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On sortable intervals of monomials

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Abstract

In 1996, in his study of Gröbner bases of toric ideals, Sturmfels introduced a sorting operator on pairs of monomials of degree d in n variables. This gave rise to the notion of *sortable sets*, namely sets B of monomials of degree d such that $B \times B$ is preserved by that operator. In this paper, we determine all lex-intervals or revlex-intervals of monomials which are sortable. The solution involves the notion of *greatest common prefix*.

1 Introduction

Throughout this paper, we shall let $S = K[x_1, \dots, x_n]$ denote the polynomial ring in the variables $x_1, \dots, x_n$ over a field K. We view S as a graded algebra endowed with the standard grading given by $\deg(x_i) = 1$ for all i. We denote by $S_{n,d}$ the set of monomials $x_1^{a_1} \cdots x_n^{a_n}$ in S of degree $a_1 + \cdots + a_n = d$.

In his study of Gröbner bases of toric ideals [5], Sturmfels introduced the following sorting operator on pairs of monomials in $S_{n,d}$.

Definition 1.1. *The operator*

$$\begin{array}{rccc} \mathrm{sort}\colon & S_{n,d} \times S_{n,d} & \longrightarrow & S_{n,d} \times S_{n,d} \\ & (u,\, v) & \longmapsto & (u',\, v') \end{array}$$

is defined as follows. For any $u, v \in S_{n,d}$, write $uv = x_{k_1} x_{k_2} \dots x_{k_{2d}}$ with non-decreasing indices $1 \leq k_1 \leq k_2 \leq \cdots \leq k_{2d} \leq n$. We then set

$$\begin{array}{rcl} u' & = & x_{k_1} x_{k_3} \dots x_{k_{2d-1}}, \\ v' & = & x_{k_2} x_{k_4} \dots x_{k_{2d}}. \end{array}$$

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Observe that u', v' still belong to $S_{n,d}$ and that $\mathrm{sort}(u, v) = \mathrm{sort}(v, u)$.

Subsets $B \subset S_{n,d}$ which behave well with respect to this operator, i.e. such that

$$\mathrm{sort}(B \times B) \subset B \times B,$$

are said to be *sortable*. These sets are of special interest, in particular because they give rise to toric ideals which have quadratic Gröbner bases and to K-algebras which are Koszul. See [1, 2, 3, 4, 5] for related information. It is difficult in general to describe families of sortable sets. Here, we shall focus on those subsets of $S_{n,d}$ which constitute an *interval* under the lexicographic or the reverse lexicographic order on $S_{n,d}$. Among them, we shall determine precisely those which are sortable.

Recall the definition of the *lexicographic order* on $S_{n,d}$. Given $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ such that $\sum_i a_i = \sum_i b_i = d$, we write

$$x_1^{a_1} \cdots x_n^{a_n} >_{lex} x_1^{b_1} \cdots x_n^{b_n}$$

if and only if the leftmost nonzero coordinate of $\mathbf{a} - \mathbf{b}$ is positive. Equivalently, let

$$u = x_{i_1} \cdots x_{i_d},\ v = x_{j_1} \cdots x_{j_d} \in S_{n,d}$$

with $i_1 \leq \cdots \leq i_d$, $j_1 \leq \cdots \leq j_d$. Then $u >_{lex} v$ if and only if the leftmost nonzero coordinate of $(i_1 - j_1, \ldots, i_d - j_d)$ is negative. Actually, we shall mostly omit the subscript and simply write $\geq$ instead of $\geq_{lex}$.

We now recall the notions of lex-segments and lex-intervals.

Definition 1.2. *For $v \in S_{n,d}$, the* lex-segment $L(v)$ *is the set of all monomials $u \in S_{n,d}$ satisfying $u \geq v$. More generally, for $v_1 \geq v_2 \in S_{n,d}$, the* lex-interval $L(v_1, v_2)$ *is the set of all monomials $u \in S_{n,d}$ satisfying $v_1 \geq u \geq v_2$.*

Here is a short description of the content of this paper. In Section 2, we establish properties of the sort operator which will help us compare the monomial pair (u, v) with $\mathrm{sort}(u, v)$. In Section 3, we define sets $B_{\mathrm{iw}}(v)$, consisting of all monomials $u \in S_{n,d}$ which are *index-wise* smaller than or equal to a given $v \in S_{n,d}$, and we show that these sets are always sortable. In Section 4, we show that *for $v \in S_{n,d}$, the lex-segment $L(v)$ is sortable if and only if $L(v)$ coincides with $B_{\mathrm{iw}}(v)$.* This necessary and sufficient condition is then turned into a simple criterion in Section 5, thereby answering the problem of characterizing sortable lex-segments originally formulated in [2]. In order to go further and study the sortability of lex-intervals, we shall need the notion of *greatest common prefix* discussed in Section 6. This allows us to determine all sortable lex-intervals in Section 7. Finally, in Section 8, we obtain analogous sortability criteria for revlex-intervals.

2 Some properties of the sort operator

We first introduce some terminology. Given a monomial $u \in S_{n,d}$, written as $u = x_{i_1} \cdots x_{i_d}$ with $1 \leq i_1 \leq \cdots \leq i_d \leq n$, we shall usually denote its *index multiset* by the corresponding capital letter, that is

$$U = [i_1, \ldots, i_d].$$

Actually, we shall rather write

$$U = [i_1 \leq \cdots \leq i_d],$$

since knowing the ordering of the elements of U is useful in the present context. Note that for $u, v \in S_{n,d}$, with index multisets U, V, the index multiset of their product $uv \in S_{n,2d}$ is given by the *multiset sum**

$$U \uplus V.$$

We now establish a few properties of the sort operator. For any $u, v \in S_{n,d}$, we shall keep the notation

$$(u', v') = \mathrm{sort}(u, v)$$

throughout the rest of this section.

Lemma 2.1. *For all $u, v \in S_{n,d}$, we have $uv = u'v'$.*

Proof. This is immediate, since the index multiset of $u'v'$ is, by construction, the same as that of uv. □

Lemma 2.2. *For all $u, v \in S_{n,d}$ such that $u \geq v$, there are only two possibilities:*

$$\begin{array}{rccccccc} \text{either} & u & \geq & u' & \geq & v' & \geq & v, \\ \text{or} & u' & \geq & u & \geq & v & \geq & v'. \end{array}$$

Proof. Assume for a contradiction that $u \geq u'$ and $v > v'$. Then we get $uv > u'v'$, in contradiction with Lemma 2.1. The proof for the other case is similar. □

In order to determine which of these alternatives the pair (u, v) satisfies, we introduce a key index $r = \delta(u, v)$ and three types of monomial pairs.

*For finite multisets $U = [i_1, \ldots, i_p], V = [j_1, \ldots, j_q]$, their multiset sum is defined as $U \uplus V = [i_1, \ldots, i_p, j_1, \ldots, j_q]$.

Definition 2.3. *Let $u, v \in S_{n,d}$ such that $u \geq v$. Denote $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \cdots x_{j_d}$ with $i_1 \leq \cdots \leq i_d$ and $j_1 \leq \cdots \leq j_d$, and set $i_0 = j_0 = 0$. We shall denote by $\delta(u, v)$ the* maximal integer r *such that $0 \leq r \leq d$ and*

$$i_0 \leq j_0 \leq i_1 \leq j_1 \leq \ldots \leq i_r \leq j_r.$$

Still with $r = \delta(u, v)$, we say that the pair (u, v) is of

- ***type 0*** *if $r = d$,*
- ***type 1*** *if $r < d$ and $j_r \leq i_{r+1} \leq i_{r+2} < j_{r+1}$,*
- ***type 2*** *if $r < d$ and $j_r \leq j_{r+1} < i_{r+1}$.*

Observe that, by construction, *every pair of monomials (u, v) with $u \geq v$ is of one of the above three types, and uniquely so.*

Proposition 2.4. *Let $u, v \in S_{n,d}$ with $u \geq v$. Then the pair (u, v) is of*

- *type 0 if and only if $(u', v') = (u, v)$;*
- *type 1 if and only if $u > u' > v' > v$;*
- *type 2 if and only if $u' > u > v > v'$.*

Proof. Denote $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \cdots x_{j_d}$ with $i_1 \leq i_2 \leq \cdots \leq j_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. If (u, v) is of type 0, i.e. if $r = d$, we have

$$i_1 \leq j_1 \leq \ldots \leq i_d \leq j_d,$$

whence $u' = u$ and $v' = v$ by definition of the sorting operator. Assume now $r < d$. We examine in turn the cases where (u, v) is of type 1 and of type 2. We shall denote by U', V' the index multiset of u', v', respectively.

Type 1. We have $j_r \leq i_{r+1} \leq i_{r+2} < j_{r+1}$. Thus, in the process of constructing (U', V'), we get

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} i_1 & \ldots & i_r & i_{r+1} & \cdots \\ j_1 & \ldots & j_r & i_{r+2} & \end{pmatrix}.$$

Since $i_{r+2} < j_{r+1}$, we have $v' > v$, and hence $u > u'$ by Lemma 2.2, as stated. Incidentally, we must have $r \leq d - 2$ in this case.

Type 2. We have $j_r \leq j_{r+1} < i_{r+1}$. Then here, in the process of constructing (U', V'), we get

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} i_1 & \ldots & i_r & j_{r+1} & \cdots \\ j_1 & \ldots & j_r & & \end{pmatrix}.$$

Since $j_{r+1} < i_{r+1}$, we have $u' > u$, whence $v > v'$ by Lemma 2.2. Incidentally, we cannot have $r = 0$ in this case, since $i_1 \leq j_1$ by the assumption $u \geq v$.

Finally, since types 0, 1 and 2 cover all possibilities, the implications proven so far are in fact equivalences. □

3 Sortable sets

Definition 3.1. *Let $B \subset S_{n,d}$ be a set of monomials of degree d in $x_1, \ldots, x_n$. We say that B is* sortable *if $B \times B$ is stable under the sort operator, i.e. if*

$$\mathrm{sort}(B \times B) \subset B \times B.$$

Here is a known class of sortable sets. Given $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let

$$S^{\mathbf{a}}_{n,d} = \{x_1^{c_1} \cdots x_n^{c_n} \in S_{n,d} \mid c_i \leq a_i \text{ for all } i\}.$$

Then $S^{\mathbf{a}}_{n,d}$ is sortable for all $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. (See Proposition 6.11 in [2, p. 105].)

We now construct a new class of sortable sets. To this end, we introduce a partial order between monomials in $S_{n,d}$, denoted $\leq_{\mathrm{iw}}$ and referred to as the *index-wise partial order.*

Definition 3.2. *Let $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \cdots x_{j_d}$ be monomials of degree d with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $j_1 \leq j_2 \leq \cdots \leq j_d$. We shall write*

$$u \leq_{\mathrm{iw}} v$$

if and only if $i_\alpha \leq j_\alpha$ for all $\alpha = 1, \ldots, d$.

Definition 3.3. *Let $v \in S_{n,d}$ be a monomial in $x_1, \ldots, x_n$ of degree d. Define*

$$B_{\mathrm{iw}}(v) = \{u \in S_{n,d} \mid u \leq_{\mathrm{iw}} v\}.$$

For instance, for any $v \in S_{n,d}$, it is plain that $B_{\mathrm{iw}}(v)$ contains x_1^d. Note that if $u, v \in S_{n,d}$ and $u \leq_{\mathrm{iw}} v$, then $u \geq v$ under the lexicographical order. This amounts to the set inclusion

$$B_{\mathrm{iw}}(v) \subset L(v).$$

(See Lemma 4.1 below).

Observe that $B_{\mathrm{iw}}(v)$ is nothing else than the set of minimal generators of the *principal Borel ideal $\langle v \rangle$ generated by* v, see e.g. [6].

We now show that the sets $B_{\mathrm{iw}}(v)$ constitute yet another class of sortable sets in $S_{n,d}$.

Proposition 3.4. *Let $v \in S_{n,d}$. Then $B_{\mathrm{iw}}(v)$ is sortable.*

Proof. Let the index multiset of v be $V = [j_1 \leq \cdots \leq j_d]$. Let $u_1, u_2 \in B_{\mathrm{iw}}(v)$, with index multisets

$$U_1 = [h_1 \leq \cdots \leq h_d], \quad U_2 = [i_1 \leq \cdots \leq i_d],$$

respectively. By hypothesis, we have

$$h_\alpha, i_\alpha \;\leq\; j_\alpha \tag{1}$$

for all $\alpha = 1, \ldots, d$. Let the index multiset of $u_1 u_2 = u_1' u_2'$ be

$$W = [k_1 \leq \cdots \leq k_{2d}],$$

where $W = U_1 \uplus U_2$ is the multiset sum of U_1, U_2. By definition of the sorting operator, the index multisets of u_1', u_2' are

$$[k_1 \leq k_3 \leq \cdots \leq k_{2d-1}], \quad [k_2 \leq k_4 \leq \cdots \leq k_{2d}],$$

respectively. We must show that $u_1', u_2' \leq_{iw} v$. Since $k_{2\beta-1} \leq k_{2\beta}$ for all β, this will follow at once from the following claim.

Claim. For all $\beta = 1, \ldots, d$, we have

$$k_{2\beta} \;\leq\; j_\beta.$$

Indeed, it follows from (1) that

$$h_\alpha, i_\alpha \;\leq\; j_\alpha \;\leq\; j_\beta$$

for all $1 \leq \alpha \leq \beta$. Thus, the multiset $W = U_1 \uplus U_2$ contains at least 2β elements which are bounded above by j_β, namely $[h_1, i_1, \ldots, h_\beta, i_\beta]$. But since $k_1, k_2, \ldots, k_{2\beta}$ are the 2β *smallest elements* in $W = U_1 \uplus U_2$, it follows that

$$k_1, \ldots, k_{2\beta} \;\leq\; j_\beta.$$

This proves the claim which, in turn, implies $u_1', u_2' \in B_{\mathrm{iw}}(v)$. $\square$

Note that the sets $B_{\mathrm{iw}}(v)$ need not coincide with the sets $S^{\mathbf{a}}_{n,d}$. Indeed, let $v = x_1 x_3 \in S_{3,2}$. Then $B_{\mathrm{iw}}(v) = \{x_1^2, x_1 x_2, x_1 x_3\}$. Thus, if we had $B_{\mathrm{iw}}(v) = S^{\mathbf{a}}_{3,2}$ for some $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{N}^3$, we would then necessarily have $\mathbf{a} = (2, 1, 1)$. But now, observe that

$$x_2 x_3 \;\in\; S^{(2,1,1)}_{3,2} \setminus B_{\mathrm{iw}}(x_1 x_3).$$

Hence $B_{\mathrm{iw}}(x_1 x_3) \neq S^{\mathbf{a}}_{3,2}$ for all $\mathbf{a} \in \mathbb{N}^3$, as claimed.

4 When is a lex-segment sortable?

In this section, we shall characterize all those lex-segments $L(v)$ which are sortable. We start by formalizing as a lemma an earlier observation comparing $B_{\mathrm{iw}}(v)$ and $L(v)$.

Lemma 4.1. *Let $v \in S_{n,d}$ be a monomial in $x_1, \dots, x_n$ of degree d. Then $B_{\mathrm{iw}}(v) \subset L(v)$.*

Proof. Let $V = [j_1 \leq \cdots \leq j_d]$ be the index multiset of v. Let $u \in S_{n,d}$ such that $u \leq_{\mathrm{iw}} v$, with index multiset $U = [i_1 \leq \cdots \leq i_d]$. Then we have

$$i_\alpha \leq j_\alpha$$

for all $\alpha = 1, \dots, d$. It follows that $u \geq v$, as desired. $\square$

Example 4.2. *Let $v = x_2^3 \in S_{3,3}$. Then we have*

$$\begin{array}{rcl} B_{\mathrm{iw}}(v) & = & \{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}, \\ L(v) & = & \{x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_1x_3^2, x_2^3\}. \end{array}$$

We are now ready to state and prove our promised characterization.

Theorem 4.3. *Let $v \in S_{n,d}$. Then $L(v)$ is sortable if and only if $L(v) = B_{\mathrm{iw}}(v)$.*

Proof.
- If $L(v) = B_{\mathrm{iw}}(v)$, then $L(v)$ is sortable by Proposition 3.4.
- Conversely, assume that $L(v) \neq B_{\mathrm{iw}}(v)$. Since $B_{\mathrm{iw}}(v) \subset L(v)$ by Lemma 4.1, there exists some $u \in L(v) \setminus B_{\mathrm{iw}}(v)$, i.e. such that $u > v$ and $u \not\leq_{\mathrm{iw}} v$. Denote $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$, and $v = x_{j_1} \cdots x_{j_d}$ with $j_1 \leq j_2 \leq \cdots \leq j_d$. Since $u > v$, there is an index $s \geq 1$ such that

$$\begin{array}{rcll} i_\alpha & = & j_\alpha & \text{for all } 1 \leq \alpha \leq s-1, \\ i_s & < & j_s. & \end{array} \tag{2}$$

Moreover, since $u \not\leq_{\mathrm{iw}} v$, there is an index $t \geq 1$ such that

$$\begin{array}{rcll} i_\beta & \leq & j_\beta & \text{for all } 1 \leq \beta \leq t-1, \\ i_t & > & j_t. & \end{array} \tag{3}$$

It follows from the definition of s, t that $s < t$. Since we do not know how j_β compares with $i_{\beta+1}$ for $s+1 \leq \beta \leq t-1$, we have no precise control on $\delta(u, v)$ and we cannot determine the type of (u, v).

To get around this problem, we shall construct a new monomial $\overline{u} \in L(v) \setminus B_{\mathrm{iw}}(v)$, by suitably increasing all indices i_β with $s+1 \leq \beta \leq t-1$, and for which the type of $(\overline{u}, v)$ will be easier to determine[†]. So, let us define

$$\bar{i}_\beta = \begin{cases} j_\beta & \text{if } s+1 \leq \beta \leq t-1, \\ i_\beta & \text{otherwise,} \end{cases}$$

and let $\overline{u} = x_{\bar{i}_1} \cdots x_{\bar{i}_d}$. We first claim that

$$\bar{i}_1 \leq \bar{i}_2 \leq \cdots \leq \bar{i}_d. \tag{4}$$

Indeed, the inequality $\bar{i}_\alpha \leq \bar{i}_{\alpha+1}$ clearly holds for all $\alpha \in [1, d-1] \setminus \{s, t-1\}$, since in those cases it reduces to either $i_\alpha \leq i_{\alpha+1}$ or $j_\alpha \leq j_{\alpha+1}$. Now, for $\alpha = s$, we have

$$\bar{i}_s = i_s < j_s \leq j_{s+1} = \bar{i}_{s+1},$$

and for $\alpha = t-1$, we have

$$\bar{i}_{t-1} = j_{t-1} \leq j_t < i_t = \bar{i}_t.$$

This proves (4). Next, we claim that $\delta(\overline{u}, v) = t-1$. Indeed, relations (2) and (3) become

$$\begin{array}{rcll} \bar{i}_\alpha & = & j_\alpha & \text{for all } 1 \leq \alpha \leq s-1, \\ \bar{i}_s & < & j_s, & \end{array} \tag{5}$$

and

$$\begin{array}{rcll} \bar{i}_\beta & = & j_\beta & \text{for all } s+1 \leq \beta \leq t-1, \\ \bar{i}_t & > & j_t, & \end{array} \tag{6}$$

respectively. As a first consequence, observe that $u \geq \overline{u} > v$. Moreover, by (5), (6) and the fact that the j_α are nondecreasing, we have

$$\bar{i}_1 = j_1 \leq \bar{i}_2 = j_2 \leq \cdots \leq \bar{i}_{t-1} = j_{t-1} \leq j_t < \bar{i}_t.$$

Thus, we have $\delta(\overline{u}, v) = t-1$ by definition of this index, and the pair $(\overline{u}, v)$ is of type 2 since $\bar{i}_{t-1} \leq j_{t-1} \leq j_t < \bar{i}_t$. It follows from Proposition 2.4 that $\overline{u}' > \overline{u} > v > v'$. Hence $L(v)$ is not sortable, as claimed. $\square$

[†]If $s+1 > t-1$, i.e. if $s = t-1$ since $s < t$, we simply end up with $\overline{u} = u$.

5 A simple criterion

We shall now derive from Theorem 4.3 a simple criterion for recognizing when a given monomial $v \in S_{n,d}$ has the property that $L(v)$ is sortable or not.

Definition 5.1. *Let w be a monomial in the variables $x_1, \dots, x_n$. If $w \neq 1$, we denote by* $\min(w)$, $\max(w)$ *the smallest, respectively the largest, index of the variables dividing w. For $w = 1$, we set* $\min(1) = \infty$ *and* $\max(1) = 0$.

For instance, if $w = x_2^5 x_3^2 x_4$, then $\min(w) = 2$ and $\max(w) = 4$.

Theorem 5.2. *Let v be a monomial in the variables $x_1, \dots, x_n$. Then $L(v)$ is not sortable if and only if v has a factor w of degree 2 in $x_2, \dots, x_{n-1}$.*

Proof. Denote $v = x_{j_1} \dots x_{j_d}$ with $j_1 \leq \dots \leq j_d$, and assume first that $L(v)$ is not sortable. By Theorem 4.3, there exists a monomial $u \in L(v) \setminus B_{iw}(v)$, i.e. satisfying $u > v$ and $u \not\leq_{iw} v$. Thus, denoting $u = x_{i_1} \dots x_{i_d}$ with $i_1 \leq \dots \leq i_d$, there exist indices $1 \leq s < t \leq d$ such that

$$\begin{array}{lcll} i_\alpha & = & j_\alpha & \text{for all } 1 \leq \alpha \leq s-1, \\ i_s & < & j_s, & \\ i_\beta & \leq & j_\beta & \text{for all } s \leq \beta \leq t-1, \\ i_t & > & j_t, & \end{array} \tag{7}$$

as in (2), (3) in the proof of Theorem 4.3. This implies

$$2 \leq j_s \leq j_t \leq n-1,$$

since $1 \leq i_s < j_s \leq j_t < i_t \leq n$ by (7). Hence, the monomial $w = x_{j_s} x_{j_t}$ is a factor of v of degree 2 in the variables $x_2, \dots, x_{n-1}$.

Conversely, assume that v has a factor w of degree 2 in $x_2, \dots, x_{n-1}$. Then there is a decomposition

$$v = v_1 x_{h_1} x_{h_2} v_2$$

with $2 \leq h_1 \leq h_2 \leq n-1$ and with v_1, v_2 possibly trivial monomials‡ satisfying $\max(v_1) \leq h_1 \leq h_2 \leq \min(v_2)$. We now set

$$u = v_1 x_{h_1-1} x_n^b,$$

where $b = d - \deg(v_1) - 1$, so that $\deg(u) = d$. Note that $u > v$. Let us now start the computation of $(u', v') = \mathrm{sort}(u, v)$. On the level of index multisets, we have

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} V_1 & h_1 - 1 & h_2 & \dots \\ V_1 & h_1 & \dots & \dots \end{pmatrix}.$$

‡Recall our conventions $\max(1) = 0$ and $\min(1) = \infty$.

Since $h_2 < n$, it follows that $u' > u$, whence also $v > v'$ by Lemma 2.2. Summarizing, we found that $L(v)$ contains u, v but not v'. Therefore $L(v)$ is not sortable, as stated. □

Corollary 5.3. *Let $v \in S_{n,d}$. Then $L(v)$ is sortable if and only if $v = x_1^a x_j x_n^b$ for some $a, b \in \mathbb{N}$ and some index j such that $1 \leq j \leq n$.* ■

Compare with Proposition 3.2 (i) of [6], a closely related statement.

6 The greatest common prefix

In order to extend our sortability criterion to arbitrary lex-intervals of monomials, we shall need the notion of *greatest common prefix* of two or more monomials. Analogous notions appear in various contexts such as computer science, combinatorics on words, computational molecular biology and braid theory.

Definition 6.1. *Let $u \in S$ be a monomial in $x_1, \dots, x_n$. A* prefix *of u is any factor w of u satisfying*

$$\max(w) \leq \min(u/w).$$

Equivalently, if $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$, a prefix *of u is a factor w of the form*

$$w = x_{i_1} \cdots x_{i_k}$$

for some $0 \leq k \leq d$.

Note that for each $0 \leq k \leq \deg(u)$, there is a *unique* prefix w of u of degree k. We now consider the case of two monomials; the extension to more monomials is straightforward.

Definition 6.2. *Let $u, v \in S$ be monomials in $x_1, \dots, x_n$. The* greatest common prefix *of u, v, denoted*

$$\operatorname{gcp}(u, v),$$

is the common prefix of u, v of highest degree.

Note that $\operatorname{gcp}(u, v)$ divides $\gcd(u, v)$, the usual greatest common divisor of u, v. For example, if $u = x_1 x_2^2 x_3$ and $v = x_1 x_2 x_3^2$, then

$$\operatorname{gcp}(u, v) = x_1 x_2, \quad \gcd(u, v) = x_1 x_2 x_3.$$

Here is an equivalent characterization of $\operatorname{gcp}(u, v)$.

Remark 6.3. *Let u, v be monomials in $x_1, \dots, x_n$. Then* $\mathrm{gcp}(u, v)$ *is the common factor w of u, v of highest degree satisfying*

$$\max(w) \leq \min(u/w), \quad \max(w) \leq \min(v/w).$$

Finally, let us observe that the notion of gcp allows a useful equivalent formulation of the lexicographical order.

Proposition 6.4. *Let $v_1 \neq v_2 \in S_{n,d}$. Let $v_0 = \mathrm{gcp}(v_1, v_2)$. Then $v_1 >_{lex} v_2$ if and only* $\min(v_1/v_0) < \min(v_2/v_0)$. ∎

7 The case of lex-intervals

Given $v_1 \geq v_2$ in $S_{n,d}$, we denote by $L(v_1, v_2)$ the *lex-interval* determined by v_1, v_2 with respect to the lexicographical order, namely

$$L(v_1, v_2) = \{u \in S_{n,d} \mid v_1 \geq u \geq v_2\}.$$

Of course, lex-segments are lex-intervals: if $v \in S_{n,d}$, then

$$L(v) = L(x_1^d, v).$$

In this section, we generalize Theorem 5.2 and determine which lex-intervals in $S_{n,d}$ are not sortable. Even though Theorem 5.2 will follow as an immediate corollary, we have treated it separately with an independent and simpler proof.

The case of arbitrary lex-intervals requires the notion of greatest common prefix introduced in the preceding section. Note $L(v, v) = \{v\}$ is sortable, since $\mathrm{sort}(v, v) = (v, v)$. Thus, we only need to examine the case where $v_1 > v_2$.

Theorem 7.1. *Let $v_1 > v_2$ be monomials in $x_1, \dots, x_n$ of degree d. Let $v_0 = \mathrm{gcp}(v_1, v_2)$. Then $L(v_1, v_2)$ is not sortable if and only if the monomial v_2/v_0 has a factor w of degree 2 such that* $\max(w) < n$.

Proof. • Assume first that v_2/v_0 has a factor $w = x_{h_1} x_{h_2}$ with $h_1 \leq h_2 < n$. Since $v_1 > v_2$, there are decompositions

$$v_1 = v_0 w_1, \quad v_2 = v_0 w_2$$

with $\max(v_0) \leq \min(w_1) < \min(w_2)$. Without loss of generality, we may assume that w is the prefix of degree 2 in w_2. Thus, we may further decompose

$$w_2 = x_{h_1} x_{h_2} \overline{w_2}$$

with $h_1 = \min(w_2)$ and $\min(\overline{w_2}) \geq h_2$. We shall now find a special monomial u in $L(v_1, v_2)$ such that $\mathrm{sort}(u, v_2)$ falls *outside* $L(v_1, v_2) \times L(v_1, v_2)$. Set

$$u = v_0 x_{h_1-1} x_n^b$$

with $b = d - \deg(v_0) - 1$, so that $\deg(u) = d$.

We first show, using Proposition 6.4, that u belongs to $L(v_1, v_2)$. Indeed, on the one hand we have

$$u > v_2,$$

since v_0 is a common prefix of u, v_2 and since $h_1 - 1 = \min(w_2) - 1$. On the other hand, in order to show

$$v_1 \geq u,$$

we need to determine the greatest common prefix of v_1, u. First note that $\min(w_1) \leq h_1 - 1$, since $\min(w_1) < \min(w_2) = h_1$. We get:

$$\mathrm{gcp}(v_1, u) = \begin{cases} v_0 & \text{if } \min(w_1) < h_1 - 1, \\ v_0 x_{h_1-1} & \text{if } \min(w_1) = h_1 - 1 \text{ and } w_1 > x_{h_1-1} x_n^b, \\ v_0 x_{h_1-1} x_n^b & \text{if } \min(w_1) = h_1 - 1 \text{ and } w_1 = x_{h_1-1} x_n^b. \end{cases}$$

In either case, we easily conclude with Proposition 6.4 that $v_1 \geq u$. Therefore $u \in L(v_1, v_2)$, as stated.

Let us now apply the sort operator to the pair (u, v_2). On the level of index multisets, we have

$$\begin{pmatrix} U' \\ V_2' \end{pmatrix} = \begin{pmatrix} V_0 & h_1 - 1 & h_2 & \dots \\ V_0 & h_1 & \dots & \dots \end{pmatrix}.$$

Since $h_2 < n$, it follows that $u' > u$, whence also $v_2 > v_2'$ by Lemma 2.2. Thus $L(v_1, v_2)$ contains u, v_2 but not v_2'. It follows that $L(v_1, v_2)$ is not sortable, as stated.

- Conversely, assume that v_2/v_0 has no factor w of degree 2 satisfying $\max(w) < n$. It follows that either $\deg(v_2/v_0) = 1$, or else

$$v_2/v_0 = x_j x_n^b$$

with $\max(v_0) \leq j \leq n$ and $b \geq 1$. We now show that, in each case, the interval $L(v_1, v_2)$ is sortable.

(1) Assume first $\deg(v_2/v_0) = 1$. Then, since $v_1 > v_2$, there are indices $\max(v_0) \leq h_1 < h_2 \leq n$ such that

$$v_1 = v_0 x_{h_1}, \quad v_2 = v_0 x_{h_2}.$$

It follows that $L(v_1, v_2) = v_0\{x_{h_1}, x_{h_1+1}, \dots, x_{h_2}\}$. But then, it is clear that for any indices i, j such that $h_1 \leq i \leq j \leq h_2$, we have

$$\operatorname{sort}(v_0x_i, v_0x_j) = (v_0x_i, v_0x_j).$$

Thus, $L(v_1, v_2)$ is sortable in this case.

(2) Assume now $v_2/v_0 = x_jx_n^b$ with $\max(v_0) \leq j \leq n$ and $b \geq 1$. Since $v_1 > v_2 = v_0x_jx_n^b$, there is a decomposition

$$v_1 = v_0x_h\overline{v_1}$$

with $\max(v_0) \leq h < j$ and $h \leq \min(\overline{v_1})$. Note that the equality $h = j$ is excluded by our assumption $\operatorname{gcp}(v_1, v_2) = v_0$. With these values of v_1, v_2, we now describe all monomials in the lex-interval $L(v_1, v_2)$.

Claim. Let $u \in S_{n,d}$. Then we have

$$v_0x_h\overline{v_1} \geq u \geq v_0x_jx_n^b$$

if and only if $u = v_0x_i\overline{u}$ for some monomial $\overline{u}$ and some index i such that $h \leq i \leq j$, and either

(i) $h = i$ and $\overline{v_1} \geq \overline{u}$, or else

(ii) $h < i \leq j$ and $\overline{u}$ is any monomial in $x_i, \dots, x_n$ of appropriate degree.

Checking the claim is straightforward and left to the reader.

We now prove that $L(v_1, v_2)$ is sortable. So let $u_1 \geq u_2 \in L(v_1, v_2)$. Since $v_1 \geq u_1 \geq u_2 \geq v_2$, there are decompositions

$$u_1 = v_0x_{i_1}\overline{u_1}, \quad u_2 = v_0x_{i_2}\overline{u_2}$$

such that $h \leq i_1 \leq i_2 \leq j$. Let us start the computation of $(u_1', u_2') = \operatorname{sort}(u_1, u_2)$. On the level of index multisets, we have

$$\begin{pmatrix} U_1' \\ U_2' \end{pmatrix} = \begin{pmatrix} V_0 & i_1 & \dots \\ V_0 & i_2 & \dots \end{pmatrix}.$$

It follows that u_2' admits $v_0x_{i_2}$ as a prefix. Recalling the equality $v_2 = v_0x_jx_n^b$, we now show that $u_2' \geq v_2$.

(i) If $i_2 < j$, then clearly $u_2' > v_2$.

(ii) If $i_2 = j$, then $u_2' \geq v_2$ since v_2, by its specific structure, is the *smallest* monomial of its degree having v_0x_j as a prefix.

It follows from Lemma 2.2 that $v_1 \geq u_1' \geq u_2' \geq v_2$, and hence that u_1', u_2' still belong to $L(v_1, v_2)$. Therefore $L(v_1, v_2)$ is sortable, as stated. □

Corollary 7.2. *The only sortable lex-intervals $L(v_1, v_2) \subset S_{n,d}$ with $v_1 > v_2$ are those such that $(v_1, v_2) = (v_0 w_1, v_0 x_j x_n^b)$, where $b = d - \deg(v_0) - 1$ and where v_0, w_1 are monomials satisfying*

$$\max(v_0) \leq \min(w_1) < j \leq n. \quad \blacksquare$$

8 From lex to revlex

Our aim here is to establish the analogue of Theorem 7.1 for intervals of monomials under the *reverse lexicographical order* on $S_{n,d}$. Instead of adapting our earlier proofs to this new setting, we shall develop tools allowing us to transfer knowledge between the lex and the revlex orders. The desired analogue will then directly follow from Theorem 7.1 using those tools.

Recall first the definition of the revlex order. Given $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ such that $\sum_i a_i = \sum_i b_i = d$, we write

$$x_1^{a_1} \cdots x_n^{a_n} >_{rev} x_1^{b_1} \cdots x_n^{b_n}$$

if and only if the *rightmost* nonzero coordinate of $\mathbf{a} - \mathbf{b}$ is *negative*. Equivalently, let

$$u = x_{i_1} \cdots x_{i_d},\ v = x_{j_1} \cdots x_{j_d} \in S_{n,d}$$

with $i_1 \leq \cdots \leq i_d$, $j_1 \leq \cdots \leq j_d$. Then $u >_{rev} v$ if and only if the rightmost nonzero coordinate of $(i_1 - j_1, \ldots, i_d - j_d)$ is negative.

Informally, the lex order *gives a premium* in priority to x_1, then to x_2 and so on, whereas the revlex order *puts a penalty* in priority to x_n, then to x_{n-1} and so on. In $S_{2,2}$ for instance, we have

$$x_1 x_3 >_{lex} x_2^2 \quad \text{but} \quad x_1 x_3 <_{rev} x_2^2.$$

Note also that $x_1 > \cdots > x_n$ for both orders.

8.1 The automorphism σ

A convenient way to compare the lex and revlex orders is through the K-algebra automorphism

$$\begin{aligned} \sigma : K[x_1, \ldots, x_n] &\longrightarrow K[x_1, \ldots, x_n] \\ x_i &\longmapsto x_{n+1-i} \ \forall i = 1, \ldots, n. \end{aligned}$$

Indeed, for any monomials $u, v \in S_{n,d}$, we have

$$u >_{lex} v \iff \sigma(u) <_{rev} \sigma(v). \tag{8}$$

Since $\sigma^{-1} = \sigma$, this equivalence may as well be written in the form

$$v >_{rev} u \iff \sigma(u) <_{lex} \sigma(v). \tag{9}$$

8.2 Revlex-intervals

We now define intervals in $S_{n,d}$ under the revlex order.

Definition 8.1. *Given $v_1 \geq_{rev} v_2$ in $S_{n,d}$, we denote by $R(v_1, v_2)$ the* revlex-interval *determined by v_1, v_2, namely*

$$R(v_1, v_2) \,=\, \{u \in S_{n,d} \mid v_1 \geq_{rev} u \geq_{rev} v_2\}.$$

Lex-intervals and revlex-intervals may be compared as follows.

Lemma 8.2. *For any $v_1 \geq_{rev} v_2$ in $S_{n,d}$, we have*

$$\sigma(R(v_1, v_2)) \,=\, L(\sigma(v_2), \sigma(v_1)).$$

Proof. Let $u \in S_{n,d}$. Then $u \in R(v_1, v_2)$ if and only if $v_1 \geq_{rev} u \geq_{rev} v_2$. Applying σ and using (9) this in turn is equivalent to

$$\sigma(v_1) \leq_{lex} \sigma(u) \leq_{lex} \sigma(v_2),$$

i.e. to $\sigma(u) \in L(\sigma(v_2), \sigma(v_1))$. It follows that $R(v_1, v_2) = \sigma^{-1}(L(\sigma(v_2), \sigma(v_1)))$, whence the stated formula. □

8.3 Sort and σ

Here we describe how sort and σ interact with each other.

Lemma 8.3. *For any $u, v \in S_{n,d}$, let $(u', v') = \mathrm{sort}(u, v)$. Then*

$$\mathrm{sort}(\sigma(u), \sigma(v)) \,=\, (\sigma(v'), \sigma(u')).$$

Proof. Denote $uv = x_{k_1} x_{k_2} \ldots x_{k_{2d}}$ with nondecreasing indices k_i. Then by construction, we have

$$\begin{aligned} u' &= x_{k_1} x_{k_3} \ldots x_{k_{2d-1}}, \\ v' &= x_{k_2} x_{k_4} \ldots x_{k_{2d}}. \end{aligned}$$

Now $\sigma(u)\sigma(v) = \sigma(uv) = x_{n+1-k_{2d}} x_{n+1-k_{2d-1}} \dots x_{n+1-k_1}$, here again with nondecreasing indices. Applying sort to the pair $(\sigma(u), \sigma(v))$, it follows that

$$\begin{array}{rcccl} \sigma(u)' & = & x_{n+1-k_{2d}} x_{n+1-k_{2d-2}} \dots x_{n+1-k_2} & = & \sigma(v') \\ \sigma(v)' & = & x_{n+1-k_{2d-1}} x_{n+1-k_{2d-3}} \dots x_{n+1-k_1} & = & \sigma(u'), \end{array}$$

as stated. □

A useful consequence is that σ preserves the sortability property.

Proposition 8.4. *Let B be a subset of $S_{n,d}$. Then B is sortable if and only if $\sigma(B)$ is sortable.*

Proof. Since $\sigma^{-1} = \sigma$, it suffices to prove one direction. So assume that B is sortable. Any element in $\sigma(B) \times \sigma(B)$ is of the form $(\sigma(u), \sigma(v))$ for some pair $(u, v) \in B \times B$. We claim that $\mathrm{sort}(\sigma(u), \sigma(v))$ still belongs to $\sigma(B) \times \sigma(B)$. Indeed, by Lemma 8.3, we have

$$\mathrm{sort}(\sigma(u), \sigma(v)) \,=\, (\sigma(v'), \sigma(u')).$$

But $(u', v') \in B \times B$ since B is sortable, whence $(\sigma(v'), \sigma(u')) \in \sigma(B) \times \sigma(B)$. This shows that $\mathrm{sort}(\sigma(u), \sigma(v)) \in \sigma(B) \times \sigma(B)$, as claimed. Therefore $\sigma(B)$ is sortable. □

8.4 The greatest common suffix

Definition 8.5. *Let $u = x_{i_1} \cdots x_{i_d}$ be a monomial in $x_1, \dots, x_n$ of degree d with $i_1 \leq \cdots \leq i_d$. Let $1 \leq k \leq d$. The* k-suffix of *u is the degree k monomial*

$$x_{i_{d+1-k}} \cdots x_{i_d}.$$

It may be characterized as the unique monomial u_0 of degree k such that u_0 divides u and $\min(u_0) \geq \max(u/u_0)$.

We now introduce the analogue for suffixes of the greatest common prefix.

Definition 8.6. *Let $u, v \in S$ be monomials in $x_1, \dots, x_n$. The* greatest common suffix *of u, v, denoted*

$$\mathrm{gcs}(u, v),$$

is the common suffix of u, v of highest degree.

Note that $\mathrm{gcs}(u, v)$ divides $\mathrm{gcd}(u, v)$, as was the case for $\mathrm{gcp}(u, v)$. Taking our earlier example, if $u = x_1 x_2^2 x_3$ and $v = x_1 x_2 x_3^2$, we have

$$\mathrm{gcs}(u, v) \,=\, x_3.$$

Moreover, $\mathrm{gcs}(u, v)$ may be characterized as the common factor w of u, v of highest degree satisfying

$$\min(w) \geq \max(u/w), \quad \min(w) \geq \max(v/w).$$

The following result shows that σ transforms gcs into gcp.

Lemma 8.7. *For all $v_1, v_2 \in S_{n,d}$, we have $\sigma(\mathrm{gcs}(v_1, v_2))=\mathrm{gcp}(\sigma(v_2), \sigma(v_1))$.*

Proof. This follows from the observation that, for any monomial v, a monomial v_0 is a suffix of v if and only if $\sigma(v_0)$ is a prefix of $\sigma(v)$. □

8.5 Sortable revlex-intervals

We are now ready to determine which revlex-intervals are sortable and which are not. The results will follow from Theorem 7.1, Corollary 7.2 and the above properties of the automorphism σ.

Theorem 8.8. *Let $v_1 >_{rev} v_2$ be monomials in $x_1, \ldots, x_n$ of degree d. Let $v_0 = \mathrm{gcs}(v_1, v_2)$. Then $R(v_1, v_2)$ is not sortable if and only if v_1/v_0 has a factor w of degree 2 such that $\min(w) > 1$.*

Proof. By Proposition 8.4, the non-sortability of $R(v_1, v_2)$ is equivalent to that of $\sigma(R(v_1, v_2))$. Now $\sigma(R(v_1, v_2)) = L(\sigma(v_2), \sigma(v_1))$ by Lemma 8.2, and the non-sortability of this lex-interval may be determined using Theorem 7.1. By Lemma 8.7, we have $\mathrm{gcp}(\sigma(v_2), \sigma(v_1)) = \sigma(v_0)$, where $v_0 = \mathrm{gcs}(v_1, v_2)$. By Theorem 7.1, $L(\sigma(v_2), \sigma(v_1))$ is non-sortable if and only if $\sigma(v_1)/\sigma(v_0)$ has a factor w' of degree 2 such that $\max(w') < n$. Let $w = \sigma(w')$. Now

$$\max(w') < n \iff \min(w) > 1.$$

Thus, applying σ, we have that $L(\sigma(v_2), \sigma(v_1))$ is non-sortable if and only if v_1/v_0 has a factor w of degree 2 such that $\min(w) > 1$. It follows that $R(v_1, v_2)$ is not sortable if and only if the latter condition holds, as stated. □

Corollary 8.9. *The only sortable revlex-intervals $R(v_1, v_2) \subset S_{n,d}$ with $v_1 >_{rev} v_2$ are those such that $(v_1, v_2) = (x_1^b x_j v_0, w_1 v_0)$, where $b = d - \deg(v_0) - 1$ and where v_0, w_1 are monomials satisfying*

$$1 \leq j < \max(w_1) \leq \min(v_0). \quad \blacksquare$$

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