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On the Consimilarity of Split Quaternions and Split Quaternion Matrices

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Abstract

In this paper, we introduce the concept of consimilarity of split quaternions and split quaternion matrices. In this regard, we examine the solvability conditions and general solutions of the equations $a\widetilde{x} = xb$ and $A\widetilde{X} = XB$ in split quaternions and split quaternion matrices, respectively. Moreover, coneigenvalue and coneigenvector are defined for split quaternion matrices. Some consequences are also presented.

1 Introduction

Hamilton introduced real quaternions that can be represented as [9]

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ q_s \in \mathbb{R}, s = 0, 1, 2, 3\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j. \tag{2}$$

It seems forthwith that multiplication of the real quaternions is not commutative owing to these ruled. So, it is not easy to work the real quaternions algebra problems. Similarly, it is well known that the main obstacle in study of the real quaternions matrices, dating back 1936 [19], is the non-commutative multiplication of the real quaternions. There are many studies on matrices of the

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real quaternions. So, Baker discussed right eigenvalues of the real quaternion matrices with a topological approach in [2]. On the other hand, Huang and So introduced on left eigenvalues of the real quaternion matrices [10]. After that Huang discussed consimilarity of the real quaternion matrices and obtained the Jordan canonical form of the real quaternion matrices down below consimilarity [11]. Jiang and Ling studied in [15] the problem of condiagonalization of the real quaternion matrices under consimilarity and gave two algebraic methods for the condiagonalization. Jiang and Wei studied the real quaternion matrix equation $X - A\widetilde{X}B = C$ by means of real representation of the real quaternion matrices [13]. Also, Jiang and Ling studied the problem of solution of the quaternion matrix equation $A\widetilde{X} - XB = C$ via real representation of a quaternions matrix [14].
After Hamilton had discovered the real quaternions, James Cockle defined, by using real quaternions, the set of split quaternions, in 1849 [5]. The split quaternions are not commutative like real quaternions. But the set of split quaternions contains zero divisors, nilpotent and nontrivial idempotent elements [16]. The split quaternions are a recently developing topic, since the split quaternions are used to express Lorentzian relations. Also, there are many studies on geometric and physical meaning of the split quaternions [16]-[17]. In [8], author studied equations $ax = xb$, $a\overline{x} = xb$ and $x^2 = a$, in algebras obtained by the Cayley-Dickson process. One of this algebras is algebra of split quaternions. In [18], authors gave eigenvalue problem of a rotation matrix in Minkowski 3 space by using split quaternions. In [6], authors investigated linear split quaternionic equations with the terms of the form axb. Alagoz *et al.* considered split quaternion matrices. They investigated the split quaternions matrices using properties of complex matrices [1]. After that Erdogdu and Ozdemir obtained method of finding eigenvalues of the split quaternions matrices. Also, they gave an extension of Gershgorin theorem for the split quaternion matrices in [7]. Zhang *et al.* studied the split quaternionic least square problem, derived two algebraic methods for finding solutions of the problems in split quaternionic mechanics [20].

2 Consimilarity of Split Quaternions

Let $\mathbb{R}$ be the real number field, $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ be complex number field, and $\mathbb{H}_S = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the split quaternion field over $\mathbb{R}$, where

$$\begin{gathered} i^2 = -1,\ j^2 = k^2 = 1 \\ ij = -ji = k, jk = -kj = -i, ki = -ik = j. \end{gathered} \tag{3}$$

The real part and the imaginary part of $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$ are defined as $\operatorname{Re} a = a_0$ and $\operatorname{Im} a = a_1 i + a_2 j + a_3 k$, respectively.

The multiplication of $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ is defined as

$$ab = \operatorname{Re} a \operatorname{Re} b + g(\operatorname{Im} a, \operatorname{Im} b) + \operatorname{Re} a \operatorname{Im} b + \operatorname{Re} b \operatorname{Im} a + \operatorname{Im} a \times \operatorname{Im} b$$

where

$$g(\operatorname{Im} a, \operatorname{Im} b) = -a_1 b_1 + a_2 b_2 + a_3 b_3,$$
$$\operatorname{Im} a \times \operatorname{Im} b = (a_3 b_2 - a_2 b_3) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k.$$

The conjugate of a split quaternion is denoted by $\overline{a}$ and it is

$$\bar{a} = a_0 - a_1 i - a_2 j - a_3 k = \operatorname{Re} a - \operatorname{Im} a.$$

The norm of a split quaternion is defined as

$$\|a\| = \sqrt{|a\overline{a}|} = \sqrt{|a_0^2 + a_1^2 - a_2^2 - a_3^2|}.$$

Also, a split quaternion a is said to be spacelike, timelike or lightlike (null), if $a\overline{a} < 0$, $a\overline{a} > 0$ or $a\overline{a} = 0$, respectively.
The linear transformation $\phi, \tau : \mathbb{H}_S \to End(\mathbb{H}_S)$, given by

$$\phi(a) : \mathbb{H}_S \to \mathbb{H}_S, \quad \phi(a)(x) = ax$$

and

$$\tau(a) : \mathbb{H}_S \to \mathbb{H}_S, \quad \tau(a)(x) = xa,$$

are called the left representation and the right representation of the algebra $\mathbb{H}_S$, respectively. We know that every associative finite-dimensional algebra A over an arbitrary K is isomorphic with a subalgebra of the matrix algebra $M_n(K)$. So we could find a faithful representation for the algebra A in the matrix algebra $M_n(K)$, [8]. For the split quaternion algebra $\mathbb{H}_S$, the mapping:

$$\phi : \mathbb{H}_S \to M_4(\mathbb{R}), \ \phi(a) = \begin{pmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \tag{4}$$

is an isomorphism between $\mathbb{H}_S$ and the algebra of matrices with the above form. The matrix $\phi(a)$ is called the left matrix representation for split quaternion $a \in \mathbb{H}_S$. In the same manner, we introduce the right matrix representation for the split quaternion a as;

$$\tau : \mathbb{H}_S \to M_4(\mathbb{R}), \ \tau(a) = \begin{pmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}, \tag{5}$$

where $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$, [16].
It is nearby to identify a split quaternion $a \in \mathbb{H}_S$ with a vector $\overrightarrow{a} \in \mathbb{R}_2^4$ (where $\mathbb{R}_2^4$ is semi-Euclidean space [17]). We will denote such identification by the symbol i.e.

$$a = a_0 + a_1 i + a_2 j + a_3 k \cong \overrightarrow{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Also, we show that conjugate, real part and imaginary part of a

$$\overline{a} \cong \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} = Ca, \quad \mathrm{Re}a = a_0 \cong a_0 e_1 \text{ and } \mathrm{Im}a \cong \overrightarrow{\mathrm{Im}a} = \begin{pmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

respectively, where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 1. *[16] If $a, b, x \in \mathbb{H}_S$ and $c \in \mathbb{R}$, then we have:*

1. $a = b \Leftrightarrow \phi(a) = \phi(b) \Leftrightarrow \tau(a) = \tau(b),$
2. $\phi(a+b) = \phi(a) + \phi(b), \quad \tau(a+b) = \tau(a) + \tau(b),$
3. $\phi(ca) = c\phi(a), \ \tau(ca) = c\tau(a),$
4. $ab = \phi(a)\overrightarrow{b}, \quad ab = \tau(b)\overrightarrow{a}, \quad \phi(a)\tau(b) = \tau(b)\phi(a),$
5. $axb = \phi(a)\tau(b)\overrightarrow{x} = \tau(b)\phi(a)\overrightarrow{x},$
6. $\phi(ab) = \phi(a)\phi(b), \ \tau(ab) = \tau(b)\tau(a),$
7. $\phi(\overline{a}) = \varepsilon(\phi(a))^T\varepsilon, \quad \tau(\overline{a}) = \varepsilon(\tau(a))^T\varepsilon, \quad \varepsilon = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix},$
8. $\phi(a^{-1}) = \phi^{-1}(a), \quad \tau(a^{-1}) = \tau^{-1}(a), \quad \|a\| \neq 0,$
9. $\phi(a^{-1}) = \begin{cases} -\frac{1}{\|a\|^2}\varepsilon(\phi(a))^T\varepsilon & if\ a\ is\ spacelike \\ \frac{1}{\|a\|^2}\varepsilon(\phi(a))^T\varepsilon & if\ a\ is\ timelike \\ There\,is\,no\,inverse & if\ a\ is\ lightlike \end{cases}$

10. $det(\phi(a)) = det(\tau(a)) = \|a\|^2$.

From Theorem 1, we get

$$ax - xb = (\phi(a) - \tau(b))\overrightarrow{x}. \tag{6}$$

The authors showed the following result about the difference $\phi(a) - \tau(b)$ in the last equation [16].

Theorem 2. *[16] Let $a = a_0 + a_1 i + a_2 j + a_3 k,\ \ b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}_S$ be given, and denote $\delta(a,b) = \phi(a) - \tau(b)$. Then*

i. If a and b are two split quaternions with $g(\mathrm{Im}a, \mathrm{Im}a) < 0,\ \ g(\mathrm{Im}b, \mathrm{Im}b) < 0$ or $g(\mathrm{Im}a, \mathrm{Im}a) > 0,\ \ g(\mathrm{Im}b, \mathrm{Im}b) > 0$ then, the determinant of $\delta(a,b)$ is

$$det(\delta(a,b)) = s^4 - 2s^2\left((\mathrm{Im}a)^2 + (\mathrm{Im}b)^2\right) + \left((\mathrm{Im}a)^2 - (\mathrm{Im}b)^2\right)^2$$

or

$$det(\delta(a,b)) = s^4 + 2s^2\left((\mathrm{Im}a)^2 + (\mathrm{Im}b)^2\right) + \left((\mathrm{Im}a)^2 - (\mathrm{Im}b)^2\right)^2$$

where $s = a_0 - b_0$. Thus $det(\delta(a,b)) = 0$ if and only if $\mathrm{Re}a = \mathrm{Re}b$ and $g(\mathrm{Im}a, \mathrm{Im}a) = g(\mathrm{Im}b, \mathrm{Im}b)$.

ii. If $a_0 \neq b_0$, or $g(\mathrm{Im}\,a, \mathrm{Im}\,a) \neq g(\mathrm{Im}\,b, \mathrm{Im}\,b)$, then $\delta(a,b)$ is non-singular and its inverse can be written as

$$\begin{aligned}\delta^{-1}(a,b) &= \phi^{-1}\left(a^2 - 2b_0 a + \|b\|\right)\left(\phi(a) - \tau\left(\bar{b}\right)\right)\\ &= \phi^{-1}\left(2(a_0 - b_0)a + \|b\| - \|a\|\right)\left(\phi(a) - \tau\left(\bar{b}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\delta^{-1}(a,b) &= \tau^{-1}\left(b^2 - 2a_0 b + \|a\|\right)\left(\phi(\bar{a}) - \tau(b)\right)\\ &= \tau^{-1}\left(2(b_0 - a_0)b + \|a\| - \|b\|\right)\left(\phi(\bar{a}) - \tau(b)\right)\end{aligned}.$$

iii. If $a_0 = b_0$ and $g(\mathrm{Im}(a), \mathrm{Im}(a)) = g(\mathrm{Im}(b), \mathrm{Im}(b))$, then $\delta(a,b)$ is singular and has a generalized inverse as follows

$$\delta^{-}(a,b) = \frac{1}{4(\mathrm{Im}a)^2}\delta(a,b) = \frac{1}{4(\mathrm{Im}a)^2}\left(\phi(\mathrm{Im}a) - \tau(\mathrm{Im}b)\right).$$

Definition 1. [16] The split quaternions a and b is said to be similar if there exists a split quaternion $p, \|p\| \neq 0$ such that $p^{-1}ap = b$. The relation, a is similar to b, is denoted $a \sim b$. Similarity is an equivalence relation on the split quaternions.

Two complex matrix $A, B \in \mathbb{C}^{n\times n}$ are complex consimilar if there exists an invertible complex matrix P such that $\overline{P}\,AP^{-1} = B$. Complex consimilarity is an equivalence relation on $C^{n\times n}$ and has been extensively studied [12]. The split quaternion holds an important place in differential geometry and structure theory of Lorentz Space [16],[17], and for this reason consimilarity relation will be defined for split quaternions.
If $a, b \in \mathbb{H}_S$, generally $\overline{ab} \neq \overline{a}\,\overline{b}$. Thus the mapping $a \to \overline{p}ap^{-1}$ is not an equivalence relation on $\mathbb{H}_S$. Thus we need to give a new definition of consimilarity of split quaternion matrices.

Definition 2. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$, then we define $\widetilde{a} = jaj = a_0 - a_1 i + a_2 j - a_3 k$. We say that $\widetilde{a}$ is the $j - conjugate$ of a.

For any $a, b \in \mathbb{H}_S$, the following equalities are easy to confirm

i. $\widetilde{(\widetilde{a})} = a$;

ii. $\widetilde{(a+b)} = \widetilde{a} + \widetilde{b}$;

iii. $\widetilde{(ab)} = \widetilde{a}\widetilde{b}$;

iv. $\overline{(\widetilde{a})} = \widetilde{(\overline{a})}$.

Definition 3. The split quaternions a and b is said to be consimilar if there exists a split quaternion $p, \|p\| \neq 0$ such that $\widetilde{p}ap^{-1} = b$. This relation is denoted $a \overset{c}{\sim} b$.

Theorem 3. *For $a, b, c \in \mathbb{H}_S$, the followings are satisfied:*

- *Reflexive: $a \overset{c}{\sim} a$;*
- *Symmetric: if $a \overset{c}{\sim} b$, then $b \overset{c}{\sim} a$;*
- *Transitive: if $a \overset{c}{\sim} b$ and $b \overset{c}{\sim} c$ then $a \overset{c}{\sim} c$.*

Proof.

- Reflexive: $\widetilde{1}\,a\,1^{-1} = a$ trivially, for $a \in \mathbb{H}_S$. So, consimilarity is reflexive.

- Symmetric: Let $\widetilde{p}ap^{-1} = b$. Since p is nonsingular, we have
$$(\widetilde{p})^{-1}bp = (\widetilde{p})^{-1}\widetilde{p}ap^{-1}p = a.$$
So, consimilarity is symmetric.
- Transitive: Let $\widetilde{p_1}ap_1{}^{-1} = b$ and $\widetilde{p_2}bp_2{}^{-1} = c$. Then
$$c = \widetilde{p_2}\widetilde{p_1}ap_1^{-1}p_2^{-1} = (\widetilde{p_2p_1})\, a(p_2p_1)^{-1}.$$
So, consimilarity is transitive.

$\square$

Then, by Theorem 3, consimilarity is an equivalence relation on $\mathbb{H}_S$.

Theorem 4. *Let $a, b \in \mathbb{H}_S$ be given. Then the linear equation*
$$ax - \widetilde{x}\, b = 0 \tag{7}$$
has a nonzero solution ($\|x\| \neq 0$), i.e., a and b are consimilar, if and only if
$$\operatorname{Re}(ja) = \operatorname{Re}(jb) \quad \textit{and} \quad g(\operatorname{Im}(ja), \operatorname{Im}(ja)) = g(\operatorname{Im}(jb), \operatorname{Im}(jb)) \neq 0. \tag{8}$$
In that case, the general solution of (7) is
$$x = \left[p + \frac{1}{(\operatorname{Im} ja)^2}(\operatorname{Im} ja)p(\operatorname{Im} jb)\right] \tag{9}$$
where $p \in \mathbb{H}_S$ is arbitrary, in particular, if $ja \neq -\overline{b}j$, i.e., $\operatorname{Im}(ja) + \operatorname{Im}(jb) \neq 0$, then the general solution of (7) can be written as
$$x = \lambda_1 (\operatorname{Im}(ja) + \operatorname{Im}(jb)) + \lambda_2 \left(\operatorname{Im}(ja)\operatorname{Im}(jb) + (\operatorname{Im}(ja))^2\right) \tag{10}$$
where λ_1, $\lambda_2 \in \mathbb{R}$ are arbitrary.

Proof. The equation (7) can be expressed as
$$ax - jxjb = 0 \Leftrightarrow jax - xjb = 0 \Leftrightarrow [\phi(ja) - \tau(jb)]\, \overrightarrow{x} = \delta(ja, jb)\, \overrightarrow{x} = 0.$$
This equation has a nonzero solution for x if and only if $det(\delta(ja, jb)) = 0$, which is equivalent to (8). In the present case, the general solution can be written as
$$x = 2\left[I_4 - \delta^{-}(ja, jb)\, \delta(ja, jb)\right] \overrightarrow{p}$$

where $\overrightarrow{p}$ is an arbitrary vector. An expression of $\delta^{-}(ja, jb)$ can be derived from Theorem 2. Thus

$$x = 2\left[I_4 - \frac{1}{4(\operatorname{Im} ja)^2}\delta(ja, jb)\,\delta(ja, jb)\right]\overrightarrow{p}$$

$$= 2\left[I_4 - \frac{1}{(\operatorname{Im} ja)^2}\Big(2(\operatorname{Im} ja)^2 I_4 - 2\phi(\operatorname{Im} ja)\,\tau(\operatorname{Im} jb)\Big)\right]\overrightarrow{p}$$

$$= \left[I_4 + \frac{1}{(\operatorname{Im} ja)^2}\phi(\operatorname{Im} ja)\,\tau(\operatorname{Im} jb)\right]\overrightarrow{p}.$$

Applying it to split quaternion form by Theorem 1 in the last equation, we have

$$x = \left[p + \frac{1}{(\operatorname{Im} ja)^2}(\operatorname{Im} ja)p(\operatorname{Im} jb)\right].$$

If $ja \neq -\bar{b}j$ in (9), then we set $p = \operatorname{Im}(ja)$ and $p = (\operatorname{Im}(ja))^2$ in (9), respectively, and (9) becomes

$$x_1 = \operatorname{Im}(ja) + \operatorname{Im}(jb)\,, \quad x_2 = (\operatorname{Im}(ja))^2 + \operatorname{Im}(ja)\operatorname{Im}(jb)\,.$$

Thus (10) is also a solution to (7) under (8). The independence of x_1 and x_2 can be seen from two simple facts that $\operatorname{Re}x_1 = 0$ and $\operatorname{Re}x_2 \neq 0$. Therefore (10) is exactly the general solution to (7), since the rank of $\delta(ja, jb)$ is two under (8).

□

3 Consimilarity of Split Quaternion Matrices

The set of $m \times n$ matrices with the split quaternion entries, which is denoted by $\mathbb{H}_S^{m\times n}$ with ordinary matrix addition and multiplication is a ring with unity. Let A^T, $\overline{A}$ and $A^* = \overline{(A^T)}$ be transpose, conjugate and transpose conjugate matrix of $A \in \mathbb{H}_S^{m\times n}$, *respectively.*

Theorem 5. *[1] For any $A \in \mathbb{H}_S^{m\times n}$ and $B \in \mathbb{H}_S^{n\times s}$, the followings statements are valid:*

i. $\left(\overline{A}\right)^T = \overline{(A^T)}$;

ii. $(AB)^* = B^* A^*$;

iii. *If A and B are nonsingular ($m = n = s$), $(AB)^{-1} = B^{-1}A^{-1}$;*

iv. *If A is nonsingular ($m = n$),* $(A^*)^{-1} = \left(A^{-1}\right)^*$;

v. *If A is nonsingular ($m = n$),* $\left(\overline{A}\right)^{-1} \neq \overline{(A^{-1})}$, *in general;*

vi. *If A is nonsingular ($m = n$),* $\left(A^T\right)^{-1} \neq \left(A^{-1}\right)^T$, *in general;*

vii. $\overline{AB} \neq \overline{A}\,\overline{B}$, *in general;*

viii. $(AB)^T \neq B^T A^T$, *in general.*

Definition 4. Let $A \in \mathbb{H}_S^{n\times n}$, then we define $\widetilde{A} = jAj$. We say that $\widetilde{A}$ is the $j-conjugate$ of A.

For any $A, B \in \mathbb{H}_S^{m\times n}$ and $C \in \mathbb{H}_S^{n\times s}$, the following equalities are easy to confirm

i. $\widetilde{\left(\widetilde{A}\right)} = A$;

ii. $\widetilde{(A+B)} = \widetilde{A} + \widetilde{B}$;

iii. $\widetilde{(AC)} = \widetilde{A}\widetilde{C}$;

iv. $\overline{\left(\widetilde{A}\right)} = \widetilde{(\overline{A})}$.

Theorem 6. *If $A \in \mathbb{H}_S^{n\times n}$, in that case*

$$A \text{ is nonsingular } \Leftrightarrow \widetilde{A} \text{ is nonsingular } \Leftrightarrow A^* \text{ is nonsingular.}$$

Furthermore, if A is nonsingular, $(A^*)^{-1} = \left(A^{-1}\right)^*$ *and* $\left(\widetilde{A}\right)^{-1} = \widetilde{(A^{-1})}$.

Proof. Since A is nonsingular, there exists an matrix $A^{-1} \in \mathbb{H}_S^{n\times n}$ so that $AA^{-1} = I_n$. Thus $AA^{-1} = I_n \Leftrightarrow jAjjA^{-1}j = I_n$ and $\left(\widetilde{A}\right)^{-1} = \widetilde{(A^{-1})}$. Similar way, we get $A^*\left(A^{-1}\right)^* = I_n$. □

Definition 5. The split quaternion matrices A and B is said to be consimilar if there exists a split quaternion P such that $\widetilde{P}AP^{-1} = B$. This relation is denoted as $A \overset{c}{\sim} B$. Consimilarity relation is an equivalence relation on $\mathbb{H}_S^{n\times n}$.

Clearly if $A \in \mathbb{C}^{n\times n}$, then $\overline{A} = \widetilde{A} = jAj$. Thus, $A \in \mathbb{C}^{n\times n}$ is consimilar to $B \in \mathbb{C}^{n\times n}$ as complex matrices if A is consimilar to B as split quaternion matrices. Then, consimilarity relation in $\mathbb{H}_S^{n\times n}$ is a natural extension of complex consimilarity in $\mathbb{C}^{n\times n}$.

Theorem 7. *If* $A, B \in \mathbb{H}_S^{n\times n}$, *then*

$$A \overset{c}{\sim} B \;\Leftrightarrow\; jA \sim jB \;\Leftrightarrow Aj \sim Bj \;\Leftrightarrow jA \sim Bj.$$

Proof. Since $A \overset{c}{\sim} B \;\Leftrightarrow$ there exists a nonsingular matrix $P \in \mathbb{H}_S^{n\times n}$ so that $\widetilde{P}AP^{-1} = jPjAP^{-1} = B$. Thus $A \overset{c}{\sim} B \;\Leftrightarrow PjAP^{-1} = jB \Leftrightarrow jA \sim jB$. Since $j^{-1}jAj = Aj$, we get $jA \sim Aj$ and $jB \sim Bj$. Therefore $jA \sim jB \Leftrightarrow Aj \sim Bj \Leftrightarrow jA \sim Bj$.

□

Definition 6. Let $A \in \mathbb{H}_S^{n\times n}$, $\lambda \in \mathbb{H}_S$. If there exists $0 \neq x \in \mathbb{H}_S^{n\times 1}$ such that

$$A\widetilde{x} = x\lambda \;\; (A\widetilde{x} = \lambda x)$$

then λ is said to be a right (left) coneigenvalues of A and x is said to be a coneigenvector of A corresponding to the right (left) coneigenvalue λ. The set of right coneigenvalues is defined as

$$\widetilde{\sigma}_r(A) = \{\lambda \in \mathbb{H}_S : A\widetilde{x} = x\lambda, \; for\ some\ x \neq 0\}.$$

The set of left coneigenvalues is similarly defined and is denoted by $\widetilde{\sigma}_l(A)$.

Recall that if $x \in \mathbb{H}_S^{n\times 1}(x \neq 0)$, and $\lambda \in \mathbb{H}_S$ satisfying $Ax = x\lambda$ $(Ax = \lambda x)$, we call x an eigenvector of A, while λ is an right (left) eigenvalue of A. We also say that x is an eigenvector corresponding to the right(left) eigenvalue λ.

Theorem 8. *Let* $A, B \in \mathbb{H}_S^{n\times n}$, *if* A *is consimilar to* B, *then* A *and* B *have the same right coneigenvalues.*

Proof. Let $A \overset{c}{\sim} B$, then, there exists a nonsingular matrix $P \in \mathbb{H}_S^{n\times n}$ such that $B = \widetilde{P}AP^{-1}$. Let $\lambda \in \mathbb{H}_S$ be a right coneigenvalue for the matrix A, then we find the matrix $0 \neq x \in \mathbb{H}_S^{n\times 1}$ such that $A\widetilde{x} = x\lambda$. Let $y = P\widetilde{x}$. Finally $By = \widetilde{P}AP^{-1}y = \widetilde{P}A\widetilde{x} = \widetilde{P}x\lambda = \widetilde{y}\lambda$. □

Theorem 9. *If* $A \in \mathbb{H}_S^{n\times n}$, *in that case* λ *is right coneigenvalue of* A *if and only if for any* $\beta \in \mathbb{H}_S$ $(\|\beta\| \neq 0)$, $\widetilde{\beta}\lambda\beta^{-1}$ *is a right eigenvalue of* A.

Proof. From $A\widetilde{x} = x\lambda$, we get $A(\widetilde{x}\beta) = x\,\widetilde{\beta}\left(\widetilde{\beta}\right)^{-1}\lambda\beta$. □

Theorem 10. *If* $A \in \mathbb{H}_S^{n\times n}$ *and* $\lambda \in \mathbb{H}_S$, *then* λ_0 *is a right coneigenvalue of* $A \;\Leftrightarrow\; j\lambda_0$ *is a right eigenvalue of* $A\,j \;\Leftrightarrow\; \lambda_0 j$ *is a right eigenvalue of* jA.

Proof. Suppose that λ_0 is right coneigenvalue of A. Then $0 \neq x \in \mathbb{H}_S^{n\times n}$ so that $A\widetilde{x} = Ajxj = x\lambda_0 \Leftrightarrow Ajx = x(\lambda_0 j) \Leftrightarrow \lambda_0 j$ is a right eigenvalue of Aj. Also, $A\widetilde{x} = x\lambda_0 \Leftrightarrow jA\widetilde{x} = jxjj\lambda_0 = \widetilde{x}j\lambda_0 \Leftrightarrow j\lambda_0$ is a right eigenvalue of jA.

□

Definition 7. [1] Let $A = A_1 + A_2 j \in \mathbb{H}_S^{n\times n}$ where $A_s \in \mathbb{C}^{n\times n}$, $s = 1, 2$. The $2n \times 2n$ matrix

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix}$$

is called the complex adjoint matrix of A and denoted χ_A.

Theorem 11. *[1] Let $A, B \in \mathbb{H}_S^{n\times n}$, then the followings are satisfied:*

i. $\chi_{A+B} = \chi_A + \chi_B$;

ii. $\chi_{AB} = \chi_A \chi_B$;

iii. *If A is nonsingular,* $(\chi_A)^{-1} = \chi_{A^{-1}}$;

iv. *In general* $\chi_{A^*} \neq (\chi_A)^*$.

Theorem 12. *For every $A \in \mathbb{H}_S^{n\times n}$,*

$$\widetilde{\sigma}_r(A) \cap \mathbb{C} = \widetilde{\sigma}(\chi_A)$$

where $\widetilde{\sigma}(\chi_A) = \left\{\lambda \in \mathbb{C} : \chi_A \overline{y} = \lambda y, \ for\ some\ 0 \neq y \in \mathbb{C}^{n\times 1}\right\}$ *is the set of coneigenvalues of* χ_A.

Proof. Let $A = A_1 + A_2 j \in \mathbb{H}_S^{n\times n}$ such that $A_s \in \mathbb{C}^{n\times n}$, $s = 1, 2$, and $\lambda \in \mathbb{C}$ be a right coneigenvalue of A. Therefore there exists nonzero column vector $x \in \mathbb{H}_S^{n\times 1}$ such that $A\widetilde{x} = x\lambda$. This implies

$$\begin{aligned} &(A_1 + A_2 j)(\overline{x_1} + \overline{x_2} j) = (x_1 + x_2 j)\lambda \\ &(A\overline{x_1} + A_2 x_2) = x_1\lambda \quad \text{and} \quad (A_2 x_1 + A_1\overline{x_2}) = x_2\overline{\lambda}. \end{aligned}$$

Using these equations, we can write

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \overline{x_2} \end{pmatrix}.$$

Therefore, the complex right coneigenvalue of the split quaternion matrix A is an equivalent to the coneigenvalue of the adjoint matrix χ_A that is $\widetilde{\sigma}_r(A) \cap \mathbb{C} = \widetilde{\sigma}(\chi_A)$.

□

4 Real Representation of Split Quaternion Matrices

Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}_S^{m\times n}$ where A_s are $m \times n$ real matrices, $s = 0, 1, 2, 3$. We define the linear transformation $\phi_A(X) = A\widetilde{X}$. Then, we can write

$$\begin{aligned}
&\phi_A(1) = A = A_0 + A_1 i + A_2 j + A_3 k\\
&\phi_A(i) = A\,\widetilde{i} = A_1 - A_0 i - A_3 j + A_2 k\\
&\phi_A(j) = A\,\widetilde{j} = A_2 + A_3 i + A_0 j + A_1 k\\
&\phi_A(k) = A\,\widetilde{k} = -A_3 + A_2 i + A_1 j - A_0 k.
\end{aligned}$$

Then, we find the real representation of the split quaternion matrix A as follows:

$$\phi_A = \begin{pmatrix} A_0 & A_1 & A_2 & -A_3 \\ A_1 & -A_0 & A_3 & A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & -A_0 \end{pmatrix} \in \mathbb{R}^{4m\times 4n}.$$

Theorem 13. *For the split quaternion matrix A, the following identities are satisfied:*

i. *If $A \in \mathbb{H}_S^{m\times n}$, then*

$$P_m^{-1}\phi_A P_n = \phi_{\widetilde{A}}, \quad Q_m^{-1}\phi_A Q_n = -\phi_A, \quad R_m^{-1}\phi_A R_n = \phi_A, \quad S_m^{-1}\phi_A S_n = -\phi_A;$$

where

$$P_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}, \; Q_m = \begin{pmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & -I_m & 0 \end{pmatrix},$$

$$R_m = \begin{pmatrix} 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \\ -I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \end{pmatrix}, \; S_m = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & -I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix},$$

ii. *If $A, B \in \mathbb{H}_S^{m\times n}$, then $\phi_{A+B} = \phi_A + \phi_B$;*

iii. *If $A \in \mathbb{H}_S^{m\times n}$, $B \in \mathbb{H}_S^{n\times r}$, in that case $\phi_{AB} = \phi_A P_n \phi_B = \phi_A \phi_{\widetilde{B}} P_r$;*

iv. *If $A \in \mathbb{H}_S^{m\times m}$, in that case A is nonsingular if and only if ϕ_A is nonsingular, $\phi_A^{-1} = P_m \phi_{A^{-1}} P_m$;*

v. *If* $A \in \mathbb{H}_S^{m\times m}$, $\phi_{\overline{A}} = \varepsilon_2 (\phi_A)^T \varepsilon_2$ *where* $\varepsilon_2 = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$;

vi. *If* $A \in \mathbb{H}_S^{m\times m}$,

$$A = A_0 + A_1 i + A_2 j + A_3 k = \frac{1}{4} \left(I_m \, iI_m \, jI_m \, kI_m \right) \phi_A \begin{pmatrix} I_n \\ iI_n \\ jI_n \\ -kI_n \end{pmatrix};$$

vii. *If* $A \in \mathbb{H}_S^{m\times m}$,

$$\widetilde{\sigma}_l (A) \cap \mathbb{C} = \sigma (\phi_A)$$

where $\sigma(\phi_A) = \{\lambda \in \mathbb{C} : \ \phi_A y = \lambda y, \ for \ some \ y \neq 0\}$ *is the set of eigenvalues of* ϕ_A.

Proof. The first six statements can be seen in an easy way. Thus we will prove vii.
Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}_S^{m\times m}$ where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times m}$ and $\lambda \in \mathbb{C}$ be a left coniegenvalue of A. Thus there exists a nonzero vector $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}_S^{m\times 1}$ such that $A\widetilde{x} = \lambda x$. This implies

$$(A_0 + A_1 i + A_2 j + A_3 k)(x_0 - x_1 i + x_2 j - x_3 k) = \lambda (x_0 + x_1 i + x_2 j + x_3 k)$$

$$\begin{aligned} &(A_0x_0 + A_1x_1 + A_2x_2 - A_3x_3) + i(A_1x_0 - A_0x_1 + A_3x_2 + A_2x_3) \\ &+j(A_2x_0 - A_3x_1 + A_0x_2 + A_1x_3) + k(A_3x_0 + A_2x_1 + A_1x_2 - A_0x_3) \\ &= \lambda x_0 + \lambda x_1 i + \lambda x_2 j + +\lambda x_3 k. \end{aligned}$$

Therefore we obtain the following equations

$$\begin{aligned} A_0x_0 + A_1x_1 + A_2x_2 - A_3x_3 &= \lambda x_0 \\ A_1x_0 - A_0x_1 + A_3x_2 + A_2x_3 &= \lambda x_1 \\ A_2x_0 - A_3x_1 + A_0x_2 + A_1x_3 &= \lambda x_2 \\ A_3x_0 + A_2x_1 + A_1x_2 - A_0x_3 &= \lambda x_3. \end{aligned}$$

Using these equations, we can write

$$\begin{pmatrix} A_0 & A_1 & A_2 & -A_3 \\ A_1 & -A_0 & A_3 & A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & -A_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus we have $\widetilde{\sigma}_l (A) \cap \mathbb{C} = \sigma (\phi (A))$.

□

5 The Split Quaternion Matrix Equation $A\widetilde{X} - XB = C$

In [3]-[4], Complex matrix equation $A\overline{X} - XB = C$ and in [14] real quaternion matrix equation $A\widetilde{X} - XB = C$ was studied by means of the theory of consimilarity. In this section, we take into consideration the split quaternion matrix equation

$$A\widetilde{X} - XB = C \tag{11}$$

by means of the real representation, where $A \in \mathbb{H}_S^{m\times m},\ B \in \mathbb{H}_S^{n\times n}$ and $C \in \mathbb{H}_S^{m\times n}$.
We define the real representation matrix equation of the split quaternion matrix equation (11) by

$$\phi_A Y - Y\phi_B = \phi_C \tag{12}$$

By (iii.) in Theorem 13, the equation (11) is equivalent to the equation

$$\phi_A \phi_X P_n - \phi_X P_n \phi_B = \phi_C. \tag{13}$$

Theorem 14. *The split quaternion matrix equation $A\widetilde{X} - XB = C$ has a solution X if and only if real matrix equation $\phi_A Y - Y\phi_B = \phi_C$ has a solution $Y = \phi_X P_n$.*

Theorem 15. *Let $A \in \mathbb{H}_S^{m\times m},\ B \in \mathbb{H}_S^{n\times n}$ and $C \in \mathbb{H}_S^{m\times n}$. The split quaternion matrix equation $A\widetilde{X} - XB = C$ has a solution $X \in \mathbb{H}_S^{m\times n}$ if and only if the matrix equation $\phi_A Y - Y\phi_B = \phi_C$ has a solution $Y \in \mathbb{R}^{4m\times 4n}$. In this case, if Y is a solution to the matrix equation $\phi_A Y - Y\phi_B = \phi_C$, we have*

$$X = \frac{1}{16}\begin{pmatrix} I_m & iI_m & jI_m & kI_m \end{pmatrix}\left(Y - Q_m^{-1}YQ_n + R_m^{-1}YR_n - S_m^{-1}YS_n\right)\begin{pmatrix} I_m \\ -iI_m \\ jI_m \\ kI_m \end{pmatrix} \tag{14}$$

is a solution to $A\widetilde{X} - XB = C$.

Proof. We show that if the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix},\ Y_{uv} \in \mathbb{R}^{m\times n},\ u, v = 1, 2, 3, 4$$

is a solution to (12), the matrix represented in (14) is a solution to (11). Since $Q_m^{-1}\phi_X Q_n = -\phi_X$, $R_m^{-1}\phi_X R_n = \phi_X$, $S_m^{-1}\phi_X S_n = -\phi_X$, and $Y = \phi_X P_n$, we have

$$\begin{aligned} &\phi_A \left(-Q_m^{-1} Y P_n Q_n\right) P_n - \left(-Q_m^{-1} Y P_n Q_n\right) P_n \phi_B = \phi_C \\ &\phi_A \left(R_m^{-1} Y P_n R_n\right) P_n - \left(R_m^{-1} Y P_n R_n\right) P_n \phi_B = \phi_C \\ &\phi_A \left(-S_m^{-1} Y P_n S_n\right) P_n - \left(-S_m^{-1} Y P_n S_n\right) P_n \phi_B = \phi_C. \end{aligned} \tag{15}$$

Last equations show that if Y is a solution to (12), then $\left(-Q_m^{-1} Y P_n Q_n\right) P_n$, $\left(R_m^{-1} Y P_n R_n\right) P_n$ and $\left(S_m^{-1} Y P_n S_n\right) P_n$ are also solutions to (12). Then the undermentioned real matrix:

$$Y' = \frac{1}{4}\left(Y - \left(Q_m^{-1} Y P_n Q_n - R_m^{-1} Y P_n R_n + S_m^{-1} Y P_n S_n\right) P_n\right) \tag{16}$$

is a solution to (12). Since $Y' = \phi_X P_n$, we easily obtain

$$Y' P_n = \phi_X = \begin{pmatrix} {Y'}_0 & {Y'}_1 & {Y'}_2 & -{Y'}_3 \\ {Y'}_1 & -{Y'}_0 & {Y'}_3 & {Y'}_2 \\ {Y'}_2 & -{Y'}_3 & {Y'}_0 & {Y'}_1 \\ {Y'}_3 & {Y'}_2 & {Y'}_1 & -{Y'}_0 \end{pmatrix},$$

where

$$\begin{aligned} &Y_0' = \tfrac{1}{4}\left(Y_{11} + Y_{22} + Y_{33} + Y_{44}\right), \quad Y_1' = \tfrac{1}{4}\left(-Y_{12} + Y_{21} - Y_{34} + Y_{43}\right), \\ &Y_2' = \tfrac{1}{4}\left(Y_{13} - Y_{24} + Y_{31} - Y_{42}\right), \quad Y_3' = \tfrac{1}{4}\left(Y_{14} + Y_{23} + Y_{32} + Y_{41}\right). \end{aligned} \tag{17}$$

Now, we construct a split quaternion matrix by using Theorem 13-vi:

$$X = Y_0' + Y_1' i + Y_2' j + Y_3' k = \frac{1}{4}\left(I_m \ \ iI_m \ \ jI_m \ \ kI_m\right) Y' \begin{pmatrix} I_n \\ -iI_n \\ jI_n \\ kI_n \end{pmatrix}.$$

Thus X is a solution to equation given by (11). □

As a special case of Theorem 15 for $C = 0$, we have the following result for consimilarity of split quaternion matrices.

Theorem 16. *Let $A \in \mathbb{H}_S^{n\times n}$, $B \in \mathbb{H}_S^{n\times n}$. If the matrix $C = 0$ for equation $A\widetilde{X} - XB = C$ and X is nonsingular, then split quaternion matrix A is consimilar to B and real matrix ϕ_A is similar to ϕ_B.*

Proof. If the matrix $C = 0$ and X is nonsingular for equation $A\widetilde{X} - XB = C$, then we get

$$A\widetilde{X} = XB \ \Rightarrow X^{-1}A\widetilde{X} = B$$

and

$$\begin{aligned} X^{-1}A\widetilde{X} = B \ &\Rightarrow \ \phi_{X^{-1}A}\phi_X P_n = \phi_B \Rightarrow \ \phi_{X^{-1}} P_n \phi_A \phi_X P_n = \phi_B \\ &\Rightarrow P_n(\phi_X)^{-1}\phi_A\phi_X P_n = \phi_B \Rightarrow (\phi_X P_n)^{-1}\phi_A\phi_X P_n = \phi_B. \end{aligned}$$

Therefore split quaternion matrix A is consimilar to B and real matrix ϕ_A is similar to ϕ_B.

□

Example: Solve the matrix equation

$$\begin{pmatrix} 1 & i \\ i & j \end{pmatrix}\widetilde{X} - X\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2i+j & 2+j-k \\ 1+i+k & 1+i-2k \end{pmatrix}$$

by using its real representation.
Real representation of given equation is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} Y - Y \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & -2 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & -2 \\ -2 & 0 & 0 & -2 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & -2 & 0 & 0 & -2 \\ -1 & 2 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

If we solve this equation, we have

$$Y = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\phi_X = Y'P_2 = \tfrac{1}{4}\left(YP_2 - \left(Q_2^{-1}YP_2Q_2 - R_2^{-1}YP_2R_2 + S_2^{-1}YP_2S_2\right)\right)$$

$$= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

From Theorem 13, we obtain

$$X = \frac{1}{16}\left(I_2 \;\; iI_2 \;\; jI_2 \;\; kI_2\right)(\phi_X)\begin{pmatrix} I_2 \\ iI_2 \\ jI_2 \\ -kI_2 \end{pmatrix} = \begin{pmatrix} i & 1+j \\ k & i+j \end{pmatrix}.$$

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