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# Division algebras with dimension $2^t$, t$\in \mathbb{N}$

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**Abstract**

In this paper we find a field such that the algebras obtained by the Cayley-Dickson process are division algebras of dimension $2^t, \forall t \in \mathbb{N}$.

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From Frobenius Theorem and from the remark given by Bott and Milnor in 1958, we know that for $n \in \{1,2,4\}$ we find the real division algebras over the real field $\mathbb{R}$. These are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$(the real quaternion algebra), $\mathbb{O}$(the real octonions algebra ). They are unitary and alternative algebras. In 1978, Okubo gave an example of a division non alternative and non unitary real algebra with dimension 8, namely the real *pseudo-octonions* algebra.(See[7]). Here we find a field such that the algebras obtained by the Cayley-Dickson process are division algebras of dimension $2^t, \forall t \in \mathbb{N}$. First of all we describe shortly the Cayley-Dickson process.

**Definition 1.** Let $U$ be an arbitrary algebra. The vector spaces morphism $\phi : U \rightarrow U$ is called an **involution** of the algebra $U$ if $\phi(\phi(x)) = x$ and $\phi(xy) = \phi(y)\phi(x), \forall x, y \in U$.

Let $U$ be a arbitrary finite dimensional algebra with unity, $1 \neq 0$, with an involution $\phi : U \rightarrow U, \phi(a) = \overline{a}$, where $a + \overline{a}$ and $a\overline{a}$ belong in $K \cdot 1$, for all $a$ in $U$. Let $\alpha \in K$, be a non zero fixed element . Over the vector space $U \oplus U$, we define the multiplication:

$$(a_1, a_2)(b_1, b_2) = \left(a_1 b_1 - \alpha \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \quad (1)$$

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In this way we obtain an algebra structure over $U \oplus U$. We denote the obtained algebra by $(U, \alpha)$ and it is called **the derivate algebra** obtained from the algebra $U$ by **Cayley-Dickson process.** It is proved easily that the algebra $U$ is isomorphic with a subalgebra of the algebra $(U, \alpha)$, and $\dim (U, \alpha) = 2 \dim U$. We denote $v = (0, 1) \in U$ and we obtain that $v^2 = -\alpha \cdot 1$, then $(U, \alpha) = U \oplus Uv$. In the next, we denote the elements of the form $\alpha \cdot 1$ by $\alpha$ and each of these elements is in $U$.

Let $x = a_1 + a_2 v \in (U, \alpha)$. Denoting $\overline{x} = \bar{a}_1 - a_2 v$, we remark that $x + \overline{x} = a_1 + \overline{a_1} \in K \cdot 1$, $x\overline{x} = a_1 \overline{a_1} + \alpha a_2 \overline{a_2} \in K \cdot 1$. The map:

$$\psi : (U, \alpha) \rightarrow (U, \alpha) \quad \psi (x) = \bar{x} \, ,$$

is an involution of the algebra $(U, \alpha)$, which extends the involution $\phi$. If $x, y \in (U, \alpha)$, we have $\overline{xy} = \overline{y} \, \overline{x}$.

For $x \in U$ we denote $t(x) = x + \overline{x} \in K$, $n(x) = x\overline{x} \in K$, and we call them **the trace**, respectively **the norm** of the element $x$ from $U$. If $z \in (U, \alpha)$, so that $z = x + yv$, then $z + \overline{z} = t(z) \cdot 1$ and $z\overline{z} = \overline{z}z = n(z) \cdot 1$, where $t(z) = t(x)$ and $n(z) = n(x) + \alpha n(y)$. From this, we have that $(z + \overline{z}) z = z^2 + \overline{z}z = z^2 + n(z) \cdot 1$, therefore

$$z^2 - t(z) z + n(z) = 0, \forall z \in (U, \alpha) \, ,$$

so that each algebra obtained by the Cayley-Dickson process is a **quadratic algebra**. We remark that all algebras obtained by this process are flexible and power-associative algebras.

If in the Cayley-Dickson process we take $U = K$, $charK \neq 2$ and the involution $\psi(x) = x$, we obtain at the step $t$ an algebra of dimension $2^t$.

At the step 0, we obtaine the field $K$. This algebra has dimension 1.

At the step 1, we obtaine $\mathbb{K}(\alpha) = (K, \alpha), \alpha \neq 0$. This algebra has dimension 2. If the polynomial $X^2 + \alpha$ is irreducible over $K$ , then $\mathbb{K}(\alpha)$ is a field, otherwise $\mathbb{K}(\alpha) = K \oplus K$ and it is a non division algebra.

At the step 2, we obtaine $\mathbb{H}(\alpha, \beta) = (\mathbb{K}(\alpha), \beta), \beta \neq 0$, the generalized quaternion algebra. This algebra has dimension 4. This is an associative algebra, but it is not a commutative algebra.

At the step 3, we obtaine $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma), \gamma \neq 0$, the generalized octonion algebra or the Cayley-Dickson algebra. This algebra has dimension 8. This algebra is alternative(i.e. $x^2 y = x(xy)$ and $yx^2 = (yx)x$) but it is a non associative and non commutative algebra.

At the step 3, we obtain the generalized sedenion algebra, which is a non-alternative algebra, but it is a flexible and power-associative algebra. This algebra has dimension 16.

**Definition 2.** Let $U$ be an arbitrary algebra over the field $K$. $U$ is called **a division algebra** if and only if $U \neq 0$ and the equations:

$$ax = b,\ ya = b, \forall a, b \in U, a \neq 0,$$

have unique solutions in $U$.

From the beginning, we remark that if we started from a finite field, $K$, we don't obtained a division algebra, for any step of the Cayley-Dickson process. Indeed, we find the quaternion algebra. This algebra is always an associative and non commutative algebra. If this algebra is a division algebra it became a finite field. By Wedderburn's theorem, this field is a commutative field, false. Then we put the problem if we get the division algebra for all steps of the Cayley-Dickson process. If the field $K$ is infinite, the answer is positive and we show that in the next.

**Remark 3.** If an algebra $U$ is a finite dimensional algebra, then U is a division algebra if and only if from the relation $xy = 0$ it results that $x$ or $y$ must be zero.

Let $X_1, X_2, ..., X_t$ be $t$ algebraically independent elements over $K$. For $i \in \{1, 2, ..., t\}$ we build the algebra $U_i$ over the field $F = K(X_1, X_2, ..., X_t)$ putting $\alpha_j = X_j$ with $j = \overline{1, t}$ . Let $U_0 = F$ and the involution $\psi(x) = x$. We prove by induction on $i$ that $U_i$ is a division algebra for $i$=$\overline{1, t}$.

**Case 1.**

For $j = 1$ we have $U_1 = U_0 \oplus U_0$ with $\alpha_1 = X_1,\ v_1 = (0, 1) \in U_1$, $x = a + bv_1$, $y = c + dv_1$, $\overline{x} = a - bv_1$ and the multiplication

$$xy = ac - \alpha_1 db + (bc + da)\, v_1.$$

If $x, y$ are nonzero elements in $U_1$ such that $xy = 0$ we have

$$ac - X_1 db = 0 \tag{2}$$

and

$$bc + da = 0. \tag{3}$$

Since $a, b, c, d$ are in $F$, from the relations (2) and (3), we have that $a, b, c, d$ are non identically zero elements. Indeed, since $x \neq 0$ and $y \neq 0$ we have the possibilities:

i) $a = c = 0$ and $b, d \neq 0 \Rightarrow b = d = 0$, false.
ii) $a = d = 0$ and $b, c \neq 0 \Rightarrow b = c = 0$, false.
iii) $b = c = 0$ and $a, d \neq 0 \Rightarrow a = d = 0$, false.
iv) $b = d = 0$ and $a, c \neq 0 \Rightarrow a = c = 0$, false.

These have the form:

$$a = f(X_1, ..., X_t), \ \ b = g(X_1, ..., X_t),$$

$$c = h(X_1, ..., X_t), \ d = r(X_1, ..., X_t), f, g, h, r \in F$$

Since we can multiply the relations (2) and (3) with the great common divisor of denominators of $f, g, h, r$, we may suppose that $f, g, h, r$ are in $K[X_1, ..., X_t]$. Replacing the elements $a, b, c, d$ in the relation (2), we have: $x_1 \mid f$ or $x_1 \mid h$.

That means $X_1 \mid ac$. It results $X_1 \mid a$ or $X_1 \mid c$ or $X_1 \mid a$ and $X_1 \mid c$.

If $X_1 \mid a$ and $X_1 \mid c$ let $i_1, i_2$ be the greatest powers such that $a = X_1^{i_1} f_1$, $c = X_1^{i_1} h_2$, $f_1, h_1 \in K[X_1, ..., X_t]$. We replace in the relation (2), and we have:

$$X_1^{i_1+i_2} f_1 h_1 - X_1 db = 0.$$

Since $i_1 + i_2 \geq 2$, we obtain that $X_1 \mid db$. We have the possibilities:

i) $X_1 \mid d$ and $X_1 \nmid b$. It results that there is a great power $i_3$ such that $d = X_1^{i_3} r_1$, $r_1 \in K[X_1, ..., X_t]$. We replace in the relations (2) and (3) and we obtain:

$$X_1^{i_1+i_2} f_1 h_1 - X_1^{i_3+1} r_1 b = 0, \tag{2.1}$$

$$b X_1^{i_2} h_1 + X_1^{i_1+i_3} f_1 r_1 = 0. \tag{3.1}$$

a) If $i_2 < i_1 + i_3$ then $X_1 \mid bh_1$. Since $X_1 \nmid b$, we obtain $X_1 \mid h_1$. It results $h_1$ identically zero, so that $c = 0$, false.

b) $i_2 = i_1 + i_3$. From the relation (2.1), we have:

$$X_1^{2i_1+i_3} f_1 h_1 - X_1^{i_3+1} r_1 b = 0.$$

It results that $X_1 \mid r_1 b$. Since $X_1 \nmid b$, we have $r_1$ identically zero, then $d = 0$, false.

c) $i_2 > i_1 + i_3$. It result $X_1 \mid f_1 r_1$, then $f_1$ or $r_1$ are identically zero, false.

ii) $X_1 \nmid d$ and $X_1 \mid b$. Let $i_4$ be the greatest power such that $b = X_1^{i_4} g_1$, $g_1 \in K[X_1, ..., X_t]$. We replace in the relations (2) and (3) and we obtain:

$$X_1^{i_1+i_2} f_1 h_1 - X_1^{i_4+1} d g_1 = 0, \tag{2.2}$$

$$X_1^{i_2+i_4} g_1 h_1 + X_1^{i_1} d f_1 = 0. \tag{3.2}$$

a) If $i_1 < i_4 + i_2$ then $X_1 \mid df_1$, therefore $f_1$ is identically zero, false.

b) $i_1 = i_4 + i_2$. Then we have $X_1^{2i_2+i_4} f_1 h_1 - X_1^{i_4+1} d g_1 = 0$, therefore $X_1 \mid dg_1$, so that $g_1$ is identically zero, false.

c) $i_1 > i_4 + i_2$, then $X_1 \mid g_1 h_1$, false.

iii) $X_1 \mid d$ and $X_1 \mid b$. We replace in the relations (2) and (3) and we have:

$$X_1^{i_1+i_2} f_1 h_1 - X_1^{i_3+i_4+1} r_1 g_1 = 0, \tag{2.3}$$

$$X_1^{i_2+i_4} g_1 h_1 + X_1^{i_1+i_3} r_1 f_1 = 0. \tag{3.3}$$

a) $i_1 + i_3 < i_4 + i_2$. Then $X_1 \mid r_1 f_1$, false.
b) $i_1 + i_3 > i_4 + i_2$. Then $X_1 \mid g_1 h_1$, false.
c) $i_1 + i_3 = i_4 + i_2$. We have:
$c_1$) $i_1 + i_2 > i_3 + i_4 + 1$, then $X_1 \mid r_1 g_1$, false.
$c_2$) $i_1 + i_2 < i_3 + i_4 + 1$, then $X_1 \mid f_1 h_1$, false.
$c_3$) $i_1 + i_2 = i_3 + i_4 + 1$. Since $i_1 + i_3 = i_4 + i_2$, we add this last two relations and we have $i_1 + 2i_2 + i_4 = 2i_3 + i_1 + i_4 + 1$, false.

Then we have $X_1 \mid a$ or $X_1 \mid c$. We suppose that $X_1 \mid a$ or $X_1 \nmid c$. From the relation (3), we have $X_1 \mid b$, therefore $X_1 \mid g$. Let $s_1, s_2$ be the greatest numbers such that $X_1^{s_1} \mid f, X_2^{s_2} \mid g$ and $X_1^{s_1+1} \nmid f, X_1^{s_2+1} \nmid g$. Then $f = X_1^{s_1} f_1$ and $g = X_1^{s_2} g_1$ with $f_1, g_1 \in K[X_1, ..., X_t]$. Replacing in the relation (2), we obtain:

$$X_1^{s_1} f_1 c - X_1^{s_2+1} d g_1 = 0. \tag{2.4}$$

We have the cases:

1) $s_1 > s_2 + 1$. We have $X_1 \mid d g_1$ then $X_1 \mid r, r = X_1 r_1, r_{1 \in K[X_1,\ldots,X_t]}$. We replace in the relation (3) and we obtain:

$$X_1^{s_2} g_1 c + X_1^{s_1+1} r_1 f_1 = 0. \tag{3.4}$$

Then $X_1 \mid g_1 c$ therefore $X_1 \mid g_1$, so that $g_1$ is identically zero, that means $b = 0$, false.

2) $s_1 < s_2 + 1$. From the relation (2.4), we have $X_1 \mid f_1 c$, it results that $X_1 \mid f_1$, then $f_1$ is identically zero therefore $a = 0$, false.

3) $s_1 = s_2 + 1$. From the relation (3.4), we obtain $X_1 \mid g_1 c$ then $g_1$ is identically zero, therefore $b = 0$, false.

We obtain that $U_1$ is a division algebra.

**Case 2.**

For $j = 2$ we have $U_2 = U_1 \oplus U_1$ with $\alpha_2 = X_2$, $v_2 = (0,1) \in U_2$, $x = a + b v_2$, $y = c + d v_2$, $\bar{x} = \bar{a} - b v_2$ and the multiplication

$$xy = ac - \alpha_2 \bar{d} b + (b\bar{c} + da)\, v_2.$$

If $x, y$ are nonzero elements in $U_2$ such that $xy = 0$ we have

$$ac - X_2 \bar{d} b = 0 \tag{4}$$

and

$$b\bar{c}+da=0. \tag{5}$$

Like in Case 1, is obviously that $a,b,c,d$ are nonzero elements. Let $\{u_1,u_2,u_3,u_4\}$ be a basis in $U_2$ over $F$. Then, we can write:

$$a=\sum_{j=1}^{4} f_j(X_1,...,X_t)u_j,\ \ b=\sum_{j=1}^{4} g_j(X_1,...,X_t)u_j,$$

$$c=\sum_{j=1}^{4} h_j(X_1,...,X_t)u_j,\ \ d=\sum_{j=1}^{4} r_j(X_1,...,X_t)u_j,$$

where the elements $f_j(X_1,...,X_t)$, $g_j(X_1,...,X_t)$, $h_j(X_1,...,X_t)$, $r_j(X_1,...,X_t)$ belong in $F$. We remark that $f_j(X_1,...,X_t)$, $g_j(X_1,...,X_t)$, $h_j(X_1,...,X_t)$, $r_j(X_1,...,X_t)$ can be chosen in $K[X_1,...,X_t]$. We replace the elements $a,b,c,d$ in the relation (4) and we obtain $X_2 \mid ac$.

Since $U_1$ is a division algebra, like in the **Case 1**, we have $X_2 \mid a$ or $X_2 \mid c$. We suppose that $X_2 \mid a$ and $X_2 \nmid c$. It results that $X_2 \mid f_j$ , $\forall j=\overline{1,4}$, therefore $f_j=X_2f_j'$. Since $X_2 \mid a$ and $X_2 \nmid c$, from the relation (5), we have $X_2 \mid b$, then $X_2 \mid g_j$ , $\forall j=\overline{1,4}$, and $g_j=X_2g_j'$. Let $s_1$ be the greatest power of $X_2$ such that $X_2^{s_1} \mid f_j$ , $\forall j=\overline{1,4}$, and $s_2$ be the greatest power of $X_2$ such that $X_2^{s_1} \mid g_j, \forall j=\overline{1,4}$. Therefore we have $X_2^s \mid f_j$, $\forall j=\overline{1,4}$, and there is an index $t_1\in\{1,2,3,4\}$ such that, $X_2^{s_1+1} \nmid f_{t_1}$. In the same way, we have $X_2^s \mid g_j$, $\forall j=\overline{1,4}$, and there is an index $t_2\in\{1,2,3,4\}$ such that, $X_2^{s_2+1} \nmid f_{t_2}$. We replace in the relation (4) and we have the cases:

1) $s_1>s_2+1$. We simplify with $X_2^{s_2+1}$, and we have that $X_2 \mid r_j, \forall j=\overline{1,4}$. Replacing in the relation (5), we simplify and, since $X_2 \nmid c$ $\exists$ $t_3$ such that $X_2 \nmid h_{t_3}$. We have $X_2^{s_{2+1}} \mid g_j, \forall j=\overline{1,4}$, then $g_j$ are identically zero for all $j\in\{1,2,3,4\}$, false.

2) $s_1<s_2+1$. We simplify by $X_2^{s_1}$, and we have $X_2 \mid f_j, \forall j=\overline{1,4}$, since $X_2 \nmid c$, false.

3) $s_1=s_2+1$. Replacing in the relation (5), we simplify with $X_2^{s_2}$, and we obtain that $X_2 \mid g_j, \forall j=\overline{1,4}$, since $X_2 \nmid c$, false.

Therefore $U_2$ is a division algebra.

**Case i.**

By the induction step, we suppose that $U_{i-1}$ is a division algebra. We have $U_i=U_{i-1}\oplus U_{i-1}$, with $\alpha_i=X_i$, $v_i=(0,1)\in U_i$, $x=a+bv_i$, $y=c+dv_i$, $\bar{x}=\bar{a}-bv_i$ and the multiplication

$$xy=ac-\alpha_i\bar{d}b+(b\bar{c}+da)\,v_i.$$

If $x, y$ are nonzero elements in $U_i$ such that $xy = 0$, we have that

$$ac - X_i \bar{d} b = 0 \tag{6}$$

and

$$b\bar{c} + da = 0. \tag{7}$$

But $a, b, c, d$ are nonzero elements, since $x \neq 0$ and $y \neq 0$.

Let $\{u_1, u_2, ..., u_q\}$ be a basis in $U_{i-1}$ over $F, q = 2^{i-1}$. Then we can write:

$$a = \sum_{j=1}^{q} f_j(X_1, ..., X_t)u_j, \ \ b = \sum_{j=1}^{q} g_j(X_1, ..., X_t)u_j,$$

$$c = \sum_{j=1}^{q} h_j(X_1, ..., X_t)u_j, \ \ d = \sum_{j=1}^{q} r_j(X_1, ..., X_t)u_j,$$

where the elements $f_j(X_1, ..., X_t), g_j(X_1, ..., X_t), h_j(X_1, ..., X_t), r_j(X_1, ..., X_t)$ belong to $F$. We remark that $f_j(X_1, ..., X_t)$, $g_j(X_1, ..., X_t)$, $h_j(X_1, ..., X_t)$, $r_j(X_1, ..., X_t) \in K[X_1, ..., X_t]$. Replacing the elements $a, b, c, d$ in the relation (6), we obtain $X_i \mid ac$

Like in the **Case 1.**, since $U_{i-1}$ is a division algebra, we have $X_i \mid a$ or $X_i \mid c$. We suppose that $X_i \mid a$ or $X_i \nmid c$. It results that $X_i \mid f_j$ , $\forall j = \overline{1, q}$, and $f_j = X_i f_j'$. Since $X_i \mid a$ or $X_i \nmid c$, by the relation (7) , we have $X_i \mid g_j$ , $\forall j = \overline{1, q}$, and $g_j = X_i g_j'$. Let $s_1$ be the greatest power of $X_i$ such that $X_i^{s_1} \mid f_j$, $\forall j = \overline{1, q}$, and $s_2$ be the greatest power of $X_i$ such that $X_i^{s_2} \mid g_j, \forall j = \overline{1, q}$. Then we have $X_i^s \mid f_j$, $\forall j = \overline{1, q}$, and there is an index $t_1$ such that $X_i^{s_1+1} \nmid f_{t_1}$. Analogously, $X_i^s \mid g_j$, $\forall j = \overline{1, q}$, and there is an index $t_2$ such that $X_i^{s_2+1} \nmid f_{t_2}$. Replacing in the relation (6) we have the cases:

1) $s_1 > s_2 + 1$. We simplify by $X_i^{s_2+1}$, and it results that $X_i \mid r_j, \forall j = \overline{1, q}$. Replacing in the relation (7) , we simplify and, since $X_i \nmid c$, $\exists \ t_3$ such that $X_i \nmid h_{t_3}$. We have $X_i^{s_2+1} \mid g_j, \forall j = \overline{1, q}$, then $g_j$ are identically zero for all $j \in 1, ..., t$, false.

2) $s_1 < s_2 + 1$. We simplify with $X_i^{s_1}$, and we have $X_i \mid f_j, \forall j = \overline{1, q}$, since $X_i \nmid c$, false.

3) $s_1 = s_2 + 1$. Replacing in the relation (7), and simplify with $X_i^{s_2}$, then we have that $X_i \mid g_j, \forall j = \overline{1, q}$, since $X_i \nmid c$, false.

Therefore the algebra $U_i$ is a division algebra and, by Cayley-Dickson process, $\dim U_i = 2^i, 1 \leq i \leq n$.

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