, z) \in \rho$, $(y, u, v) \in \rho$, then $(x, u, v) \in \rho$.

(ii) If $(x, y, z) \in \rho$, $(z, u, v) \in \rho$, then $(x, y, u) \in \rho$, $(x, y, v) \in \rho$, $(x, u, v) \in \rho$, $(y, u, v) \in \rho$.

(iii) If $(x, y, z) \in \rho$, $(u, z, v) \in \rho$, then $(x, y, v) \in \rho$.

Lemma 3. *If ρ is an n-ary preordering, then any projection $\rho_{1,i,n}$, $2 \leq i \leq n-1$, is a 3-ary preordering.*

Proof. The reflexivity is immediately. Set an arbitrary i in $\{2, \ldots, n-1\}$. We prove that $\rho_{1,i,n}$ is 3-transitive in three steps (see the Example 3):

(i) If $(x, y, z) \in \rho_{1,i,n}$ and $(y, u, v) \in \rho_{1,i,n}$, then there exist $a_1, \ldots, a_{n-3}$, $b_1, \ldots, b_{n-3} \in H$ such that $(x, a_1, \ldots, a_{i-2}, y, a_{i-1}, \ldots, a_{n-3}, z) \in \rho$ and $(y, b_1, \ldots, b_{i-2}, u, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$; by the n-transitivity of the relation ρ it follows that $(x, b_1, \ldots, b_{i-2}, u, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$, that is $(x, u, v) \in \rho_{1,i,n}$.

(ii) If $(x, y, z) \in \rho_{1,i,n}$ and $(z, u, v) \in \rho_{1,i,n}$, then there exist $a_1, \ldots, a_{n-3}$, $b_1, \ldots, b_{n-3} \in H$ such that $(x, a_1, \ldots, a_{i-2}, y, a_{i-1}, \ldots, a_{n-3}, z) \in \rho$ and $(z, b_1, \ldots, b_{i-2}, u, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$; by the n-transitivity of the relation ρ it follows that:
$(x, b_1, \ldots, b_{i-2}, u, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$, that is $(x, u, v) \in \rho_{1,i,n}$,
$(y, b_1, \ldots, b_{i-2}, u, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$, that is $(y, u, v) \in \rho_{1,i,n}$,
$(x, a_1, \ldots, a_{i-2}, y, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$, that is $(x, y, v) \in \rho_{1,i,n}$,
$(x, a_1, \ldots, a_{i-2}, y, a_{i-1}, \ldots, a_{n-3}, u) \in \rho$, that is $(x, y, u) \in \rho_{1,i,n}$.

(iii) If $(x, y, z) \in \rho_{1,i,n}$ and $(u, z, v) \in \rho_{1,i,n}$, then there exist $a_1, \ldots, a_{n-3}$, $b_1, \ldots, b_{n-3} \in H$ such that $(x, a_1, \ldots, a_{i-2}, y, a_{i-1}, \ldots, a_{n-3}, z) \in \rho$ and $(u, b_1, \ldots, b_{i-2}, z, b_{i-1}, \ldots, b_{n-3}, v) \in \rho$; by the n-transitivity it follows that: $(x, a_1, \ldots, a_{i-2}, y, a_{i-1}, \ldots, a_{n-3}, v) \in \rho$, that is $(x, y, v) \in \rho_{1,i,n}$.

□

Lemma 4. *Let ρ be an n-ary relation on H such that $\rho_{1,n} = H \times H$.*
If ρ is strongly symmetric and n-transitive, then ρ is an n-equivalence on H.

Proof. Set x arbitrary in H. Since $\rho_{1,n} = H \times H$, there exist $u_1, \ldots, u_{n-2} \in H$ such that $(x, u_1, \ldots, u_{n-2}, x) \in \rho$ and by the symmetry it follows that $(u_1, \ldots, u_{n-2}, x, x) \in \rho$. Using the n-transitivity, $(x, u_2, \ldots, u_{n-2}, x, x) \in \rho$. Again by the symmetry, we obtain that $(u_2, \ldots, u_{n-2}, x, x, x) \in \rho$ and therefore, by the n-transitivity, that $(x, u_3, \ldots, u_{n-2}, x, x, x) \in \rho$ and so on; finally it results that $(x, x, \ldots, x) \in \rho$, for any $x \in H$, so ρ is an n-equivalence on H.
□

We also obtain the following two immediate corollaries.

Corollary 5. *If ρ is an n-ary equivalence, then any projection $\rho_{1,i,n}$, $2 \leq i \leq n-1$, is a 3-ary equivalence.*

Corollary 6. *If ρ is a strongly symmetric n-ary relation, then, for any $i \neq j \in \{2, \ldots, n-1\}$, the following equivalence holds:*

$$(x, y, z) \in \rho_{1,i,n} \Longleftrightarrow (x, y, z) \in \rho_{1,j,n}.$$

There are esentially two types of operations with n-ary relations useful to describe information manipulations in databases: the first type permits the construction of new databases (union, intersection, diference, Cartesian product) and the second type may be characterized as operations that are motivated by the information manipulations (projection, join).

In the following we recall the definitions of the projection and the join.

Definition 7. Let ρ be an n-ary relation on a nonempty set H and $k < n$. The $(i_1, \ldots, i_k)$*-projection* of ρ, denoted by $\rho_{i_1,\ldots,i_k}$, is a k-ary relation on H defined by: if $(a_1, \ldots, a_n) \in \rho$, then $(a_{i_1}, \ldots, a_{i_k}) \in \rho_{i_1,\ldots,i_k}$.

Definition 8. Let ρ be an m-ary relation on a nonempty set H, λ an n-ary relation on the same set H. The *join* of ρ and λ, denoted by $J_p(\rho, \lambda)$, where $1 \leq p < m, 1 \leq p < n$, is an $(m + n - p)$-relation on H that consists of all $(m + n - p)$-tuples $(a_1, \ldots, a_{m-p}, c_1, \ldots, c_p, b_1, \ldots, b_{n-p})$ such that $(a_1, \ldots, a_{m-p}, c_1, \ldots, c_p) \in \rho$ and $(c_1, \ldots, c_p, b_1, \ldots, b_{n-p}) \in \lambda$.

Example 9. On $H = \{1, 2, 3, 4, 5, 6\}$ we define the 5-ary relation ρ and the ternary relation λ:
$\rho = \{(1, 1, 3, 4, 1), (2, 3, 5, 2, 1), (3, 3, 4, 5, 1),$
$\lambda = \{(4, 1, 6), (5, 1, 2), (3, 4, 3), (2, 2, 2)\}.$
The $\rho_{2,4,5}$ projection is the ternary relation $\rho_{2,4,5} = \{(1, 4, 1), (3, 2, 1), (3, 5, 1)\}$; the join $J_2(\rho, \lambda)$ is the 6-ary relation $J_2(\rho, \lambda) = \{(1, 1, 3, 4, 1, 6), (3, 3, 4, 5, 1, 2)\}$.

3 Hypergroups associated with n-ary relations

In the paper [19] M. Ştefănescu proposes, as a problem, the study of the hyperstructure associated with a ternary relation as follows:

$$(a, b, c) \in \rho \text{ if and only if } c \in a \circ b.$$

In this section we consider this problem, but in a little different form.

Let ρ be a ternary relation on H. For any $x, y \in H$, we define the hyperproduct

$$x \circ_\rho y = \{z \mid z \in H, (x, z, y) \in \rho\}.$$

We notice that $\langle H, \circ_\rho \rangle$ is a hypergroupoid if and only if, for any $x, y \in H$, there exists $z \in H$ such that $(x, z, y) \in \rho$, that is the projection $\rho_{1,3}$ is the total relation, i.e. $\rho_{1,3} = H \times H$.

Now we generalize the definition of the hyperproduct associated with a ternary relation to the case of n-ary relation, $n \geq 3$, defined on a nonempty set H.

Definition 10. Let ρ be an n-ary relation on H. For any $i \in \{2, \ldots, n-1\}$, using the projections $\rho_{1,i,n}$, we define the hyperproducts:

$$x \circ_i y = \{z \in H \mid (x, z, y) \in \rho_{1,i,n}\}$$

and

$$x \circ_\rho y = \{z \in H \mid (x, z, y) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n}\} = \bigcup_{i=2}^{n-1} x \circ_i y.$$

We notice that $\langle H, \circ_\rho \rangle$ is a hypergroupoid if and only if the projection $\rho_{1,n}$ is the total relation, i.e. $\rho_{1,n} = H \times H$.

In the following we determine neccessary conditions such that the hypergroupoid $\langle H, \circ_\rho \rangle$ is a hypergroup or a join space.

Proposition 11. *Let ρ be an n-ary relation on H. Then $\langle H, \circ_\rho \rangle$ is a quasihypergroup if and only if $\rho_{1,n} = H \times H$ and there exist i, j, with $2 \leq i, j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n} = H \times H$.*

Proof. We know that a hypergroupoid $\langle H, \circ \rangle$ is a quasihypergroup if and only if, for any $x \in H$, $x \circ H = H = H \circ x$. Therefore $\langle H, \circ_\rho \rangle$ is a quasihypergroup iff , for any $x, y \in H$, there exist $z, t \in H$, such that $y \in x \circ_\rho z \cap t \circ_\rho x$, that is $(x, y, z) \in \bigcup_{i=2}^{n-1} \rho_{1,i,n} \ni (t, y, x)$. It follows that $\langle H, \circ_\rho \rangle$ is a quasihypergroup iff there exist i, j, with $2 \leq i, j \leq n-1$, such that $(x, y, z) \in \rho_{1,i,n}$ and $(t, y, x) \in \rho_{1,j,n}$, so $\rho_{1,i} = \rho_{j,n} = H \times H$. □

Proposition 12. *Let ρ be an n-ary relation on H such that $\rho_{1,n} = H \times H$. If ρ is a preordering, then $\langle H, \circ_\rho \rangle$ is the total hypergroup.*

Proof. We have to prove that, for any $x, y, z \in H$, $z \in x \circ_\rho y$.
Set $x, y, x \in H$. Since $\rho_{1,n} = H \times H$, there exist $a_1, a_2, \ldots, a_{n-2} \in H$, such that $(z, a_1, \ldots, a_{n-2}, y) \in \rho$. By the reflexivity, $(z, z, \ldots, z) \in \rho$ and then, by the n-transitivity, it follows that $(z, \ldots, z, y) \in \rho$. Again since $\rho_{1,n} = H \times H$, there exist $c_1, \ldots, c_{n-2} \in H$ such that $(x, c_1, \ldots, c_{n-2}, z) \in \rho$. Using the n-transitivity for $(z, \ldots, z, y) \in \rho$ and $(x, c_1, \ldots, c_{n-2}, z) \in \rho$, we obtain that $(x, z, \ldots, z, y) \in \rho$, so $z \in x \circ_\rho y$. □

Proposition 13. *Let ρ be an n-ary relation on H such that $\rho_{1,n} = H \times H$. If ρ is n-ary transitive and strongly symmetric, then $\langle H, \circ_\rho \rangle$ is the total hypergroup.*

Proof. By Lemma 4, ρ is an n-equivalence on H and by the previous proposition, $\langle H, \circ_\rho \rangle$ is the total hypergroup. □

Proposition 14. *Let ρ be a ternary relation on H such that $\rho_{1,3} = \rho_{1,2} = H \times H$ or $\rho_{1,3} = \rho_{2,3} = H \times H$. If ρ is 3-transitive, then $\langle H, \circ_\rho \rangle$ is the total hypergroup.*

Proof. We prove that ρ is reflexive and then, by Proposition 12, it results the conclusion. We suppose that $\rho_{1,3} = \rho_{1,2} = H \times H$; thus, for any $x \in H$, there exist $a_x, c_x \in H$ such that $(x, a_x, x) \in \rho$ and $(a_x, x, c_x) \in \rho$. Then by the 3-transitivity, $(x, x, c_x) \in \rho$. Again, since $\rho_{1,3} = H \times H$ it follows that there exists $u_x \in H$ such that $(c_x, u_x, x) \in \rho$. Using the 3-transitivity for $(x, x, c_x), (c_x, u_x, x) \in \rho$, we get $(x, x, x) \in \rho$, that is ρ is a 3-preordering on H. $\square$

Corollary 15. *Let ρ be an n-ary relation on H $(n \leq 4)$ such that there exist i, j, with $2 \leq i, j \leq n-1$, such that $\rho_{1,i} = \rho_{j,n} = \rho_{1,n} = H \times H$. If ρ is n-transitive, then $\langle H, \circ_\rho \rangle$ is the total hypergroup.*

Proof. If ρ is n-transitive, then the ternary relations $\lambda = \rho_{1,i,n}$ and $\alpha = \rho_{1,j,n}$ are 3-transitive such that $\lambda_{1,2} = \lambda_{1,3} = H \times H$ and $\alpha_{2,3} = \alpha_{1,3} = H \times H$. By Proposition 14 it follows that the hypergroupoids $\langle H, \circ_i \rangle$ and $\langle H, \circ_j \rangle$ are total hypergroups. therefore, for any $x, y \in H$, we have $x \circ_\rho y = \bigcup_{l=2}^{n-1} x \circ_{\rho_{1,l,n}} y \supseteq x \circ_{\rho_{1,i,n}} y = H$, so $x \circ_\rho y = H$, that is, $\langle H, \circ_\rho \rangle$ is the total hypergroup. $\square$

Let ρ be a ternary relation on H. We denote the join relation $J_2(\rho, \rho)$ by α.
We recall that $J_2(\rho, \rho)$ is a 4-relation such that

$$(x, y, z, t) \in J_2(\rho, \rho) \Longleftrightarrow (x, y, z), (y, z, t) \in \rho.$$

Using projections of $J_2(\rho, \rho)$, we give a neccessary condition for a hypergroupoid $\langle H, \circ_\rho \rangle$ to be a semihypergroup.

Proposition 16. *Let ρ be a reflexive and symmetric ternary relation on H. If $\rho \not\subset \alpha_{1,2,4}$ or $\rho \not\subset \alpha_{1,3,4}$, then the hyperoperation "$\circ_\rho$" is not associative.*

Proof. We suppose that $\rho \not\subset \alpha_{1,2,4}$; then there exists $(x, y, z) \in \rho \setminus \alpha_{1,2,4}$ and thus, for any $a \in H$,

$$(x, y, a, z) \notin J_2(\rho, \rho). \tag{1}$$

Since ρ is reflexive and $(x, y, z) \in \rho$, it follows that $y \in y \circ_\rho y \subset (x \circ_\rho z) \circ_\rho y$. To prove that "$\circ_\rho$" is not associative, it is enough to show that $y \notin x \circ_\rho (z \circ_\rho y)$, that is, for any $a \in z \circ_\rho y$, $y \notin x \circ_\rho a$.

Let's suppose by the absurd that there exists $\bar{a} \in z \circ_\rho y$, with $y \in x \circ_\rho \bar{a}$. Then, $(z, \bar{a}, y) \in \rho$ (and by the symmetry $(y, \bar{a}, z) \in \rho$) and $(x, y, \bar{a}) \in \rho$. Therefore it results that $(x, y, \bar{a}, z) \in J_2(\rho, \rho)$, which is a contradiction with (1). Similarly if we suppose $\rho \not\subset \alpha_{1,3,4}$. $\square$

Corollary 17. *Let ρ be a reflexive and symmetric ternary relation on H. If $\langle H, \circ_\rho \rangle$ is a semihypergroup, then $\rho \subset \alpha_{1,2,4} \cap \alpha_{1,3,4}$.*

In the following we give a similar result for the n-ary relations. Let ρ be an n-ary relation on H and α be the join relation $J_2(\rho, \rho)$; then α is a $(2n-2)$-relation on H.

Proposition 18. *Let ρ be a reflexive and symmetric n-ary relation on H which satisfies the condition:*

$$(S) : (x, a_1, \ldots, a_{n-2}, y) \in \rho \Longleftrightarrow (x, a_{\sigma(1)}, \ldots, a_{\sigma(n-2)}, y) \in \rho,$$

for any permutation σ of the set $\{1, 2, \ldots, n-2\}$.
If, there exists $j \in \{2, \ldots, n-1\}$ such that $\rho_{1,j,n} \not\subset \alpha_{1,n-1,2n-2}$ or $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2}$, then the hyperoperation "$\circ_\rho$" is not associative.

Proof. First we suppose there exists $j \in \{2, \ldots, n-1\}$ such that $\rho_{1,j,n} \not\subset \alpha_{1,n,2n-2}$. Then there exists $(x, y, z) \in \rho_{1,j,n}$ such that $(x, y, z) \notin \alpha_{1,n,2n-2}$. Therefore, $y \in x \circ_\rho z$ and since ρ is reflexive, it follows that $y \in y \circ_\rho (x \circ_\rho z)$. We prove that $y \notin (y \circ_\rho x) \circ_\rho z$, and thus the hyperoperation "$\circ_\rho$" is not associative.
Let's suppose by the absurd that $y \in (y \circ_\rho x) \circ_\rho z$; then there exists $a \in y \circ_\rho x$ such that $y \in a \circ_\rho z$. It follows that $(y, \ldots, a, \ldots, x) \in \rho$ and $(a, \ldots, y, \ldots, z) \in \rho$. By the simmetry and the condition (S), we obtain that $(x, \ldots, a, y) \in \rho$ and $(a, y, \ldots, z) \in \rho$ and therefore $(x, \ldots, a, y, \ldots, z) \in J_2(\rho, \rho)$, that is $(x, y, z) \in \alpha_{1,n,2n-2}$ which contradicts our supposition.

We proceed similarly if we suppose there exists $j \in \{2, \ldots, n-1\}$ such that $\rho_{1,j,n} \not\subset \alpha_{1,n-1,2n-2}$: one can prove that $y \in (x \circ_\rho z) \circ_\rho y$, but $y \notin x \circ_\rho (z \circ_\rho y)$, so the hyperoperation "$\circ_\rho$" is not associative. □

4 An example of reduced hypergroup

It may happen that the hyperoperation "$\circ$" does not discriminate between a pair of elements of H, when two elements play interchangeable roles with respect to the hyperoperation. On a hypergroupoid $\langle H, \circ \rangle$, the following three equivalence relations, called *the operational equivalence, the inseparability* and *the essential indistinguishability*, respectively, may be defined (see [10]):

- $x \sim_o y \Longleftrightarrow x \circ a = y \circ a$ and $a \circ x = a \circ y$, for any $a \in H$;
- $x \sim_i y \Longleftrightarrow$ for $a, b \in H$, we have $x \in a \circ b \Longleftrightarrow y \in a \circ b$;
- $x \sim_e y \Longleftrightarrow x \sim_o y$ and $x \sim_i y$.

For any $x \in H$, let $\widehat{x}_o$, $\widehat{x}_i$ and $\widehat{x}_e$, respectively, denote the equivalence classes of x with respect to the relations $\sim_o, \sim_i$ and $\sim_e$.

We say that a hypergroupoid $\langle H, \circ \rangle$ is *reduced* if and only if, for any $x \in H$, $\widehat{x}_e = \{x\}$.

Let H be a nonempty set such that $H = \bigcup_{i \in I} H_i$, where $\{H_i\}_{i \in I}$ is a family of subsets of H indexed by a set I. We define on H the binary relation ρ^b as follows

$$(x, y) \in \rho^b \Longleftrightarrow \exists i \in I : x, y \in H_i.$$

Using the join operation we obtained the ternary relation ρ defined as:

$$\begin{aligned} \rho = J_1(\rho^b, \rho^b) \quad &= \{(x, z, y) \mid (x, z) \in \rho^b \wedge (z, y) \in \rho^b\} = \\ &= \{(x, z, y) \mid \exists i, j \in I : x, z \in H_i \wedge z, y \in H_j\}. \end{aligned}$$

As in the previous section, we consider the hyperproduct on H defined by

$$x \circ_\rho y = \{z \mid (x, z, y) \in \rho\}. \tag{2}$$

It is obvious that

$$x \circ_\rho y = x \circ_\rho x \cap y \circ_\rho y$$

and if we denote $x \circ_\rho x$ by H_x, then

$$x \circ_\rho y = H_x \cap H_y.$$

We notice that $\langle H, \circ_\rho \rangle$ is a partial hypergroupoid and it is a hypergroupoid if and only if, for any $i \neq j \in I$, $H_i \cap H_j \neq \emptyset$. We are interested in finding conditions such that the hypergroupoid $\langle H, \circ_\rho \rangle$ is reduced.

Proposition 19. *Let $\langle H, \circ_\rho \rangle$ be a partial hypergroupoid associated with a ternary relation as in* (2)*. For any $x, y \in H$, $x \sim_e y$ if and only if $H_x = H_y$.*

Proof. First we notice that, for any $a, b \in H$, $a \in H_b$ if and only if $b \in H_a$.

Let x, y be in H such that $x \sim_o y$, that is $x \circ_\rho a = y \circ_\rho a$, for any $a \in H$; thus $x \circ_\rho x = x \circ_\rho y = y \circ_\rho x = y \circ_\rho y$ and then $H_x = H_y$.
Conversely, set $x, y \in H$ such that $H_x = H_y$; then, for any $a \in H$, we have $H_x \cap H_a = H_y \cap H_a$, therefore $x \circ_\rho a = y \circ_\rho a$, so $x \sim_o y$.

Now set $x, y \in H$ such that $x \sim_i y$, thus $x \in a \circ_\rho b$ if and only if $y \in a \circ_\rho b$. Let $z \in H_x$, then $x \in H_z = z \circ_\rho z$ and since $x \sim_i y$, it follows that $y \in H_z$, thus $z \in H_y$. So $H_x \subset H_y$ and similarly one can prove that $H_y \subset H_x$.
Conversely, set $x, y \in H$ such that $H_x = H_y$. If $x \in a \circ_\rho b = H_a \cap H_b$, with $a, b \in H$, then $a, b \in H_x = H_y$ and therefore $y \in H_a \cap H_b = a \circ_\rho b$, so $x \sim_i y$. $\square$

Proposition 20. *Let $\{H_i\}_{i\in I}$ be a partition of a nonempty set H. Then the family of the equivalence classes respected to the equivalence $\sim_e$ on $\langle H, \circ_\rho\rangle$ coincides with the family $\{H_i\}_{i\in I}$. Moreover, the partial hypergroupoid $\langle H, \circ_\rho\rangle$ is reduced if and only if , for any $i \in I$, $\mid H_i \mid= 1$.*

Proof. Let $\{H_i\}_{i\in I}$ be a partition of H. For any $x \neq y \in H$, there exist and are unique $i_0, j_0 \in I$ such that $x \in H_{i_0}$ and $y \in H_{j_0}$. Since $\{H_i\}_{i\in I}$ is a partition of H, it follows that $H_x = H_{i_0}$ and $H_y = H_{j_0}$ and then

$$x \sim_e y \Longleftrightarrow H_x = H_y \Longleftrightarrow \exists! i_0 \in I : x, y \in H_{i_0},$$

so the family of the equivalence classes respected to "$\sim_e$" is $\{H_i\}_{i\in I}$.

Moreover, $\langle H, \circ_\rho\rangle$ is reduced if and only if, for any $x \neq y \in H$, $x \not\sim_e y$, which is equivalent with $H_x = \{x\}$, for any $x \in H$, so $\mid H_i \mid= 1$, for any $i \in I$. □

5 Conclusions and future work

Many connections between hypergroups and binary relations have been considered and investigated. In this paper we associated, firstly with a ternary relation, and secondly with an n-ary relation, a partial hypergroupoid. We determined neccessary conditions such that the obtained hyperstructure is a hypergroup. Let ρ be a ternary relation on H. For any $x, y \in H$, we define the hyperproduct

$$x \circ_\rho y = \{z \mid z \in H, (x, z, y) \in \rho\}.$$

We notice that we can do the association also in the other direction: given a hypergroupoid $\langle H, \circ\rangle$, we define the ternary relation ρ on H setting $(x, y, z) \in \rho$ if and only if $y \in x \circ z$. This is the unique ternary relation obtained in this way. We can not say the same thing in the case of n-ary relation, $n \geq 4$.

The operations on databases (such union, intersection, Cartesian product, projection, join) can be extended to similar operations on n-ary relations that permit to construct new hypergroupoids. It would be interesting to determine some relationships between the hypergroupoids associated with two n-ary relations ρ and λ and the hypergroupoids associated with the union, intersection, join, Cartesian product of ρ and λ.

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