An. Şt. Univ. Ovidius Constanţa Vol. 16(2), 2008, 83–90

Notes on linear combinations of two tripotent, idempotent, and involutive matrices that commute

Halim ÖZDEMİR and Murat SARDUVAN

Abstract

The aim of this paper is to provide alternate proofs of all the results of our previous paper [2] in the particular case when the given two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ in the linear combination $\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2$ commute.

1 Introduction and Preliminaries

Let $\mathcal{C}$ and $\mathcal{C}_{m,n}$ denote the sets of complex numbers and $m \times n$ complex matrices. Moreover, $\mathcal{C}^*$ will mean $\mathcal{C} \setminus \{0\}$.

Now, consider a linear combination of the form

$$\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2, \tag{1.1}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_{n,n}$ are nonzero matrices and $c_1, c_2 \in \mathcal{C}^*$.

The aim of this paper is to provide alternate proofs of all the results of our previous paper [2] in the particular case that $\mathbf{A}_1$ and $\mathbf{A}_2$ in (1.1) are commuting matrices, i.e. $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$.

Recall that a matrix $\mathbf{B} \in \mathcal{C}_{n,n}$ is said to be similar to a matrix $\mathbf{A} \in \mathcal{C}_{n,n}$ if there exists a nonsingular matrix $\mathbf{P} \in \mathcal{C}_{n,n}$ such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Key Words: Involutive matrix; Idempotent matrix; Tripotent matrix; Similar matrix; Diagonalization.
Mathematics Subject Classification: 15A18; 15A57
Received: May, 2008
Accepted: October, 2008

If a matrix $\mathbf{A} \in \mathcal{C}_{n,n}$ is similar to a diagonal matrix, then $\mathbf{A}$ is said to be diagonalizable. It is clear that the spectrums of an idempotent, a tripotent, and an involutive matrices are contained in $\{0,1\}$, $\{-1,0,1\}$, and $\{-1,1\}$, respectively. Then, any tripotent or idempotent or involutive matrix is diagonalizable [1, Corollary 3.3.10]. Two diagonalizable matrices $\mathbf{A}$, $\mathbf{B} \in \mathcal{C}_{n,n}$ are said to be simultaneously diagonalizable if there is a single similarity matrix $\mathbf{P} \in \mathcal{C}_{n,n}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{BP}$ are both diagonal. Two diagonalizable matrices are commuting if and only if they are simultaneously diagonalizable [1, Theorem 1.3.12]. Hence, we can write $\mathbf{A}$ of the form (1.1) as $\mathbf{P}\left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right)\mathbf{P}^{-1}$ with $\mathbf{\Lambda}$ and $\mathbf{M}$ diagonal matrices, where the main diagonal entries of $\mathbf{\Lambda}$ and $\mathbf{M}$ are the eigenvalues of $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Consequently, direct calculations show that a linear combination of the form (1.1) is a tripotent matrix, an idempotent matrix, and an involutive matrix if and only if

$$\left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right)^3 - \left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right) = \mathbf{0},$$
$$\left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right)^2 - \left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right) = \mathbf{0},$$

and

$$\left(c_1\mathbf{\Lambda} + c_2\mathbf{M}\right)^2 = \mathbf{I},$$

respectively. By carrying out necessary arrangements, we obtain equivalently

$$\left(c_1\lambda_i + c_2\mu_i\right)\left(c_1\lambda_i + c_2\mu_i - 1\right)\left(c_1\lambda_i + c_2\mu_i + 1\right) = 0, \tag{1.2}$$

$$\left(c_1\lambda_i + c_2\mu_i\right)\left(c_1\lambda_i + c_2\mu_i - 1\right) = 0, \tag{1.3}$$

and

$$\left(c_1\lambda_i + c_2\mu_i - 1\right)\left(c_1\lambda_i + c_2\mu_i + 1\right) = 1, i = 1, 2, \ldots, n, \tag{1.4}$$

where λ_i and μ_i are diagonal entries of $\mathbf{\Lambda}$ and $\mathbf{M}$, respectively. We may, without loss of generality, assume that any multiple eigenvalues of $\mathbf{A}_1$ and $\mathbf{A}_2$ occur contiguously on the main diagonal of $\mathbf{\Lambda}$ and $\mathbf{M}$, respectively.

2 Main Results

Let $\mathbf{A}_1$, $\mathbf{A}_2$ be tripotent matrices. Firstly we consider the trivial case where $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$. If $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$, say $\mathbf{A}_1 = c\mathbf{A}_2$ for $c \in \mathcal{C}^*$, then

$$\mathbf{A}_1 = \mathbf{A}_1^3 = \left(c\mathbf{A}_2\right)^3 = c^3\mathbf{A}_2^3 = c^3\mathbf{A}_2 = c^2c\mathbf{A}_2 = c^2\mathbf{A}_1.$$

Since $\mathbf{A}_1 \neq \mathbf{0}$, we obtain $1 = c^2$. Due to the fact that an involutive matrix is always a tripotent matrix, we also obtain $1 = c^2$ in case $\mathbf{A}_1$ and $\mathbf{A}_2$ are involutive matrices. Since $c_1\mathbf{A}_1 + c_2\mathbf{A}_2 = \left(cc_1 + c_2\right)\mathbf{A}_2$ is tripotent (or involutive), a similar argument proves $(cc_1 + c_2) \in \{ -1, 0, 1\}$. Thus, the case where $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$ is also excluded from further considerations in Theorems 2.1-2.4. Now we may give the main results.

Theorem 2.1 [2, Theorem 2.1] *For nonzero $c_1, c_2 \in \mathcal{C}^*$ and involutive matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_{n,n}$ such that $\mathbf{A}_1 \neq \pm\mathbf{A}_2$ and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$, let $\mathbf{T}$ be their linear combination of the form*

$$\mathbf{T} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2. \tag{2.1}$$

Then the matrix $\mathbf{T}$ of the form (2.1) is tripotent if and only if

$$(c_1, c_2) \in \{(-1/2, -1/2) , (-1/2, 1/2) , (1/2, -1/2) , (1/2, 1/2)\} .$$

Proof. Solving (1.2) for possible pairs of (λ_i, μ_i) we obtain the values and equations in Table 1 which is arranged according to the pairs of (c_1, c_2) commonly fulfilling (1.2).

Table 1: All possible (c_1, c_2) pairs associated with (λ_i, μ_i) pairs in Theorem 2.1

Cases	(c_1, c_2)	(λ_i, μ_i)
I	$\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$ $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$
II(a)	$c_1 = -c_2$ or $c_1 = -c_2 + 1$ or $c_1 = -c_2 - 1$	$(1, 1)$, $(-1, -1)$
II(b)	$c_1 = c_2$ or $c_1 = c_2 + 1$ or $c_1 = c_2 - 1$	$(1, -1)$, $(-1, 1)$

Moreover, the matrix $\mathbf{T}$ of the form (2.1) satisfies $\mathbf{T}^3 = \mathbf{T}$ if and only if

$$\left(c_1^3 + 3c_1c_2^2 - c_1\right) \mathbf{A}_1 + \left(c_2^3 + 3c_1^2c_2 - c_2\right) \mathbf{A}_2 = \mathbf{0}. \tag{2.2}$$

Combining the pairs of (c_1, c_2) in Table 1 with (2.2), the following results are obtained: $\mathbf{T}$ of the form (2.1) is always tripotent for Case I. For Cases II(a)-(b), we obtain simply either the case that $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$ (which has been excluded from considerations in Theorem 2.1) or Case I (which means that Cases II(a)-(b) contain Case I), again. Hence, the proof is complete. ■

Theorem 2.2 [2, Theorem 2.2 (a)] *For nonzero $c_1, c_2 \in \mathcal{C}^*$ and involutive matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_{n,n}$ such that $\mathbf{A}_1 \neq \pm\mathbf{A}_2$ and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$, let $\mathbf{P}$ be their linear combination of the form*

$$\mathbf{P} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2. \tag{2.3}$$

Then the matrix $\mathbf{P}$ of the form (2.3) is idempotent if and only if:

(a) $(c_1, c_2) = (\ -1/2, -1/2)$ *and* $-\mathbf{A}_1 - \mathbf{A}_2 = \mathbf{I} + \mathbf{A}_1\mathbf{A}_2$,
(b) $(c_1, c_2) = (\ 1/2, 1/2)$ *and* $\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I} + \mathbf{A}_1\mathbf{A}_2$,
(c) $(c_1, c_2) = (\ -1/2, 1/2)$ *and* $-\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I} - \mathbf{A}_1\mathbf{A}_2$,
(d) $(c_1, c_2) = (\ 1/2, -1/2)$ *and* $\mathbf{A}_1 - \mathbf{A}_2 = \mathbf{I} - \mathbf{A}_1\mathbf{A}_2$.

Proof. Solving (1.3) for possible pairs of (λ_i, μ_i) we obtain the values in Table 2 which is arranged according to the pairs of (c_1, c_2) commonly fulfilling (1.3).

Table 2: All possible (c_1, c_2) pairs associated with (λ_i, μ_i) pairs in Theorem 2.2

Cases	(c_1, c_2)	(λ_i, μ_i)
I	$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$(1,-1)$, $(-1,1)$, $(-1,-1)$
II	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$(1,1)$, $(1,-1)$, $(-1,1)$
III	$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$(1,1)$, $(-1,1)$, $(-1,-1)$
IV	$\left(\frac{1}{2}, -\frac{1}{2}\right)$	$(1,1)$, $(1,-1)$, $(-1,-1)$

Moreover, the matrix $\mathbf{P}$ of the form (2.3) satisfies $\mathbf{P}^2 = \mathbf{P}$ if and only if

$$\left(c_1^2 + c_2^2\right)\mathbf{I} + 2c_1c_2\mathbf{A}_1\mathbf{A}_2 - c_1\mathbf{A}_1 - c_2\mathbf{A}_2 = \mathbf{0}. \tag{2.4}$$

Combining the pairs of (c_1, c_2) in Table 2 with (2.4), the following results are obtained: $\mathbf{P} = \frac{1}{2}(\mathbf{I} + \mathbf{A}_1\mathbf{A}_2)$ for Cases I and II, which are parts (a) and (b), and $\mathbf{P} = \frac{1}{2}(\mathbf{I} - \mathbf{A}_1\mathbf{A}_2)$ for Cases III and IV, which are parts (c) and (d). These complete the proof. ■

Theorem 2.3 [2, Theorem 2.3] *For nonzero* $c_1, c_2 \in \mathcal{C}^*$ *and nonzero tripotent matrices* $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{C}_{n,n}$ *such that* $\mathbf{T}_1 \neq \pm\mathbf{T}_2$ *and* $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$*, let* $\mathbf{A}$ *be their linear combination of the form*

$$\mathbf{A} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2. \tag{2.5}$$

Then the matrix $\mathbf{A}$ *of the form* (2.5) *is involutive if and only if:*

(a) $(c_1, c_2) = (1, 1)$ *or* $(c_1, c_2) = (-1, -1)$ *and* $\mathbf{T}_1^2 + 2\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are not involutive,*
(b) $(c_1, c_2) = (1, -1)$ *or* $(c_1, c_2) = (-1, 1)$ *and* $\mathbf{T}_1^2 - 2\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are not involutive,*
(c) $(c_1, c_2) = (2, 1)$ *or* $(c_1, c_2) = (-2, -1)$ *and* $4\mathbf{T}_1^2 + 4\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_1$ *is not involutive,*
(d) $(c_1, c_2) = (2, -1)$ *or* $(c_1, c_2) = (-2, 1)$ *and* $4\mathbf{T}_1^2 - 4\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_1$ *is not involutive,*
(e) $(c_1, c_2) = (1, 2)$ *or* $(c_1, c_2) = (-1, -2)$ *and* $\mathbf{T}_1^2 + 4\mathbf{T}_1\mathbf{T}_2 + 4\mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_2$ *is not involutive,*
(f) $(c_1, c_2) = (1, -2)$ *or* $(c_1, c_2) = (-1, 2)$ *and* $\mathbf{T}_1^2 - 4\mathbf{T}_1\mathbf{T}_2 + 4\mathbf{T}_2^2 = \mathbf{I}$ *and* $\mathbf{T}_2$ *is not involutive.*

Proof. Solving (1.4) for possible pairs of (λ_i, μ_i) we obtain the values and equations in Table 3 which is arranged according to the pairs of (c_1, c_2) commonly fulfilling (1.4).

Table 3: All possible (c_1, c_2) pairs associated with (λ_i, μ_i) pairs in Theorem 2.3

Cases	(c_1, c_2)	(λ_i, μ_i)
I	$(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$	$(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$
II(a)	$(2,-1)$, $(-2,1)$	$(0,1)$, $(0,-1)$, $(1,1)$, $(-1,-1)$
II(b)	$(2,1)$, $(-2,-1)$	$(0,1)$, $(0,-1)$, $(1,-1)$, $(-1,1)$
III(a)	$(1,-2)$, $(-1,2)$	$(1,0)$, $(-1,0)$, $(1,1)$, $(-1,-1)$
III(b)	$(1,2)$, $(-1,-2)$	$(1,0)$, $(-1,0)$, $(1,-1)$, $(-1,1)$
IV(a)	$c_1 = -c_2 + 1$ or $c_1 = -c_2 - 1$	$(1,1)$, $(-1,-1)$
IV(b)	$c_1 = c_2 - 1$ or $c_1 = c_2 + 1$	$(1,-1)$, $(-1,1)$

From the pairs of (λ_i, μ_i) in Table 3, it is evidently seen that $\mathbf{T}_1$ and $\mathbf{T}_2$ are not involutive in Case I, $\mathbf{T}_1$ is not involutive in Cases II (a)-(b), and $\mathbf{T}_2$ is not involutive in Cases III (a)-(b).

Moreover, the matrix $\mathbf{A}$ of the form (2.5) satisfies $\mathbf{A}^2 = \mathbf{I}$ if and only if

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^2 = \mathbf{I}. \tag{2.6}$$

Substituting the pairs of (c_1, c_2) in Case I, in Cases II (a)-(b), and in Cases III (a)-(b) into (2.6) we obtain $\mathbf{T}_1^2 \pm 2\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$, $4\mathbf{T}_1^2 \pm 4\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2^2 = \mathbf{I}$, and $\mathbf{T}_1^2 \pm 4\mathbf{T}_1\mathbf{T}_2 + 4\mathbf{T}_2^2 = \mathbf{I}$, which establish the parts (a),(b), the parts (c),(d), and the parts (e),(f), respectively. Finally, in the Cases IV (a),(b) it is clear that $\mathbf{T}_1$ and $\mathbf{T}_2$ are involutive. So, from (2.6), $\mathbf{T}_1$ is a scalar multiple of $\mathbf{T}_2$. Hence, the proof is complete. ■

Theorem 2.4 [2, Theorem 2.4 (a)] *For nonzero $c_1, c_2 \in \mathcal{C}^*$ and involutive matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_{n,n}$ such that $\mathbf{A}_1 \neq \pm\mathbf{A}_2$ and $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$, let $\mathbf{A}$ be their linear combination of the form*

$$\mathbf{A} = c_1\mathbf{A}_1 + c_2\mathbf{A}_2. \tag{2.7}$$

Then there is no situation for which the matrix $\mathbf{A}$ of the form (2.7) *is an involutive matrix.*

Proof. Solving (1.4) for possible pairs of (λ_i, μ_i) we obtain the values and equations in Table 4 which is arranged according to the pairs of (c_1, c_2) commonly fulfilling (1.4).

Table 4: All possible (c_1, c_2) pairs associated with (λ_i, μ_i) pairs in Theorem 2.4

Cases	(c_1, c_2)	(λ_i, μ_i)
I	$c_1 = -c_2 + 1$ or $c_1 = -c_2 - 1$	$(1,1)$, $(-1,-1)$
II	$c_1 = c_2 - 1$ or $c_1 = c_2 + 1$	$(1,-1)$, $(-1,1)$

Moreover, the matrix $\mathbf{A}$ of the form (2.7) satisfies $\mathbf{A}^2 = \mathbf{I}$ if and only if

$$\left(c_1^2 + c_2^2 - 1\right)\mathbf{I}+2c_1c_2\mathbf{A}_1\mathbf{A}_2 = \mathbf{0}. \tag{2.8}$$

From (2.8), it is easily seen that $\mathbf{A}_1$ is a scalar multiple of $\mathbf{A}_2$ for Cases I-II in Table 4 which contradict with the assumptions. So the proof is complete. ■

Theorem 2.5 [2, Theorem 2.5 (a)] *For nonzero $c_1, c_2 \in \mathcal{C}^*$ and nonzero idempotent matrices $\mathbf{P}_1, \mathbf{P}_2 \in \mathcal{C}_{n,n}$ such that $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$, let $\mathbf{A}$ be their linear combination of the form*

$$\mathbf{A} = c_1\mathbf{P}_1 + c_2\mathbf{P}_2. \tag{2.9}$$

Then the matrix $\mathbf{A}$ of the form (2.9) is involutive if and only if:

(a) $(c_1, c_2) = (-1, -1)$ *or* $(c_1, c_2) = (1, 1)$ *or* $(c_1, c_2) = (-1, 1)$ *or* $(c_1, c_2) = (1, -1)$ *and* $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{I}$,
(b) $(c_1, c_2) = (1, -2)$ *or* $(c_1, c_2) = (-1, 2)$ *and* $\mathbf{P}_1 = \mathbf{I}$,
(c) $(c_1, c_2) = (2, -1)$ *or* $(c_1, c_2) = (-2, 1)$ *and* $\mathbf{P}_2 = \mathbf{I}$.

Proof. Solving (1.4) for possible pairs of (λ_i, μ_i) we obtain the values in Table 5 which is arranged according to the pairs of (c_1, c_2) commonly fulfilling (1.4).

Table 5: All possible (c_1, c_2) pairs associated with (λ_i, μ_i) pairs in Theorem 2.5

Cases	(c_1, c_2)	(λ_i, μ_i)
I	$(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$	$(0, 1)$, $(1, 0)$, $(1, 1)$
II	$(1, -2)$, $(-1, 2)$	$(1, 0)$, $(1, 1)$
III	$(2, -1)$, $(-2, 1)$	$(0, 1)$, $(1, 1)$

Moreover, the matrix $\mathbf{A}$ of the form (2.9) satisfies $\mathbf{A}^2 = \mathbf{I}$ if and only if

$$c_1^2\mathbf{P}_1 + 2c_1c_2\mathbf{P}_1\mathbf{P}_2 + c_2^2\mathbf{P}_2 = \mathbf{I}. \tag{2.10}$$

Substituting the pairs of (c_1, c_2) in Case I into (2.10) we obtain $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$. Combining $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0}$ with (2.10) we obtain $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{I}$, which leads to part (a). From the pairs of (λ_i, μ_i) in Table 5, it is evidently seen that $\mathbf{P}_1 = \mathbf{I}$ in Case II and $\mathbf{P}_2 = \mathbf{I}$ in Case III, which lead to parts (b) and (c), respectively. ■

Acknowledgements The authors are very grateful to the anonymous referee for the constructive comments and helpful suggestions which have improved the presentation of this paper.

References

[1] R.A. Horn and C.R. Johnson, *Matrix Analysis.* Cambridge University Press, Cambridge, UK, 1985.

[2] M. Sarduvan and H. Özdemir, *On linear combinations of two tripotent, idempotent, and involutive matrices*, Appl. Math. Comput., **200**(2008), 401–406.

Department of Mathematics, Sakarya University, TR54187 Sakarya, Turkey
e-mail:hozdemir@sakarya.edu.tr