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A computational method to find an approximate analytical solution for fuzzy differential equations

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Abstract

In this paper, we introduce a computational method to find an approximate analytical solution for fuzzy differential equations. At first, variational iteration method (VIM) is used to solve the crisp problem then with the extension principle we find the fuzzy approximation solution. Examples are given , including linear and nonlinear fuzzy first-order differential equations.

1 Introduction

In this paper, we will consider the first-order ordinary differential equation

$$\frac{dy}{dt} = f(t, y, k), \qquad y(0) = c \tag{1}$$

where $k = (k_1, \ldots, k_n)$ is a vector of constants, and t is in some interval (closed and bounded) I which contains zero. We assume that f satisfies conditions [4, 13] so that Eq. (1) has an unique solution $y = g(t, k, c)$, for $t \in I$, $k \in K \subset \Re^n$, $c \in C \subset \Re$. Let I_1, be an interval for the y-values and set $R = I \times I_1$, a region in $\Re^2$. Well-known sufficient conditions for Eq. (1) to have a unique solution are, given any $k \in K$ and $c \in C$: (1) $(0, c)$ is in R, (2) f is continuous in R (k is held fixed), and (3) $\frac{\partial f}{\partial y}$ is continuous in R. If these

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conditions are satisfied, then there is a unique solution $y = g(t, k, c)$ for $t \in I^*$. Since zero will belong to I^* we will assume that $I^* = I$. We will also assume that g is continuous on $I \times K \times C$. The values of the k_i and c are uncertain and we will model this uncertainty by substituting triangular fuzzy numbers for the k_i and c in Eq. (1). Then, we wish to solve for y which will now be a fuzzy function. The approximate analytical solution for fuzzy y is the topic of this paper.

The paper is organized as follows. In Section 2, we presents the basic notations. The VIM is defined in Section 3. The fourth section presents Buckly-Feuring solution and Sikkala derivative. In section 5, VIM is applied for two nonlinear fuzzy initial value problem in crisp case.

2 Notations and preliminaries

We place a bar over a capital letter to denote a fuzzy subset of $\Re^n$. So, $\bar{Y}$, $\bar{K}$, $\bar{C}$, etc. all represent fuzzy subsets of $\Re^n$ for some n. We write $\mu_{\bar{A}}(x)$, a number in $[0, 1]$, for the membership function of $\bar{A}$ evaluated at $x \in \Re^n$. Define $\bar{A} \leq \bar{B}$ when $\mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(x)$ for all x. An γ-cut of $\bar{A}$, written $\bar{A}(\gamma)$, is defined as $\{x|\mu_{\bar{A}}(x) \geq \gamma\}$, for $0 < \gamma \leq 1$. We separately specify $\bar{A}(0)$ as the closure of the union of all the $\bar{A}(\gamma)$ for $0 < \gamma \leq 1$.

We adopt the general definition of a fuzzy number given in [5]. A triangular fuzzy number $\bar{N}$ is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $\mu_{\bar{N}}(x)$ is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify $\bar{N}$ as (a_1, a_2, a_3). We will write: (1) $\bar{N} > 0$ if $a_1 > 0$, (2) $\bar{N} \geq 0$ if $a_1 \geq 0$, (3) $\bar{N} < 0$ if $a_3 < 0$; and (4) $\bar{N} \leq 0$ if $a_3 \leq 0$. The γ-cut of any fuzzy number is always a closed and bounded interval. Let $\bar{K} = (\bar{K}_1, \ldots, \bar{K}_n)$ be a vector of triangular fuzzy numbers and let $\bar{C}$ be another triangular fuzzy number. Substitute $\bar{K}$ for k and $\bar{C}$ for c in Eq. (1) and we get

$$\frac{d\bar{Y}}{dt} = f(t, \bar{Y}, \bar{K}), \qquad \bar{Y}(0) = \bar{C} \tag{2}$$

assuming we have adopted some definition for the derivative of the unknown fuzzy function $\bar{Y}(t)$. We wish to find an approximate Eq. (2) for $\bar{Y}(t)$ and have $\bar{Y}(t)$ a fuzzy number for each t in I . In general, we use the notation $\frac{d\bar{Y}}{dt}$ for the derivative of a fuzzy function $\bar{Y}$, although we have not yet defined this derivative.

Definition 2.1 *We represent an arbitrary fuzzy number by an ordered pair of functions* $\bar{N}(\gamma) = [N_1(\gamma), N_2(\gamma)]$, $0 \leq \gamma \leq 1$, *which satisfy the following requirements [11]:*

(a) $N_1(\gamma)$ *is a bounded left continuous nondecreasing function over* $[0, 1]$,

(b) $N_2(\gamma)$ *is a bounded left continuous nonincreasing function over* $[0,1]$,

(c) $N_1(\gamma) \leq N_2(\gamma)$, $0 \leq \gamma \leq 1$.

Definition 2.2 *For arbitrary fuzzy numbers* $\bar{N}(\gamma) = [N_1(\gamma), N_2(\gamma)]$ *and* $\bar{Z}(\gamma) = [Z_1(\gamma), Z_2(\gamma)]$ *the quantity*

$$D(\bar{N}, \bar{Z}) = \max\{\sup_{0\leq\gamma\leq1} |N_1(\gamma) - Z_1(\gamma)|, \sup_{0\leq\gamma\leq1} |N_2(\gamma) - Z_2(\gamma)|\}$$

is the distance between $\bar{N}$ *and* $\bar{Z}$.

3 Variational iteration method

The VIM [6, 7, 8], which is a modified general lagrange multiplier method [9], has been shown to solve effectively, easily and accurately, a large class of nonlinear problem with approximations which converge rapidly to accurate solutions.

We consider the following general differential equation

$$Ly(t) + Ny(t) = g(t)$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous or forcing term. According to the VIM, we can construct a correction functional as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\{Ly_n(\tau) + N\tilde{y}_n(\tau) - g(\tau)\}d\tau$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, $y_0(t)$ is an initial approximation with possible unknowns, and $\tilde{y}_n$ is considered as restricted variation [9], i.e. $\delta\tilde{y}_n = 0$.

For first-order initial value problem (1), by the above method its correction functional can be written down as follows

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\{\frac{dy_n(\tau)}{d\tau} - f(\tilde{y}_n(\tau), \tau, k)\}d\tau.$$

Making the above correction functional stationary, notice that $\delta y(0) = 0$,

$$\begin{aligned} \delta y_{n+1}(t) &= \delta y_n(t) + \delta \textstyle\int_0^t \lambda\{y'_n(\tau) - f(\tilde{y}_n(\tau), \tau, k)\}d\tau \\ &= \delta y_n(t) + \lambda(\tau)\delta y_n(\tau)|_{\tau=t} + \textstyle\int_0^t \lambda'(\tau)\delta y_n(\tau)\, d\tau = 0 \end{aligned}$$

thus, we obtain the following stationary conditions

$$\begin{cases} \delta y_n : 1+\lambda(\tau)|_{\tau=t}=0 \\ \delta y_n : \lambda'(\tau)|_{\tau=t}=0. \end{cases}$$

The lagrange multiplier, therefore, can be readily identified $\lambda = -1$ and the following iteration formula can be obtained

$$\begin{aligned} y_{n+1}(t) &= y_n(t) - \int_0^t \{y_n'(\tau) - f(y_n(\tau), \tau, k)\}d\tau \\ y_0(0) &= c. \end{aligned} \tag{3}$$

Notice that it is not necessary to take $y_n(\tau)$ as restricted in $f(y_n(\tau), \tau, k)$. For obtaining better λ in different IVPs one can use the methods introduced in [6, 7, 8].

3.1 Buckly-Feuring solution and Seikkala derivative

Let $\bar{K}(\gamma) = K_1(\gamma) \times \cdots \times K_n(\gamma)$ and $\Phi(\gamma) = \bar{K}(\gamma) \times \bar{C}(\gamma)$, for $0 \le \gamma \le 1$. Assume that $\Phi(0) \subset \bar{K} \times \bar{C}$ so that g will be continuous on $I \times \Phi(\gamma)$ for all γ. Buckly-Feuring first fuzzify the crisp solution $y = g(t, k, c)$ to obtain $\bar{Y}(t) = g(t, \bar{K}, \bar{C})$ using the extension principle [1]. Alternatively, they get γ-cuts as follows [2, 3]:

$$\bar{Y}(t,\gamma) = [y_1(t,\gamma), y_2(t,\gamma)] \tag{4}$$

with

$$y_1(t,\gamma) = \min\{g(t,k,c)|k \in \bar{K}(\gamma), c \in \bar{C}(\gamma)\} \tag{5}$$

and

$$y_2(t,\gamma) = \max\{g(t,k,c)|k \in \bar{K}(\gamma), c \in \bar{C}(\gamma)\} \tag{6}$$

for $t \in I$ and $\gamma \in [0,1]$. Still another equivalent procedure to determine $\bar{Y}(t)$ is to first specify, for $0 \le \gamma \le 1$, and $t \in I$

$$\Omega(\gamma) = \{g(t,k,c)|(k,c) \in \Phi(\gamma)\}$$

and then define the membership function of $\bar{Y}(t)$ as follows

$$\mu_{\bar{Y}(t)}(x) = sup\{\gamma | x \in \Omega(\gamma)\}.$$

Theorem 3.1 *1.* $\bar{Y}(t,\gamma) = \Omega(\gamma)$ *for all* $\gamma \in [0,1]$, $t \in I$,

2. $\bar{Y}(t)$ *is a fuzzy number for all* $t \in I$ *[1].*

Assume that $y_i(t,\gamma)$ is differentiable with respect to $t \in I$ for each $\gamma \in [0,1]$, $i = 1,2$. Denote the partial of $y_i(t,\gamma)$ with respect to t as $y_i'(t,\gamma)$, $i = 1,2$. Let

$$\Gamma(t,\gamma) = (y_1'(t,\gamma), y_2'(t,\gamma)) \tag{7}$$

for all $t \in I$, $\gamma \in [0,1]$. If $\Gamma(\gamma)$ defines the γ-cuts of a fuzzy number for each $t \in I$ then $\bar{Y}(t)$ is differentiable and write

$$\frac{d\bar{Y}(t,\gamma)}{dt} = \Gamma(t,\gamma) = (y_1'(t,\gamma), y_2'(t,\gamma)) \tag{8}$$

for all $t \in I$, $\gamma \in [0,1]$. Notice, that Eq. (8) is just the derivative (with respect to t) of Eq. (4). So, Eq. (8) could be written $\frac{d\bar{Y}(t,\gamma)}{dt}$. Sufficient conditions for $\Gamma(t,\gamma)$ to define the γ-cuts of a fuzzy number are [5, 10]

(i) $y_1'(t,\gamma)$ and $y_2'(t,\gamma)$ are continuous on $I \times [0,1]$,

(ii) $y_1'(t,\gamma)$ is an increasing function of γ for each $t \in I$,

(iii) $y_2'(t,\gamma)$ is an decreasing function of γ for each $t \in I$,

(iv) $y_1'(t,1) \le y_2'(t,1)$ for all $t \in I$.

Now, for $\bar{Y}(t)$ to be a solution to the FIVP it is needed that $\frac{d\bar{Y}(t)}{dt}$ exists but also Eq. (2) must hold. To check Eq. (2) one must first compute $f(t,\bar{Y},\bar{K})$. γ-cuts of $f(t,\bar{Y},\bar{K})$ can be found as follows

$$f(t,\bar{Y},\bar{K},\gamma) = [f_1(t,\gamma), f_2(t,\gamma)]$$

with

$$f_1(t,\gamma) = \min\{f(t,y,k) | y \in \bar{Y}(t,\gamma), k \in \bar{K}(\gamma)\}$$
$$f_2(t,\gamma) = \max\{f(t,y,k) | y \in \bar{Y}(t,\gamma), k \in \bar{K}(\gamma)\}$$

for $t \in I$, $\gamma \in [0,1]$. We will say that $\bar{Y}$ is a solution to Eq. (2) if $\frac{d\bar{Y}(t)}{dt}$ exists and

$$y_1'(t,\gamma) = f_1(t,\gamma) \tag{9}$$

$$y_2'(t,\gamma) = f_2(t,\gamma) \tag{10}$$

$$y_1'(0,\gamma) = c_1(\gamma) \tag{11}$$

$$y_2'(0,\gamma) = c_2(\gamma) \tag{12}$$

where $\bar{C}(\gamma) = (c_1(\gamma), c_2(\gamma))$.

Let $\bar{X}(t)$ is a fuzzy number for each $t \in I$. Also, let $\bar{X}(t,\gamma) = [x_1(t,\gamma), x_2(t,\gamma)]$ and write $x_i'(t,\gamma)$ with respect to t, $i = 1,2$. Assume these partial always exist in this section. The Seikkala derivative of $\bar{X}(t)$, written $SD\bar{X}(t)$, was defined in [12]. This definition is as follows: if $[x_1'(t,\gamma), x_2'(t,\gamma)]$ are γ-cuts of a fuzzy number for each $t \in I$, then $SD\bar{X}(t)$ exists and $SD\bar{X}(t,\gamma) = [x_1'(t,\gamma), x_2'(t,\gamma)]$. Notice that this is the definition of the derivative of a fuzzy function that it used in this section. That is, if $\frac{d\bar{Y}(t,\gamma)}{dt}$ exists, then $SD\bar{X}(t,\gamma) = \frac{d\bar{Y}(t,\gamma)}{dt}$. Also, $SD\bar{X}(t)$ is a fuzzy number for all $t \in I$. The Buckley-Feuring solution, written BFS, to the FIVP, was defined in this section. To review those results let $BFS = \bar{Y}(t)$. Then (i) $\bar{Y}(t) = g(t, \bar{K}, \bar{C})$ (Eqs. (4)-(6)), (ii) $SD\bar{Y}(t)$ exists (Eq. (7) defines a fuzzy number for all t) and (iii) $SD\bar{Y}(t) = f(t, \bar{Y}(t), \bar{K})$ and $\bar{Y}(0) = \bar{C}$ (Eqs. (9-12)). Therefore obtain the following results regarding $BFS = \bar{Y}(t)$.

Theorem 3.2 *Assume $SD\bar{Y}(t)$ exists for $t \in I$. Then $BFS = \bar{Y}(t)$ if*

$$\frac{\partial f}{\partial y} > 0, \qquad \frac{\partial g}{\partial c} > 0 \tag{13}$$

and

$$(\frac{\partial f}{\partial k_i})(\frac{\partial g}{\partial k_i}) > 0 \tag{14}$$

$i = 1, \ldots, n$. If Eq. (13) does not hold or Eq. (14) dose not hold for some i, then $\bar{Y}(t)$ dose not solve the FIVP [1].

4 Examples

Throughout this section $y_i'(t,\gamma)$, $i = 1, 2$, are continuous and we will assume that $I = [0, M]$, for some $M > 0$. We use the following strategy: (i) find $y_n(t,k,c)$ with VIM, It is an approximation of $y(t) = g(t,k,c)$, the solution of Eq. (1), then fuzzify it to $\bar{Y}(t) = y_n(t, \bar{K}, \bar{C})$ by extension principle; (ii) checking conditions (13) and (14) for $y_n(t,k,c)$; (iii) is $\bar{Y}(t)$ a fuzzy number? (conditions (i)-(iv) in Section 4); (iv) fuzzify $f(t,y,k)$ to a fuzzy function $f(t, \bar{Y}, \bar{K})$ by extension principle, where $\bar{Y}(t) = y_n(t, \bar{K}, \bar{C})$. Since we approximate $g(t,k,c)$ by $y_n(t,k,c)$, when $y(t) = y_n(t,k,c)$ is extended to fuzzy case ($\bar{Y}(t) = y_n(t, \bar{K}, \bar{C})$), then $\bar{Y}(t) = y_n(t, \bar{K}, \bar{C})$ usually dose not satisfy in Eqs. (9)-(12). We calculate distance between $f(t, \bar{Y}, \bar{K})$ and $SD\bar{Y}(t)$ with metric D.

Example 4.1 *Consider the initial value problem*

$$y'(t) = k_1 y^2(t) + k_2, \qquad y(0) = 0, \quad t \in I = [0, 0.5] \tag{15}$$

where $k_i > 0$ for $i = 1, 2$.

The approximation solution by VIM (3) with $y_0(0) = 0$ is

$$g(t,k,0) \approx y_3(t,k,0) = k_2 t + \frac{1}{3}k_1 k_2^2 t^3 + \frac{2}{15}k_1^2 k_2^3 t^5 + \frac{1}{63}k_1^3 k_2^4 t^7.$$

Calculating $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial k_1}$, $\frac{\partial f}{\partial k_2}$, $\frac{\partial y_3}{\partial k_1}$ and $\frac{\partial y_3}{\partial k_2}$, we can see that the conditions (13) and (14) are satisfied so we have a BFS. Now we consider the corresponding FIVP with $\bar{K}_i > 0$, $i = 1,2$. Using $\frac{\partial y_3}{\partial k_i} \geq 0$ we obtain $\bar{Y}(t) = y_3(t,\bar{K},0)$, so the γ-cuts corresponding to $\bar{Y}(t)$ are

$$y_1(t,\gamma) = k_{21}(\gamma)t + \tfrac{1}{3}k_{11}(\gamma)(k_{21}(\gamma))^2 t^3 + \tfrac{2}{15}(k_{11}(\gamma))^2(k_{21}(\gamma))^3 t^5 + \tfrac{1}{63}(k_{11}(\gamma))^3(k_{21}(\gamma))^4 t^7$$
$$y_2(t,\gamma) = k_{22}(\gamma)t + \tfrac{1}{3}k_{12}(\gamma)(k_{22}(\gamma))^2 t^3 + \tfrac{2}{15}(k_{12}(\gamma))^2(k_{22}(\gamma))^3 t^5 + \tfrac{1}{63}(k_{12}(\gamma))^3(k_{22}(\gamma))^4 t^7$$

where $\bar{K}_i(\gamma) = [k_{i1}(\gamma), k_{i2}(\gamma)]$, for $i = 1,2, 0 \leq \gamma \leq 1$. The γ-cuts of $SD\bar{Y}(t)$, for $0 \leq \gamma \leq 1$ are (differential respect to t)

$$y_1'(t,\gamma) = k_{21}(\gamma) + k_{11}(\gamma)(k_{21}(\gamma))^2 t^2 + \tfrac{2}{3}(k_{11}(\gamma))^2(k_{21}(\gamma))^3 t^4 + \tfrac{1}{9}(k_{11}(\gamma))^3(k_{21}(\gamma))^4 t^6$$
$$y_2'(t,\gamma) = k_{22}(\gamma) + k_{12}(\gamma)(k_{22}(\gamma))^2 t^2 + \tfrac{2}{3}(k_{12}(\gamma))^2(k_{22}(\gamma))^3 t^4 + \tfrac{1}{9}(k_{12}(\gamma))^3(k_{22}(\gamma))^4 t^6.$$

Due to $\frac{dk_{i1}(\gamma)}{d\gamma} > 0$ and $\frac{dk_{i2}(\gamma)}{d\gamma} < 0$, $i = 1,2$, $0 \leq \gamma \leq 1$, therefore $y_1'(t,\gamma)$ is increasing and $y_2'(t,\gamma)$ is decreasing, for $t \in I$. On the other hand, it is clear that $y_1'(t,1) \leq y_2'(t,1)$, for $t \in I$ i.e. $[y_1'(t,\gamma), y_2'(t,\gamma)]$ are γ-cuts of a fuzzy number.

For practical results we set $\bar{K}_1(\gamma) = \bar{K}_2(\gamma) = [\gamma, -\gamma + 2]$, $0 < \gamma \leq 1$. First we obtain $f(t,\bar{Y},\bar{K})$ by extension principle where $\bar{Y}(t) = y_3(t,\bar{K},0)$, then we compare $SD\bar{Y}(t)$ with $f(t,\bar{Y},\bar{K})$ by metric D. Results are presented in Table 1.

Table 1

t	$D(SD\bar{Y}(t), y_3(t,\bar{K},0))$	$D(SD\bar{Y}(t), y_{10}(t,\bar{K},0))$
0	0	0
0.1	0.0000347	6.6613×10^{-16}
0.2	0.0023487	1.3389×10^{-12}
0.3	0.0292973	2.8442×10^{-9}
0.4	0.1869826	8.8726×10^{-7}
0.5	0.8402922	0.000115

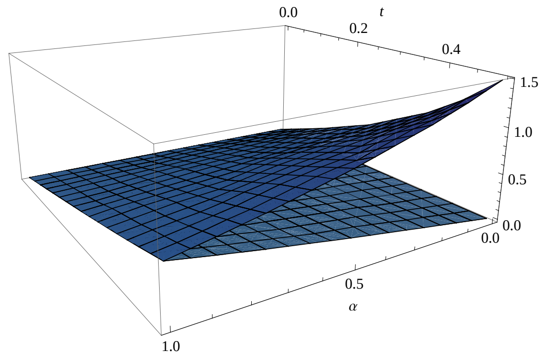

Figure 1. $y_3(t, \bar{K}, 0)$

Example 4.2 *Let $c > 0$ and consider the following initial value problem*

$$\frac{dy}{dt} = t^6 y^3(t) + t^3 y(t) + t^2, \qquad y(0) = c, \;\; t \in I = [0, 0.5] \tag{16}$$

The approximation solution by VIM (3) with $y_0(0) = c$ is

$$g(t, c) \approx y_1(t, c) = c + \frac{1}{3}t^3 + \frac{1}{4}ct^4 + \frac{1}{7}c^3 t^7.$$

Since $\frac{\partial f}{\partial y} > 0$ and $\frac{\partial y_1}{\partial c} > 0$ then condition (13) and (14) are satisfy then we have a BFS. We consider the corresponding FIVP, using $\frac{\partial y_1}{\partial c} > 0$ we obtain $\bar{Y}(t) = y_1(t, \bar{C})$ and γ-cut corresponding to $\bar{Y}(t)$

$$y_1(t, \gamma) = c_1(\gamma) + \tfrac{1}{3}t^3 + \tfrac{1}{4}c_1(\gamma)t^4 + \tfrac{1}{7}c_1^3(\gamma)t^7$$
$$y_2(t, \gamma) = c_2(\gamma) + \tfrac{1}{3}t^3 + \tfrac{1}{4}c_2(\gamma)t^4 + \tfrac{1}{7}c_2^3(\gamma)t^7$$

where $\bar{C}(\gamma) = (c_1(\gamma), c_2(\gamma))$. Then γ-cuts of $SD\bar{Y}(t)$ are

$$y_1'(t, \gamma) = t^2 + c_1(\gamma)t^3 + c_1^3(\gamma)t^6$$
$$y_2'(t, \gamma) = t^2 + c_2(\gamma)t^3 + c_2^3(\gamma)t^6.$$

Similar Example 1, since $\frac{dc_1(\gamma)}{d\gamma} > 0$ and $\frac{dc_2(\gamma)}{d\gamma} < 0$, $0 \le \gamma \le 1$, therefore $y_1'(t, \gamma)$ is increasing and $y_2'(t, \gamma)$ is decreasing, for $t \in I$. On the other hand, it is clear that $y_1'(t, 1) \le y_2'(t, 1)$, for $t \in I$ i.e. $[y_1'(t, \gamma), y_2'(t, \gamma)]$ are γ-cuts of a fuzzy number.

For practical results we set $\bar{C}(\gamma) = [\gamma, -\gamma + 2]$. First we obtain $f(t, \bar{Y}, 0)$ by extension principle where $\bar{Y}(t) = y_1(t, \bar{C})$, then we compare $SD\bar{Y}(t)$ with $f(t, \bar{Y}, 0)$ by metric D. Results are presented in Table 2.

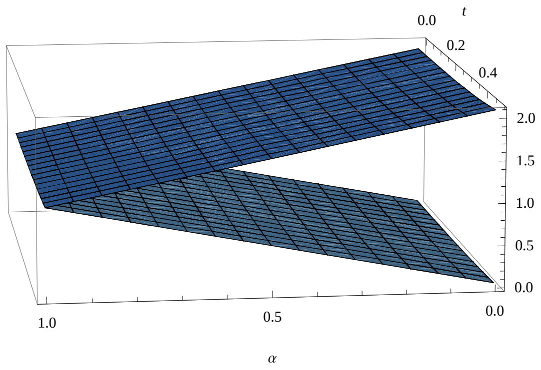

Figure 2. $y_3(t, \bar{C})$

Table 2

t	$D(SD\bar{Y}(t), y_1(t, \bar{C}))$	$D(SD\bar{Y}(t), y_2(t, \bar{C}))$
0	0	0
0.1	3.88049×10^{-7}	5.4986×10^{-12}
0.2	0.0000305	7.2271×10^{-9}
0.3	0.0004762	6.4883×10^{-7}
0.4	0.0041061	0.000022
0.5	0.0262132	0.0004614

5 Conclusion

In this paper, we present an analytical approximate solution to FIVP. Using VIM we solve the crisp problem then with the extension principle we find the fuzzy approximation solution.

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