Univalence conditions for a general integral operator

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Abstract

For analytic functions in the open unit disk U, we define a new general integral operator. The main object of the this paper is to study this new integral operator and to determine univalence conditions of it. Several corollaries of the main results are also considered.

1   Introduction

Let $\mathcal{A}$ denote the class of function of the form:

$$f(z) = z + \sum_{i=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$

Further, by S we shall denote the class of functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

In [9], Silverman define the class G_b.

For $0 < b \leq 1$, he considered the class:

$$G_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, z \in \mathcal{U} \right\} \tag{2}$$

Let $\alpha_i \in \mathbb{C}^*$, for all $i = 1, 2, ...n$, $n \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$, with $\mathrm{Re}\gamma > 0$.
We consider $I_{\gamma,\alpha_i} : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ be the integral operator defined by:

$$I_{\gamma,\alpha_i}(f_1, ...f_n, g_1, ...g_n)(z) = \tag{3}$$

$$\left\{ \gamma \int_0^z t^{\gamma-1} \left(\frac{f_1(t)}{t} \right)^{1-\frac{1}{\alpha_1}} (g_1'(t))^{\frac{1}{\alpha_1}} .... \left(\frac{f_n(t)}{t} \right)^{1-\frac{1}{\alpha_n}} (g_n'(t))^{\frac{1}{\alpha_n}} dt \right\}^{\frac{1}{\gamma}}$$

In the present paper we study the univalence conditions for the integral operator $I_{\gamma,\alpha_i}(f_1, ...f_n, g_1, ...g_n)$, when $g_i \in G_{b_i}$ and $f_i \in \mathcal{A}$ for all $i = 1, ..., n$. Also, the univalence of some integral operators has been studied by other authors like V. Pescar and D. Breaz in work [8]. In order to derive our main results, we have to recall here the following univalence criteria.

Theorem 1. [5] *Let* $f \in \mathcal{A}$ *and* $\gamma \in \mathbb{C}$. *If* $\mathrm{Re}\gamma > 0$ *and*

$$\frac{1 - |z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, *then the integral operator*

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt \right)^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Theorem 2. [6] *Let* $\delta \in \mathbb{C}$ *with* $\mathrm{Re}\delta > 0$. *If* $f \in \mathcal{A}$ *satisfies*

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, *then, for any complex* γ *with* $\mathrm{Re}\gamma \geq \mathrm{Re}\delta$, *the integral operator*

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)dt \right)^{\frac{1}{\gamma}}$$

is in the class S.

Theorem 3. [7] *Let* γ *be a complex number,* $\mathrm{Re}\gamma > 0$, *and* c *a complex number,* $|c| \leq 1$, $c \neq -1$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left| c|z|^{2\gamma} + \left(1 - |z|^{2\gamma} \right) \frac{z f''(z)}{\gamma f'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t) dt \right)^{\frac{1}{\gamma}}$$

is in the class S.

Also, we need the following general Schwarz Lemma

Lemma 1. *(General Schwarz-Lemma)* [4] *Let the function f be regular in the disk $\mathcal{U}_R = \{ z \in \mathbb{C} : |z| < R \}$, with $|f(z)| < M$, M fixed. If f has one zero with multiplicity greater then m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \qquad z \in \mathcal{U}_R. \tag{4}$$

the equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2   Univalence conditions for $I_{\gamma,\alpha_i}(f_1, ...f_n, g_1, ...g_n)(z)$

First, we prove:

Theorem 4. *Let $\gamma \in \mathbb{C}$ and $\alpha_i \in \mathbb{C}^*$ for all $i = 1, ...n$, with*

$$\mathrm{Re}\,\gamma \geq \sum_{i=1}^n \frac{1}{|\alpha_i|}(|\alpha_i - 1| + 2b_i + 1) \tag{5}$$

If for all $i = 1, ..., n$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$, $f_i \in \mathcal{A}$, and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1, z \in \mathcal{U} \quad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1, z \in \mathcal{U} \tag{6}$$

then the integral operator $I_{\gamma,\alpha_i}(f_1, ...f_n, g_1, ...g_n)$, defined by (3) is in the class S.

Proof. We define:

$$h(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{1 - \frac{1}{\alpha_i}} (g_i'(t))^{\frac{1}{\alpha_i}} \right] dt,$$

so, that, obviously, we get:

$$h'(z) = \prod_{i=1}^{n} \left[\left(\frac{f_i(z)}{z} \right)^{1-\frac{1}{\alpha_i}} (g_i'(z))^{\frac{1}{\alpha_i}} \right] \tag{7}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left[\left(1 - \frac{1}{\alpha_i} \right) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \frac{1}{\alpha_i} \frac{zg_i''(z)}{g_i'(z)} \right] \tag{8}$$

$$= \sum_{i=1}^{n} \left[\left(1 - \frac{1}{\alpha_i} \right) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \frac{1}{\alpha_i} \left(\frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1 \right) + \frac{1}{\alpha_i} \left(\frac{zg_i'(z)}{g_i(z)} - 1 \right) \right]$$

Since $g_i \in G_{b_i}; 0 < b_i \leq 1$ for all $i = 1, ... n$, from (2) and (6) we obtain:

$$\left| \frac{zh''(z)}{h'(z)} \right|$$

$$\leq \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} \left| \frac{zg_i''(z)}{g_i'(z)} - \frac{zg_i'(z)}{g_i(z)} + 1 \right| + \frac{1}{|\alpha_i|} \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right]$$

$$\leq \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \left| \frac{zg_i'(z)}{g_i(z)} \right| + \frac{1}{|\alpha_i|} \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right]$$

$$\leq \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \left(\left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + 1 \right) + \frac{1}{|\alpha_i|} \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right]$$

$$= \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i + \frac{1}{|\alpha_i|} \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \right]$$

$$= \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} (b_i + 1) \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \right]$$

$$\leq \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} + \frac{1}{|\alpha_i|} (b_i + 1) + \frac{1}{|\alpha_i|} b_i \right]$$

$$\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} (|\alpha_i - 1| + 2b_i + 1), \tag{9}$$

which shows that:

$$\frac{1 - |z|^{2\text{Re}\gamma}}{\text{Re}\gamma} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\text{Re}\gamma}}{\text{Re}\gamma} \left(\sum_{i=1}^{n} \frac{1}{|\alpha_i|} (|\alpha_i - 1| + 2b_i + 1) \right)$$

$$\leq \frac{1}{\text{Re}\gamma} \left(\sum_{i=1}^{n} \frac{1}{|\alpha_i|} (|\alpha_i - 1| + 2b_i + 1) \right) \leq 1$$

Applying Theorem 1 for the function h, we prove that the integral operator $I_{\gamma, \alpha_i}(f_1, ... f_n, g_1, ... g_n)$ is in the class S. $\qquad \square$

Let $\alpha_i = 1$ for all $i = 1, 2, ...n$ in the Theorem 4, we have:

Corollary 1. *Let $\gamma \in \mathbb{C}$ with*

$$\mathrm{Re}\gamma \geq \sum_{i=1}^{n}(2b_i + 1) \tag{10}$$

If for all $i = 1, ..., n$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$ and

$$\left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1, \ z \in \mathcal{U}, \tag{11}$$

then the integral operator

$$I_\gamma(g_1, ..., g_n)(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \prod_{i=1}^{n} (g_i'(t)) \, dt \right\}^{\frac{1}{\gamma}} \tag{12}$$

is in the class S.

Let $n = 1$, $\alpha_1 = \alpha$, $b_1 = b$ and $f_1 = f$, $g_1 = g$ in Theorem 4, we have:

Corollary 2. *Let $\gamma \in \mathbb{C}$ and $\alpha \in \mathbb{C}^*$ with*

$$\mathrm{Re}\gamma \geq \frac{1}{|\alpha|}(|\alpha - 1| + 2b + 1) \tag{13}$$

If $f \in \mathcal{A}$, $g \in G_b$; $0 < b \leq 1$ and

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \ \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1, \ z \in \mathcal{U}, \tag{14}$$

then the integral operator defined by

$$I_{\gamma,\alpha}(f, g)(z) = \left\{ \gamma \int_0^z t^{\gamma-1+\frac{1}{\alpha}} \left(\frac{g'(t)}{f(t)} \right)^{\frac{1}{\alpha}} \frac{f(t)}{t} \, dt \right\}^{\frac{1}{\gamma}} \tag{15}$$

is in the class S.

Using Theorem 2 and Schwarz Lemma, we prove:

Theorem 5. *Let $\alpha_i \in \mathbb{C}^*$, $M_i \geq 1$, $N_i \geq 1$, for all $i = 1, ..., n$ and $\delta \in \mathbb{C}$ with*

$$\mathrm{Re}\delta \geq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left[|\alpha_i - 1|(2M_i + 1) + (b_i + 1)(2N_i + 1) + b_i \right] \tag{16}$$

If for all $i = 1, ..., n$, $f_i \in \mathcal{A}$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$ satisfy

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \ z \in \mathcal{U} \qquad (17)$$

and

$$|f_i(z)| \leq M_i, \ |g_i(z)| \leq N_i, \ z \in \mathcal{U}, \ i = 1, ...n,$$

then for any complex number γ with $\mathrm{Re}\gamma \geq \mathrm{Re}\delta$, the integral operator $I_{\gamma,\alpha_i}(f_1, ...f_n, g_1, ...g_n)$ defined by (3) is in the class S.

Proof. From the proof of Theorem 4, we have:

$$\left| \frac{zh''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|}(b_i + 1) \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \right]$$

$$(18)$$

Thus, we obtain:

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left| \frac{zh''(z)}{h'(z)} \right|$$

$$\leq \frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \frac{1}{|\alpha_i|}(b_i + 1) \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + \frac{1}{|\alpha_i|} b_i \right]$$

$$\leq \frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right) + \right.$$

$$\left. + \frac{1}{|\alpha_i|}(b_i + 1) \left(\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) + \frac{1}{|\alpha_i|} b_i \right]$$

Since $|f_i(z)| \leq M_i$, $|g_i(z)| \leq N_i$, $z \in \mathcal{U}$, $i = 1, ...n$ and each f_i, each g_i satisfy conditions (17) for all $i = 1, ..., n$, then applying general Schwarz Lemma, we have:

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left| \frac{zh''(z)}{h'(z)} \right| \leq$$

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| M_i + M_i + 1 \right) + \right.$$

$$\left. + \frac{1}{|\alpha_i|}(b_i + 1) \left(\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| N_i + N_i + 1 \right) + \frac{1}{|\alpha_i|} b_i \right]$$

$$\leq \frac{1}{\mathrm{Re}\delta} \sum_{i=1}^{n} \left[\frac{|\alpha_i - 1|}{|\alpha_i|}(2M_i + 1) + \frac{1}{|\alpha_i|}(b_i + 1)(2N_i + 1) + \frac{1}{|\alpha_i|} b_i \right]$$

$$\leq \frac{1}{\mathrm{Re}\delta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left[|\alpha_i - 1|(2M_i + 1) + (b_i + 1)(2N_i + 1) + b_i \right], \ z \in \mathcal{U},$$

and from the hypothesis of the Theorem 5, we get:

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, z \in \mathcal{U}$$

Applying Theorem 2 for the function h, we prove that

$$I_{\gamma,\alpha_i}(f_1, ..., f_n, g_1, ...g_n) \in S.$$

$\square$

Let $\alpha_i = 1$ for all $i = 1, ..., n$ in Theorem 5, we have:

Corollary 3. *Let $N_i \geq 1$, for all $i = 1, ..., n$ and $\delta \in \mathbb{C}$ with*

$$\mathrm{Re}\delta \geq \sum_{i=1}^{n} \left[(b_i + 1)(2N_i + 1) + b_i \right] \tag{19}$$

If for all $i = 1, ..., n$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$ satisfy

$$\left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \ z \in \mathcal{U} \tag{20}$$

and $|g_i(z)| \leq N_i$, $z \in \mathcal{U}$, $i = 1, 2, ..., n$, then for any complex number γ with $\mathrm{Re}\gamma \geq \mathrm{Re}\delta$, the integral operator $I_\gamma(g_1, ..., g_n)$ defined by (12) is in the class S.

Let $n = 1$, $\alpha_1 = \alpha$, $b_1 = b$, $M_1 = M$, $N_1 = N$ and $f_1 = f$, $g_1 = g$ in Theorem 5, we have:

Corollary 4. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$ and $\delta \in \mathbb{C}$ with*

$$\mathrm{Re}\delta \geq \frac{1}{|\alpha|} \left[|\alpha - 1|(2M + 1) + (b + 1)(2N + 1) + b \right] \tag{21}$$

If $f \in \mathcal{A}$, $g \in G_b$; $0 < b \leq 1$ satisfy

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1, \ z \in \mathcal{U} \tag{22}$$

and

$$|f(z)| \leq M, |g(z)| \leq N, \ z \in \mathcal{U},$$

then for any complex γ with $\mathrm{Re}\gamma \geq \mathrm{Re}\delta$, the integral operator $I_{\gamma,\alpha}$ defined by (15) is in the class S.

Theorem 6. *Let $\alpha_i \in \mathbb{C}^*$ for all $i = 1, 2, ...n$ and $\gamma \in \mathbb{C}$ with*

$$\operatorname{Re}\gamma \geq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(|\alpha_i - 1| + 2b_i + 1 \right) \tag{23}$$

and let $c \in \mathbb{C}$ be such that:

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\gamma} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(|\alpha_i - 1| + 2b_i + 1 \right) \tag{24}$$

If for all $i = 1, ...n$, $f_i \in \mathcal{A}$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$, and

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1 \ z \in \mathcal{U} \tag{25}$$

then the integral operator $I_{\gamma,\alpha_i}(f_1, ..., f_n, g_1, ...g_n)$ defined by (3) is in the class S.

Proof. From (9) we deduce that

$$\left| c|z|^{2\gamma} + \left(1 - |z|^{2\gamma}\right) \frac{zh''(z)}{\gamma h'(z)} \right| \leq |c| + \left| \frac{1 - |z|^{2\gamma}}{\gamma} \right| \left| \frac{zh''(z)}{h'(z)} \right|$$

$$\leq |c| + \left| \frac{1 - |z|^{2\gamma}}{\gamma} \right| \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(|\alpha_i - 1| + 2b_i + 1 \right)$$

$$\leq |c| + \frac{1}{|\gamma|} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(|\alpha_i - 1| + 2b_i + 1 \right)$$

$$\leq |c| + \frac{1}{\operatorname{Re}\gamma} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(|\alpha_i - 1| + 2b_i + 1 \right) \leq 1$$

Finally, by applying Theorem 3, we get that $I_{\gamma,\alpha_i}(f_1, ..., f_n, g_1, ..., g_n) \in S$ □

Let $\alpha_i = 1$ for all $i = 1, ..., n$ in Theorem 6, we have

Corollary 5. *Let $\gamma \in \mathbb{C}^*$ with*

$$\operatorname{Re}\gamma \geq \sum_{i=1}^{n} (2b_i + 1) \tag{26}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\gamma} \sum_{i=1}^{n} (2b_i + 1) \tag{27}$$

If for all $i = 1, ..., n$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$ and

$$\left| \frac{z g_i'(z)}{g_i(z)} - 1 \right| < 1, \ z \in \mathcal{U} \tag{28}$$

then the integral operator $I_\gamma(g_1, g_2, ..., g_n)$ defined by (12) is in the class S.

Let $n = 1$, $\alpha_1 = \alpha$, $b_1 = b$ and $f_1 = f$, $g_1 = g$ in Theorem 6, then we have:

Corollary 6. *Let $\alpha \in \mathbb{C}^*$ and $\gamma \in \mathbb{C}$ with*

$$\mathrm{Re}\gamma \geq \frac{1}{|\alpha|} \left(|\alpha - 1| + 2\mathrm{b} + 1 \right) \tag{29}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\gamma} \frac{1}{|\alpha|} \left(|\alpha - 1| + 2b + 1 \right) \tag{30}$$

If $f \in \mathcal{A}$, $g \in G_b$; $0 < b \leq 1$ satisfy

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1, \ \left| \frac{z g'(z)}{g(z)} - 1 \right| < 1 \ z \in \mathcal{U} \tag{31}$$

then the integral operator $I_{\gamma, \alpha}(f_1, ..., f_n, g_1, ...g_n)$ defined by (15) is in the class S.

Theorem 7. *Let $\alpha_i \in \mathbb{C}^*$, $M_i \geq 1$, $N_i \geq 1$ for all $i = 1, ..., n$ and $\gamma \in \mathbb{C}$ with*

$$\mathrm{Re}\gamma \geq \sum_{i=1}^n \frac{1}{|\alpha_i|} \left(|\alpha_i - 1|(2\mathrm{M}_i + 1) + (\mathrm{b}_i + 1)(2\mathrm{N}_i + 1) + \mathrm{b}_i \right) \tag{32}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\gamma} \sum_{i=1}^n \frac{1}{|\alpha_i|} \left(|\alpha_i - 1|(2M_i + 1) + (b_i + 1)(2N_i + 1) + b_i \right) \tag{33}$$

If for all $i = 1, ..., n$, $f_i \in \mathcal{A}$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$ satisfy

$$\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| < 1, \ \left| \frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1 \right| < 1, \ z \in \mathcal{U} \tag{34}$$

then the integral operator $I_{\gamma, \alpha_i}(f_1, ..., f_n, g_1, ..., g_n)$ defined by (3) is in the class S.

Proof. From the proof of the Theorem 5 we have

$$\left| c \left| z \right|^{2\gamma} + \left(1 - \left| z \right|^{2\gamma} \right) \frac{zh''(z)}{\gamma h'(z)} \right| \leq$$

$$\left| c \right| + \frac{1}{\text{Re}\gamma} \sum_{i=1}^{n} \frac{1}{\left| \alpha_i \right|} \left(\left| \alpha_i - 1 \right| (2M_i + 1) + (b_i + 1)(2N_i + 1) + b_i \right), z \in \mathcal{U}$$

which, from the hypothesis of the Theorem 7, we get:

$$\left| c \left| z \right|^{2\gamma} + \left(1 - \left| z \right|^{2\gamma} \right) \frac{zh''(z)}{\gamma h'(z)} \right| \leq 1$$

Applying Theorem 6, for the function h, we prove that

$$I_{\gamma, \alpha_i}(f_1, ..., f_n, g_1, ..., g_n) \in S.$$

$\square$

Let $\alpha_i = 1$ for all $i = 1, ..., n$ in Theorem 4, we have:

Corollary 7. *Let $N_i \geq 1$ for all $i = 1, ..., n$ and $\gamma \in \mathbb{C}$ with*

$$\text{Re}\gamma \geq \sum_{i=1}^{n} \left[(b_i + 1)(2N_i + 1) + b_i \right] \tag{35}$$

and let $c \in \mathbb{C}$ be such that

$$\left| c \right| \leq 1 - \frac{1}{\left| \text{Re}\gamma \right|} \sum_{i=1}^{n} \left[(b_i + 1)(2N_i + 1) + b_i \right] \tag{36}$$

If for all $i = 1, ..., n$, $g_i \in G_{b_i}$; $0 < b_i \leq 1$, satisfy

$$\left| \frac{z^2 g_i'(z)}{\left[g_i(z) \right]^2} - 1 \right| < 1, \ z \in \mathcal{U}, \tag{37}$$

then the integral operator $I_\gamma(g_1, ..., g_n)$ defined by(12) is in the class S.

Let $n = 1$, $\alpha_1 = \alpha$, $b_1 = b$, $M_1 = M$ and $f_1 = f$, $g_1 = g$ in Theorem 7, then we have

Corollary 8. *Let $\alpha \in \mathbb{C}^*$, $M \geq 1$, $N \geq 1$ and $\gamma \in \mathbb{C}$ with*

$$\text{Re}\gamma \geq \frac{1}{\left| \alpha \right|} \left[\left| \alpha - 1 \right| (2M + 1) + (b + 1)(2N + 1) + b \right] \tag{38}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\gamma} \left[|\alpha - 1|(2M+1) + (b+1)(2N+1) + b \right] \qquad (39)$$

If $f \in \mathcal{A}$, $g \in G_b$;$0 < b \leq 1$ satisfy:

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1, \quad \left| \frac{z^2 g'(z)}{[g(z)]^2} - 1 \right| < 1 \ z \in \mathcal{U}, \qquad (40)$$

then the integral operator $I_{\gamma,\alpha}(f,g)$ defined by (15) is in the class S.

Remark 1. *If $\beta_i = \frac{1}{\alpha_i}$ and $g_i = f_i$, for all $i = 1, ..., n$, in relation (3), we get the integral operator*

$$I_\gamma^{\beta_i}(f_1, ...f_n, g_1, ...g_n)(z) =$$

$$\left\{ \gamma \int_0^z t^{\gamma-1} \left(\frac{f_1(t)}{t} \right)^{1-\beta_1} (f_1'(t))^{\beta_1} .... \left(\frac{f_n(t)}{t} \right)^{1-\beta_n} (f_n'(t))^{\beta_n} dt \right\}^{\frac{1}{\gamma}},$$

studied by B. A. Frasin in [3].

Remark 2. *For $\beta_i = 1 - \frac{1}{\alpha_i}$ and $g_i'(t) = 1$ for all $i = 1, ..., n$, in relation (3), we get the integral operator $G(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt \right\}^{\frac{1}{\gamma}}$, $z \in \mathcal{U}$ studied by D. Breaz and N. Breaz in [1].*

Remark 3. *If instead of $1 - \frac{1}{\alpha_i}$ we consider α_i, $g_i'(t) = 1$, and if instead of $f_i(t)$ we have $\mathcal{D}_\lambda^{n,\gamma} f_i(t)$ for all $i = 1, ..., n$ in relation (3), we get the integral operator*

$$I_\beta^{n,\gamma}(f_1, ..., f_m) : \mathcal{A}^m \to \mathcal{A},$$

$$I_\beta^{n,\gamma}(f_1, f_2, ..., f_m)(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^m \left(\frac{\mathcal{D}_\lambda^{n,\gamma} f_i(t)}{t} \right)^{\alpha_i} dt \right\}^{\frac{1}{\beta}}, \ z \in \mathcal{U},$$

where $\mathcal{D}_\lambda^{n,\gamma}$ is the generalized Al-Oboudi differenatial operator, defined and studied by S. Bulut in [2].

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