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Weak geodomination in graphs

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Abstract

A pair x, y of vertices in a nontrivial connected graph G is said to geodominate a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x-y$ geodesic of G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. In this paper we study weak geodomination in a graph G.

1 Introduction

For two vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an $x-y$ geodesic. A vertex v is said to lie in an $x-y$ geodesic P if v is an internal vertex of P. The closed interval $I[x, y]$ consists of x, y and all vertices lying in some $x-y$ geodesic of G, while for $S \subseteq V(G)$,

$$I[S] = \cup_{x,y \in S} I[x, y].$$

A set S of vertices in a graph G is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g(G)$-set, [1,2].

G. Chartrand, F. Harary, H. C. Swart and P. Zhang in [2] studied geodetic concepts from the point of view of domination. Geodetic sets and the geodetic number were referred to as geodominating sets and geodomination number that we adopt in this paper.

A pair x, y of vertices in a nontrivial connected graph G is said to *geodominate* a vertex v of G if either $v \in \{x, y\}$ or v lies in an $x-y$ geodesic of

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G. A set S of vertices of G is a geodominating set if every vertex of G is geodominated by some pair of vertices of S. A vertex of G is *link− complete* if the subgraph induced by its neighborhood is complete. It is easily seen that any link-complete vertex belongs to any geodominating set.

R. Muntean and P. Zhang continued the study of geodomination in [5,6]. They introduce some new conditions on geodomination. For a graph G and an integer $k \geq 1$, a vertex v of G is $k-$ *geodominated* by a pair x, y of distinct vertices in G if v is geodominated by x, y and $d(x, y) = k$. A set S of vertices of G is a k-geodominating set of G if each vertex v in $V(G)\backslash S$ is k-geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-geodominating set of G is its $k-$ *geodomination number* $g_k(G)$. A pair x, y of vertices in G is said to *openly geodominate* a vertex v of G if $v \neq x, y$ and v is geodominated by x and y. A set S is an *open geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is openly geodominated by some pair of vertices of S. The minimum cardinality of an open geodominating set of G is its *open−geodomination number* $og(G)$. A k-geodominating set of cardinality $g_k(G)$ is called a $g_k(G)$-set of G and an open geodominating set of cardinality $og(G)$ is called an $og(G)$-set.

In [3,4] we studied connected geodomination and perfect geodomination in a graph G. Here we study weak, open weak and k-weak geodomination in a graph G. First we give some definitions, notations and properties which will be necessary in our paper. All graphs in this paper are connected and we denote the Cartesian product of two graphs G, H by $G \times H$ and it is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to $(u^{'}, v^{'})$ if and only if (1) $u = u^{'}$ and $vv^{'} \in E(H)$, or (2) $v = v^{'}$ and $uu^{'} \in E(G)$. This graph has $|V(G)|$ copies of H as rows and $|V(H)|$ copies of G as columns. In this paper for an edge $e = \{u, v\}$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e a pendant edge and u a pendant vertex.

2 Weak geodomination

In this section we introduce and study weak geodomination in a graph G.

Definition 2.1 *A pair of vertices x, y in a connected graph G weakly geodominate a vertex v if one of the following holds : 1) $v \in \{x, y\}$ or*

2) v lies in an x-y geodesic L of G and further L is the only x-y geodesic of G of length $d(x, y)$.

A set S of vertices of G is a weak geodominating set if every vertex of G is weakly geodominated by some pair of vertices of S. The minimum cardinality of a weak geodominating set is the weak geodomination number $g_w(G)$.

We refer a $g_w(G)$-set to a weak geodominating set of size $g_w(G)$. By definition the inequality $g_w(G) \geq g(G)$ is obvious. Also for complete graphs we have $g_w(K_n) = g(K_n) = n$.

Observation 2.2 *Let T be a tree with n vertices and l leaves, then $g_w(T) = g(T) = l$.*

Proof. The result follows from that a pendant vertex belongs to each geodominating set and there is exactly one geodesic between any pair of vertices in a tree.□

So as a particular result $g_w(P_n) = 2$ and $g_w(K_{1,n}) = \begin{cases} 2 & n = 1 \\ n & n \geq 2 \end{cases}$.

Observation 2.3 *I) $g_w(C_n) = \begin{cases} 4 & n = 4 \\ 3 & n \neq 4 \end{cases}$.*

II) If $\min\{m, n\} \geq 2$ then $g_w(K_{m,n}) = m + n$.

Proof. I) Let $V(C_n) = \{v_1, v_2, ..., v_n\}$. Let n be odd, then $g_w(C_n) \geq g(C_n) = 3$ and further the weak geodominating set $\{v_1, v_{\frac{n+1}{2}}, v_{\frac{n+1}{2}+1}\}$ implies that $g_w(C_n) = 3$. Now let n be even. It is obvious that $g(C_4) = 4$. Let $n \geq 6$, It is easily seen that no $g(C_n)$-set is a weak geodominating set, hence $g_w(C_n) \geq 3$. On the other hand $\{v_1, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}\}$ is a weak geodominating set. So for $n \geq 6$, $g_w(C_n) = 3$. The other statement is similarly verified.□

Proposition 2.4 *1) If $\min\{m, n\} \geq 2$ then $g_w(P_m \times P_n) = 2\min\{m, n\}$,*

2) $g_w(K_m \times K_n) = mn$,

3) $g_w(K_2 \times C_n) = \begin{cases} 8 & n = 4 \\ 6 & n \neq 4 \end{cases}$,

4) For $m \geq 2$, $g_w(K_m \times G) = mg_w(G)$.

Proof. We only prove 4. The other parts similarely verified. Let

$V(K_m \times G) = \{(1, v_1), (1, v_2), ..., (1, v_n), (2, v_1), (2, v_2), ..., (2, v_n), ..., (m, v_1), (m, v_2), ..., (m, v_n)\}$ where (k, v_i) is adjacent to (l, v_i) for $i = 1, 2, ..., n$, $\{k, l\} \subseteq \{1, 2, ..., m\}$ and the subgraph induced by $\{(k, v_1), (k, v_2), ..., (k, v_n)\}$ is a copy G_k of G. Let S be a $g_w(K_m \times G)$-set and for $k = 1, 2, ..., m$ let $X_k = S \cap \{(k, v_1), (k, v_2), ..., (k, v_n)\}$, then $X_k \neq \emptyset$. For any three integers i, j, k the $(k, v_i) - (k, v_j)$ geodesic lies in G_k, so there is no vertex of $X_{k'}$ that is weakly geodominated by two vertices of X_k for $k \neq k'$. Also since for any two vertices $x \in X_k, y \in X_{k'}$ with $k \neq k'$ there is two $x - y$ geodesic with length $d(x, y)$, then there is no vertex of X_k that is weakly geodominated by a pair (x, y) with $x \in X_k, y \in X_{k'}$.

Suppose that $|X| = \min\{|X_i|, i = 1, 2, ..., m\}$ then X is a weak geodominating set for G. Since $|X| \leq \frac{|S|}{m}$ then $g_w(K_m \times G) \geq mg_w(G)$.

Now let $V(G) = \{v_1, v_2, ..., v_n\}$. Let S' be a $g_w(G)$-set and let $S' = \{v_{i1}, v_{i2}, ..., v_{it}\}$, then $\{(1, v_{i1}), (1, v_{i2}), ..., (1, v_{it}), (2, v_{i1}), (2, v_{i2}), ..., (2, v_{it}), ..., (m, v_{i1}), (m, v_{i2}), ..., (m, v_{it})\}$ is a weak geodominating set for $K_m \times G$ which implies that $g_w(K_m \times G) \leq mg_w(G)$. $\square$

Theorem 2.5 *For any two positive integers a, b with $b \geq a + 1$ there exists a connected graph G with $|V(G)| = b, g_w(G) = a$.*

Proof. Let G be a graph obtained from $K_{1,a}$ by subdividing an edge xy to $xw_1w_2...w_{b-a-1}y$. Then $|V(G)| = b, g_w(G) = a$.

The following result shows that the weak geodomination number is affected by adding a pendant vertex.

Proposition 2.6 *Let G' be a graph obtained from G by adding a pendant vertex, then $g_w(G) \leq g_w(G') \leq 1 + g_w(G)$ and these bouns are sharp.*

Proof. Let G' be a graph obtained from G by adding a pendant vertex x and joining x to $y \in V(G)$. First any weak geodominating set for G' is a weak geodominating set for G, so $g_w(G) \leq g_w(G')$. Now let S be a weak geodominating set for G, then $S \cup \{x\}$ is a weak geodominating set for G'. Hence the result follows. The sharpness of the above bounds follows from Observation 2.2 and the fact that the complete graph K_n has no proper weak geodominating set.$\square$

3 Open (and k)- weak geodomination

In this section we introduce and study *open weak* geodomination as well as $k - weak$ geodomination in graphs.

Definition 3.1 *A pair x, y of vertices in a graph G open weakly geodominate a vertex v of G if $v \neq x, y$ and v is weakly geodominated by x and y. A set S is an open weakly geodominating set of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is open weakly geodominated by some pair of vertices of S. The minimum cardinality of an open weak geodominating set of G is its $open - weak$ geodomination number $og_w(G)$.*

An open weak geodominating set of cardinality $og_w(G)$ is called an $og_w(G)$-set. It follows from definition that for a graph G of order $n \geq 2$ $og_w(G) \geq g_w(G)$ and $2 \leq og_w(G) \leq n$.

The following has a straightforward proof and we left it out.

Observation 3.2 *1) If T is a tree with n vertices and l leaves, then $og_w(T) = l$,*

2) $g_w(P_n) = 2$,

3)$og_w(K_{1,n}) = n$,

4) $og_w(C_n) = \min\{n, 6 - \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor\}$,

5) $og_w(K_n) = g_w(K_n) = g(K_n) = n$,

6) $og_w(P_m \times P_n) = mn$,

7) $og_w(K_n \times K_n) = n^2$,

8) If a graph G has a proper open weak geodominating set then $|V(G)| \geq 3$,

9) If a graph G has $k \geq 2$ link-complete vertices, and $g_w(G) = k$, then $og_w(G) = k$,

10) For any graph G, $og_w(G \times K_m) = mog_w(G)$.

For a graph G and an integer $k \geq 1$, we say that a vertex v of G is $k-weakly$ geodominated by a pair x, y of distinct vertices in G if v is weakly geodominated by x, y and $d(x, y) = k$. A set S of vertices of G is a k-weak geodominating set of G if each vertex v in $V(G)\backslash S$ is k-weakly geodominated by some pair of distinct vertices of S. The minimum cardinality of a k-weak geodominating set of G is its *$k-$weak geodomination number* $g_{kw}(G)$.

A $k-$weak geodominating set of cardinality $g_{kw}(G)$ is called a g_{kw}-set. By definition any k-weak geodominating set is both a k-geodominating set and a weak geodominating set.

Let $k \geq 1$. A set S of vertices of G is an *open $k-$weak geodominating set* of G if for each vertex v, either (1) v is link-complete and $v \in S$ or (2) v is open weakly geodominated by some pair x, y of vertices of S with $d(x, y) = k$. The minimum cardinality of an open k-weak geodominating set of G is its *open $k-$weak geodomination number* $og_{kw}(G)$.

It follows from definition that for a graph G, $g_{1w}(G) = og_{1w}(G) = |V(G)|$. Also for any integer $k \geq 2$, $2 \leq og_{kw}(G) \leq |V(G)|$. The following has a simple proof and we left it.

Proposition 3.3 *1)If $k > diam(G)$, then $g_{kw}(G) = og_{kw}(G) = |V(G)|$, and for $k \geq 2$,*

2) $og_{kw}(G) \geq g_{kw}(G) \geq g_w(G)$,

3) $g_{kw}(G) \leq og_{kw}(G) \leq 3(og_{kw}(G))$,

4) $g_{kw}(K_{m,n}) = og_{kw}(K_{m,n}) = m + n$,

5) $g_{kw}(K_n \times K_n) = og_{kw}(K_n \times K_n) = n^2$,

6) If G' is a graph obtained from G by adding a pendant vertex, then $g_{kw}(G') \leq 1 + g_{kw}(G)$.

Proposition 3.4 *For two positive integers a, b, k with $b = (a-1)k + 1$ and $a \geq 2$, there exists a connected graph G with $|V(G)| = b$ and $g_{kw}(G) = a$.*

Proof. Let G be a graph obtained from $K_{1,a-1}$ by subdividing each of its edges $k-1$ times, then $|V(G)| = b$ and $g_{kw}(G) = a$.□

Any k-geodominating set of a path is a k-weak geodominating set. So by Lemma 3 of [5] we have the following

Proposition 3.5 *1)* $g_{2w}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

2) For each integer k *with* $3 \le k \le n-2$, $g_{kw}(P_n) = \left\lfloor \frac{n}{k} \right\rfloor + l$, *where*

$$l = \begin{cases} 1, & n \equiv 1(modk) \\ 2, & n \equiv 0,2(modk) \\ 3, & otherwise \end{cases}.$$

Proposition 3.6 *Let* $2 \le k \le \left\lfloor \frac{n}{2} \right\rfloor$, *then*

$$g_{kw}(C_n) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & n \equiv 0,1(modk),\ n \neq 2k \\ n, & n = 2k, \\ \left\lceil \frac{n}{k} \right\rceil + 1, & otherwise \end{cases}.$$

Proof. Let C_n be the n-cycle, $n \ge 3$, with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, ..., n-1\} \cup \{v_n v_1\}$. If $n = 2k$, there is no proper k-weak geodominating set for C_n, so we may let $n \neq 2k$. It is obvious that for each integers $n \ge 3$ and $k \ge 2$, $g_{kw}(C_n) \ge g_k(C_n) \ge \left\lceil \frac{n}{k} \right\rceil$. On the other hand the k-weak geodominating sets

$S = \{v_1, v_{k+1}, v_{2k+1}, ..., v_{(\lfloor \frac{n}{k} \rfloor - 1)k+1}\}$ for $n \equiv 0(modk)$, $n \neq 2k$ and

$S = \{v_1, v_{k+1}, v_{2k+1}, ..., v_{\lfloor \frac{n}{k} \rfloor k+1}\}$ for $n \equiv 1(modk)$ imply that $g_{kw}(C_n) = \left\lceil \frac{n}{k} \right\rceil$ when $n \equiv 0, 1(modk)$.

From now on let $n \neq 0, 1(modk)$. We show that no subset of G with size $\left\lceil \frac{n}{k} \right\rceil$ is a k-weak geodominating set. Suppose that S is a k-weak geodominating set for C_n with size $\left\lceil \frac{n}{k} \right\rceil$. It is easily seen that there is a vertex $v \in V(G) \backslash S$ which is not k-geodominated by two vertices of S which is a contradiction, hence $g_{kw}(C_n) \ge \left\lceil \frac{n}{k} \right\rceil + 1$. On the other hand let $n = \left\lfloor \frac{n}{k} \right\rfloor k + l$, $2 \le l < k$ and let $T = \{v_1, v_{k+1}, v_{2k+1}, ..., v_{\lfloor \frac{n}{k} \rfloor k+1}, v_{k-l+1}\}$, then T is a k-weak geodominating set. Hence $g_{kw}(C_n) = \left\lceil \frac{n}{k} \right\rceil + 1$.□

Note that for $k > \left\lfloor \frac{n}{2} \right\rfloor$, $g_{kw}(C_n) = n$.

References

[1] G. Chartrand, F. Harary and P. Zhang, *Geodetic sets in graphs*, Discuss. Math., Graph Theory, **20**(2000), 129-138.

[2] G. Chartrand, F. Harary, H. C.Swart and P. Zhang, *Geodomination in graphs*, Bull. ICA **31**(2001), 51-59.

[3] D. A. Mojdeh and N. Jafari Rad, *Connected geodomination in graphs*, Journal of Discrete Mathematical Science and Cryptography, **9** (2006), 177–186.

[4] D. A. Mojdeh and N. Jafari Rad, *(k-) perfect geodominating sets in graphs*, Opusc. Math., **27** (2007), 51-57.

[5] R. Muntean and P. Zhang, *k-geodomination in graphs*, Ars Comb., **63**(2002), 33-47.

[6] R. Muntean and P. Zhang, *On geodomination in graphs*, Congr. Numer., **143**(2000), 161-174.

[7] D. B. West, *Introduction to Graph Theory*, (2nd edition), Prentice-Hall, U.S.A, 2001.

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