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ON THE DETERMINANT OF AN HODGE-DE RHAM LIKE OPERATOR

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Abstract

Let (M, g) be a closed (i.e. compact without boundary), orientable, n-dimensional smooth Riemannian manifold, $C^\infty(M)$ the real algebra of smooth real functions on M, $A^k(M)$ the $C^\infty(M)$-module of smooth differential k-forms, $0 \leq k \leq n$, and $\mathbf{h}$ a pointwise nonsingular tensor field of type (1,1) on M with vanishing Nijenhuis tensor. For such $\mathbf{h}$ there is an associated exterior derivative $d_{\mathbf{h}}^{(k)} : A^k(M) \to A^{k+1}(M)$ having an adjoint $\delta_{\mathbf{h}}^{(k+1)} : A^{k+1}(M) \to A^k(M)$ with respect to the usual global inner product, so that one can define a (strongly) elliptic self-adjoint second order differential operator $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$, which is a generalization of the Hodge-de Rham operator (see [4]). In this paper we shall discuss the smooth dependence on the Riemannian metric of the determinant of this $\mathbf{h}$-dependent Hodge-de Rham operator.

1 Introduction

Let (M, g) be a closed (i.e. connected, compact, and without boundary), oriented, n-dimensional smooth Riemannian manifold, $C^\infty(M)$ the real algebra of smooth real functions on M and $A^k(M)$ the $C^\infty(M)$-module of smooth differential k-forms, $0 \leq k \leq n$. A tensor field $\mathbf{h}$ of type $(1, 1)$ on M, which can be conceived as a $C^\infty(M)$-linear mapping $\mathbf{h} : A^1(M) \to A^1(M)$, induces $C^\infty(M)$-linear mappings $\mathbf{h}^{(q)} : A^k(M) \to A^k(M)$ for any nonnegative integer q, and the $\mathbf{h}^{(q)}$ are defined by setting $\mathbf{h}^{(q)} := 0$ if $q > k$, and

$$\mathbf{h}^{(q)}(\omega^1 \wedge \ldots \wedge \omega^k) :=$$

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$$= \frac{1}{(k-q)!q!} \sum_{\sigma \in \mathcal{S}_k} sgn(\sigma)[\mathbf{h}(\omega^{\sigma(1)}) \wedge \ldots \wedge \mathbf{h}(\omega^{\sigma(q)})] \wedge \omega^{\sigma(q+1)} \wedge \ldots \wedge \omega^{\sigma(k)}$$

if $0 \leq q \leq k$, where $\omega^i \in A^1(M)$, $1 \leq i \leq k$, σ runs through all permutations of $\{1, \ldots, k\}$, and $sgn(\sigma)$ denotes the sign of the permutation σ. The transformation $\mathbf{h}^{(0)}$ is taken to be the identity mapping on $A^k(M)$.

If $\mathbf{h}$ is a pointwise nonsingular tensor field of type $(1,1)$ on M with vanishing Nijenhuis derivation, an $\mathbf{h}$-dependent exterior derivation $d_{\mathbf{h}}^{(k)} : A^k(M) \to A^{k+1}(M)$ can be defined by setting $d_{\mathbf{h}}^{(k)} := \mathbf{h}^{(1)} \circ d^{(k)} - d^{(k)} \circ \mathbf{h}^{(1)}$, where $d^{(k)} : A^k(M) \to A^{k+1}(M)$ is the usual exterior differential. When $\mathbf{h}$ is the identity, $d_{\mathbf{h}}^{(k)}$ coincides with $d^{(k)}$. Let $\delta^{(k+1)} : A^{k+1}(M) \to A^k(M)$ be the adjoint of $d^{(k)}$ relative to the usual inner product induced by g. If $\mathbf{h}_t^{(1)}$ denotes the transpose of $\mathbf{h}^{(1)}$, the mapping $\delta_{\mathbf{h}}^{(k+1)} : A^{k+1}(M) \to A^k(M)$, defined by setting $\delta_{\mathbf{h}}^{(k+1)} := \delta^{(k+1)} \circ \mathbf{h}_t^{(1)} - \mathbf{h}_t^{(1)} \circ \delta^{(k+1)}$, is the adjoint of $d_{\mathbf{h}}^{(k)}$ with respect to the usual inner product. The $\mathbf{h}$-dependent Hodge-de Rham operator $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$, defined by setting $\Delta_{\mathbf{h}}^{(k)} := d_{\mathbf{h}}^{(k-1)} \circ \delta_{\mathbf{h}}^{(k)} + \delta_{\mathbf{h}}^{(k+1)} \circ d_{\mathbf{h}}^{(k)}$, is elliptic and self-adjoint, and it reduces to the usual Hodge-de Rham operator $\Delta^{(k)}$ in the case when $\mathbf{h}$ is the identity (see [4]).

2 Spectral properties of the h-dependent Hodge-de Rham operators

In what follows we assume that $\mathbf{h}$ is a nonsingular tensor field of type $(1,1)$ on M with vanishing Nijenhuis derivation. Since $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ is an elliptic second order differential operator, formally self-adjoint and formally positive, and (M,g) is a closed, smooth Riemannian manifold, the following statements are valid (see also [2]).

Theorem 2.1 *(i) For each $k \in \{0, 1, \ldots, n\}$, there exists a discrete spectral resolution $\{\omega_{\mathbf{h};j}^{(k)}, \lambda_{\mathbf{h};j}^{(k)}\}_{j \in \mathbb{N}}$ of the operator $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$, where $\omega_{\mathbf{h};j}^{(k)} \in A^k(M)$ for any $j \in \mathbb{N}$, that is $\{\omega_{\mathbf{h};j}^{(k)}\}_{j \in \mathbb{N}}$ is a complete orthonormal system in the real Hilbert space $L^2(A^k(M)) = W^{0,2}(A^k(M))$ such that $\Delta_{\mathbf{h}}^{(k)} \omega_{\mathbf{h};j}^{(k)} = \lambda_{\mathbf{h};j}^{(k)} \omega_{\mathbf{h};j}^{(k)}$ for any $j \in \mathbb{N}$. Moreover, $\lambda_{\mathbf{h};j}^{(k)} \in [0, +\infty)$ for any $j \in \mathbb{N}$, each eigenspace of $\Delta_{\mathbf{h}}^{(k)}$ is finite dimensional, and $0 \in \mathbb{R}$ is an eigenvalue of $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ if and only if the k-th Betti number $\beta_k(M) \neq 0$;*

(ii) For any $m \in \mathbb{N}$, there exists $\ell(m) \in \mathbb{N}^*$ and a constant $C_m > 0$, such that

$$||\omega_{\mathbf{h};j}^{(k)}||_{\infty,m} \leq C_m (1 + (\lambda_{\mathbf{h};j}^{(k)})^{\ell(m)}),$$

where

$$||\omega_{\mathbf{h};j}^{(k)}||_{\infty,m} := \sup_{x\in M} |j_m(\omega_{\mathbf{h};j}^{(k)})_x|_{J^m(\Lambda^k T^* M)_x},$$

$j_m(\omega_{\mathbf{h};j}^{(k)})_x$ *denoting the* m*-jet of the smooth section* $\omega_{\mathbf{h};j} \in C^\infty(\Lambda^k T^* M)$ *at the point* $x \in M$*;*

(iii) *If one arrange the eigenvalues of* $\Delta_{\mathbf{h}}^{(k)}$ *such that*

$$0 \leq \lambda_{\mathbf{h};0}^{(k)} \leq \lambda_{\mathbf{h};1}^{(k)} \leq \dots,$$

then there exist real constants $C(k) > 0$ *and* $\varepsilon(k) > 0$ *such that* $\lambda_{\mathbf{h};j}^{(k)} \geq C(k) j^{\varepsilon(k)}$ *if* $j \geq j_0$ *is sufficiently large;*

(iv) *Let* $a_j^{(k)} := \langle \theta, \omega_{\mathbf{h};j}^{(k)} \rangle \in \mathbb{R}$*,* $j \in \mathbb{N}$*, be the Fourier coefficients associated to* $\theta \in L^2(A^k(M))$*. If* $\theta \in A^k(M)$*, then*

$$\sum_{j\in\mathbb{N}} |a_j^{(k)}| \lambda_{\mathbf{h};j}^{(k)} < +\infty$$

and the series $\sum_{j\in\mathbb{N}} |a_j^{(k)}| \lambda_{\mathbf{h};j}^{(k)}$ *tends to* θ *uniformly with respect to the norm* $\|\cdot\|_{\infty,i}$ *for any* $k \in \{0, 1, \dots, n\}$*.*

For an initial smooth differential k-form $\theta \in A^k(M)$, let us consider the heat equation associated to $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ with the initial condition θ:

$$(\frac{\partial}{\partial t} + \Delta_{\mathbf{h}}^{(k)})\omega(x,t) = 0, \tag{1}$$

$$\lim_{t\searrow 0} \omega(x,t) = \theta(x), \tag{2}$$

where $x \in M$, $t \in (0, +\infty)$. Let $(e^{-t\Delta_{\mathbf{h}}^{(k)}})(\theta)$ be the unique solution of the evolution equation (1) that satisfies the initial condition (2). The linear operator $e^{-t\Delta_{\mathbf{h}}^{(k)}} : A^k(M) \to A^k(M)$ can be extended to a compact, self-adjoint operator from $L^2(A^k(M))$ into $L^2(A^k(M))$, denoted also by $e^{-t\Delta_{\mathbf{h}}^{(k)}}$. Let $\{\omega_{\mathbf{h};j}^{(k)}, \lambda_{\mathbf{h};j}^{(k)}\}$ be a discrete spectral resolution of $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ and let us consider the Fourier expansion of $\theta \in A^k(M)$:

$$\theta = \sum_{j\in\mathbb{N}} a_j^{(k)} \omega_{\mathbf{h};j}^{(k)}, \quad \text{where} \quad a_j^{(k)} := \langle \theta, \omega_{\mathbf{h};j}^{(k)} \rangle, \quad j \in \mathbb{N}.$$

Let us define $\omega_{\mathbf{h};j}^{*(k)} \in (A^k(M))^*$ by $\omega_{\mathbf{h};j}^{*(k)}(\varpi) := \langle \varpi, \omega_{\mathbf{h};j}^{(k)} \rangle$, $\varpi \in A^k(M)$, such that $\|\omega_{\mathbf{h};j}^{*(k)}\| = \|\omega_{\mathbf{h};j}^{(k)}\|$. With these notations, the following statement is valid.

Theorem 2.2 *The following two series converge uniformly in the C^ℓ-topology for any $\ell \in \mathbb{N}$ if $t \geq \delta > 0$:*

$$E_{\mathbf{h}}^{(k)}(x,y,t) := \sum_{j\in\mathbb{N}} e^{-t\lambda_{\mathbf{h};j}^{(k)}} \omega_{\mathbf{h};j}^{(k)}(x) \otimes \omega_{\mathbf{h};j}^{*(k)}(y) \in Hom(\Lambda^k(T_y^*M), \Lambda^k(T_x^*M))$$

and

$$(e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta))(x,t) := \sum_{j\in\mathbb{N}} e^{-t\lambda_{\mathbf{h};j}^{(k)}} a_j^{(k)} \omega_{\mathbf{h};j}^{(k)}(x) := \int_M (E_{\mathbf{h}}^{(k)}(x,y,t))\theta(y) d\mu_g(y),$$

where $x, y \in M$, and μ_g denotes the canonical measure on M associated to g.

Let us define the matrix of Fourier coefficients of the bounded linear operator

$$e^{-t\Delta_{\mathbf{h}}^{(k)}} : L^2(A^k(M)) \to L^2(A^k(M))$$

by

$$\begin{aligned} a_{ij}(E_{\mathbf{h}}^{(k)}) : &= \langle e^{-t\Delta_{\mathbf{h}}^{(k)}}(\omega_{\mathbf{h};i}^{(k)}), \omega_{\mathbf{h};j}^{(k)} \rangle = \\ &= \int_{x\in M} \int_{y\in M} (E_{\mathbf{h}}^{(k)}(x,y,t))(\omega_{\mathbf{h};i}^{(k)}(y) | \omega_{\mathbf{h};j}^{(k)}(x)) d\mu_g(x) d\mu_g(y), \end{aligned}$$

$i, j \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$, where $(\ |\ )$ denotes the pointwise inner product on $A^k(M)$ defined by using the fiber metric on $\Lambda^k T^* M$ induced by g.

Theorem 2.3 (i) *With the previous notations the following equalities*

$$\begin{aligned} Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) : &= \int_{x\in M} Trace_{\Lambda^k(T_x^*M)}(E_{\mathbf{h}}^{(k)}(x,x,t)) d\mu_g(x) = \\ &= \sum_{j\in\mathbb{N}} a_{jj}(E_{\mathbf{h};j}^{(k)}) = \sum_{j\in\mathbb{N}} e^{-t\lambda_{\mathbf{h};j}^{(k)}} \end{aligned}$$

are valid for each $k \in \{0, 1, \ldots, n\}$, that is the continuous linear operator $e^{-t\Delta_{\mathbf{h}}^{(k)}} : L^2(A^k(M)) \to L^2(A^k(M))$ is a trace class operator;

(ii) *If $(A(M), d_{\mathbf{h}})$ is the elliptic cochain complex previously defined, then*

$$\sum_{k=0}^{n} (-1)^k Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) = Index(A(M), d_{\mathbf{h}}) = \chi(M),$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M.

Finally, let us discuss the asymptotics of $Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}})$ as $t \searrow 0$ and relate these asymptotics to $Index(A(M), d_{\mathbf{h}})$.

For each $t \in (0,1)$ and $k \in \{0,1,\ldots,n\}$, the restriction of $E_{\mathbf{h}}^{(k)}(\cdot,\cdot,t)$ on the diagonal of $M \times M$ admits an asymptotic expansion. More precisely, the following statement is valid.

Lemma 2.4 *For each $k \in \{0,1,\ldots,n\}$ and each $m \in \mathbb{N}$, there exists a smooth mapping $e_m(\cdot, \Delta_{\mathbf{h}}^{(k)}) : M \to End(\Lambda^k(T^*M))$ such that*

(i) $e_m(x, \Delta_{\mathbf{h}}^{(k)}) \in End(\Lambda^k(T_x^*M))$ depends functorially on a finite number of jets of the symbol of the second order differential operator $\Delta_{\mathbf{h}}^{(k)}$;

(ii) $E_{\mathbf{h}}^{(k)}(x,x,t) \sim \sum_m e_m(x, \Delta_{\mathbf{h}}^{(k)}) t^{\frac{m-n}{2}}$ as $t \searrow 0$, for any $x \in M$;

(iii) $e_m(x, \Delta_{\mathbf{h}}^{(k)}) = 0$ for any $x \in M$ and any m odd.

For the proof, we refer to [5], Lemma 1.8.2.

Let
$$M \ni x \mapsto a_m(x, \Delta_{\mathbf{h}}^{(k)}) := Trace(e_n(x, \Delta_{\mathbf{h}}^{(k)})) \in \mathbb{R}$$
and
$$a_m(\Delta_{\mathbf{h}}^{(k)}) := \int_{x \in M} a_m(x, \Delta_{\mathbf{h}}^{(k)}) d\mu_g(x) \in \mathbb{R}, \tag{3.3}$$
the invariant scalar functions and the numerical invariants respectively associated to the differential operator $\Delta_{\mathbf{h}}^{(k)}$, $k \in \{0,1,\ldots,n\}$.

Lemma 2.5 *(i) The invariants $e_m(x, \Delta_{\mathbf{h}}^{(k)})$ and $a_m(x, \Delta_{\mathbf{h}}^{(k)})$, m even, can be expressed by local formulas that are homogeneous of order m in the jets of the total symbol of $\Delta_{\mathbf{h}}^{(k)}$ for each $k \in \{0,1,\ldots,n\}$;*

(ii) $a_m(x, \Delta_{\mathbf{h}}^{(k)}) = 0$ for each m odd and any $k \in \{0,1,\ldots,n\}$;

(iii) $Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) \sim \sum_m a_m(\Delta_{\mathbf{h}}^{(k)}) t^{\frac{m-n}{2}}$ as $t \searrow 0$, for each $k \in \{0,1,\ldots,n\}$.

For the proof of statement (i) we refer to [5], Lemma 1.8.3(c), while the statement (ii) is an immediate consequence of Lemma 2.4(iii). Statement (iii) is an immediate consequence of the definition of $Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}})$ [see Theorem 2.3(i)] and of Lemma 2.4(ii).

3 The determinant of an h-dependent Hodge-de Rham operator

In this Section we assume that the k-th Betti number $\beta_k(M)$ of M is equal to zero, so we do not must to deal with the 0-spectrum. Let $\mathcal{R}$ be a complementary region to a cone about positive real axis lying in the right half plane

$\mathbb{C}$, cone that includes the symbolic spectrum of $\Delta_{\mathbf{h}}^{(k)}$, so that the intersection of this symbolic spectrum with $\mathcal{R}$ is empty. We normalize the choice of $\mathcal{R}$ so that the boundary γ is a curve about the positive real axis which consists of a portion of a large circle around the origin and of two rays lying in the right half plane. We orient γ in a clockwise fashion. If $\lambda \in \mathcal{R}$ and $|\lambda|$ is large, then $(\Delta_{\mathbf{h}}^{(k)} - \lambda)$ is invertible. Since $\Delta_{\mathbf{h}}^{(k)}$ is positive definite, we may assume $Re(\lambda) > 0$ for $\lambda \in \gamma$. Choose the branch of λ^s, $s \in \mathbb{C}$, defined on the right half plane so $1^s = 1$. The L^2-norm of the operator $(\Delta_{\mathbf{h}}^{(k)} - \lambda)^{-1}$ is uniformly bounded in $L^2(A^k(M))$. For $Re(s) > 1$, the integral

$$(\Delta_{\mathbf{h}}^{(k)})^{-s} := (2\pi i)^{-1} \int_\gamma \lambda^{-s} (\Delta_{\mathbf{h}}^{(k)} - \lambda)^{-1} d\lambda$$

converges to a bounded operator on $L^2(A^k(M))$.

Let $\{\omega_{\mathbf{h};j}^{(k)}, \lambda_{\mathbf{h};j}^{(k)}\}_{j \in \mathbb{N}}$ be a discrete spectral resolution of $\Delta_{\mathbf{h}}^{(k)}$ [see Theorem 2.1(i)], and

$$E_{\mathbf{h}}^{(k)}(x, y, s, \Delta_{\mathbf{h}}^{(k)}) := \sum_j (\lambda_{\mathbf{h};j}^{(k)})^{-s} \omega_{\mathbf{h};j}^{(k)}(x) \otimes \omega_{\mathbf{h};j}^{*(k)}(y) \tag{3}$$

the kernel of $(\Delta_{\mathbf{h}}^{(k)})^{-s}$. Theorem 2.1 implies the existence of real constants $\delta > 0$ and $\ell(m) > 0$ such that

$$\lambda_{\mathbf{h};j}^{(k)} \geq j^\delta \text{ and } ||\omega_{\mathbf{h};j}^{(k)}||_{\infty,m} \leq (\lambda_{\mathbf{h};j}^{(k)})^{\ell(m)}, \tag{4}$$

whence it follows that (3) converges absolutely and uniformly in the $||\cdot||_{\infty,m}$-norm, for $Re(s)$ large.

Definition 3.1 *We define the generalized zeta function*

$$\zeta(s, \Delta_{\mathbf{h}}^{(k)}) := trace_{L^2}((\Delta_{\mathbf{h}}^{(k)})^{-s}) = \sum_j (\lambda_{\mathbf{h};j}^{(k)})^{-s} =$$

$$= \int_M Trace(E_{\mathbf{h}}^{(k)}(x, x, s, \Delta_{\mathbf{h}}^{(k)})) d\mu_g(x).$$

Using relations (4), one can show that it converges uniformly and absolutely for $Re(s)$ large.

We use the Mellin transform to relate the zeta function to the heat kernel. From Lemma 2.5(iii), Theorem 2.3(i) and the identity

$$\int_0^\infty t^{s-1} e^{-\lambda_{\mathbf{h}}^{(k)} t} dt = (\lambda_{\mathbf{h}}^{(k)})^{-s} \Gamma(s),$$

where $\Gamma : \{s \in \mathbb{C} |\ Re(s) > 0\} \to \mathbb{C}$, $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$, we obtain the following theorem.

Theorem 3.2 *(i) If $Re(s)$ is large, then*

$$\zeta(s, \Delta_{\mathbf{h}}^{(k)}) = \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}})dt;$$

(ii) $\Gamma(s)\zeta(s, \Delta_{\mathbf{h}}^{(k)})$ has a meromorphic extension to $\mathbb{C}$ with isolated simple poles at $s = \frac{n-m}{2}$ for $m \in \mathbb{N}$ and

$$Res_{s=\frac{n-m}{2}} \Gamma(s)\zeta(s, \Delta_{\mathbf{h}}^{(k)}) = a_m(\Delta_{\mathbf{h}}^{(k)});$$

(iii) Let $\Delta_{\mathbf{h}}^{(k)}(\epsilon)$ be a smooth one parameter family of such operators. Then $\zeta(s, \Delta_{\mathbf{h}}^{(k)}(\epsilon))$ is smooth with respect to (s, ϵ) away from the exceptional values at $s = \frac{n-m}{2}$.

Remark 3.3 *Because by the determinant of the* **h***-dependent Hodge-de Rham operator $\Delta_{\mathbf{h}}^{(k)}$ we understand the product of all eigenvalues, considered with their multiplicity, that is*

$$det(\Delta_{\mathbf{h}}^{(k)}) = \prod_{j=1}^{\infty} \lambda_{\mathbf{h},j}^{(k)},$$

by Theorem 3.2 *one can define a regularization of it, using the generalized zeta function, namely*

$$det_{\zeta}(\Delta_{\mathbf{h}}^{(k)}) = e^{\left(-\frac{d}{ds}\Big|_{s=0} \zeta(s, \Delta_{\mathbf{h}}^{(k)})\right)}. \tag{5}$$

Note that this is formally correct since if there were only a finite number of eigenvalues we would have

$$\frac{d}{ds}\left(\sum_j (\lambda_{\mathbf{h},j}^{(k)})^{-s}\right) = -\sum_j (\lambda_{\mathbf{h},j}^{(k)})^{-s} \log(\lambda_{\mathbf{h},j}^{(k)})$$

which takes the value $-\log(\prod_j \lambda_{\mathbf{h},j}^{(k)})$ at $s = 0$.

In what follows, we endow the set $\mathfrak{M}(M)$ of all smooth Riemannian metrics on M with the structure of a Fréchet manifold (see [1] and [7]). In the paper [6] it is proven the following result.

Corollary 3.4 *If M is a closed, n-dimensional smooth manifold, and* **h** *a nonsingular tensor field of type $(1,1)$ on M with vanishing Nijenhuis derivation, then for each fixed $t \in (0, \infty)$, the real functions*

$$Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}(\cdot)}) = \sum_{j=1}^{\infty} e^{\lambda_{\mathbf{h},j}^{(k)}(\cdot)t} : \mathfrak{M}(M) \to \mathbb{R}$$

are smooth with respect to the canonical Fréchet manifold structure considered on $\mathfrak{M}(M)$.

Therefore, by relation (5), by Theorem 3.2 and by Corollary 3.4 one obtains

Corollary 3.5 *If* M *is a closed,* n*-dimensional smooth manifold, and* $\mathbf{h}$ *a nonsingular tensor field of type* $(1,1)$ *on* M *with vanishing Nijenhuis derivation, then for each fixed* $t \in (0,\infty)$*, the real functions*

$$det_\zeta(\Delta_{\mathbf{h}}^{(k)}(\cdot)) = e^{\left(-\frac{d}{ds}\big|_{s=0}\zeta(s,\Delta_{\mathbf{h}}^{(k)}(\cdot))\right)} : \mathfrak{M}(M) \to \mathbb{R}$$

are smooth with respect to the canonical Fréchet manifold structure considered on $\mathfrak{M}(M)$.

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