Determinant Identities and the Geometry of Lines and Circles

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Abstract

The focus of this note is the nontrivial determinant identities which typically underlie the complex analytic proofs of all the results in the plane geometry of lines and circles. After setting up a basic dictionary relating lines and circles to complex determinants we derive such identities in connection with four geometry problems: the Steiner line, a variant of Euler's nine-point circle, the Johnson-Tzitzeica circles, and an extension of a certain geometry problem, proposed at the 52nd International Mathematical Olympiad, Amsterdam 2011.

1. Introduction

When it comes to solving any geometry problem there is no substitute for the elegance and concision of a purely geometric argument. Yet, there are situations when the complexity of a problem dictates that an analytic approach is more desirable. Even regardless such complexity, almost always analytic methods can prove to be more revealing that initially intended. One only has to be open to introspection.

The purpose of this note is to reinforce such an analytic approach when dealing with geometric problems about lines and circles in a Euclidean plane. Certainly, there is a huge inventory of fascinating problems of this type in the

multi-millennial history of classical Euclidean geometry. Our starting point is the elementary observation that any geometric information involving lines and circles can be conveniently packaged in analytic information via determinant theory. Then, most of the time solving such a problem merely requires concluding that certain determinants vanish, given the vanishing of others. The novel observation here is that all these determinants are typically related by some nontrivial identity, which is valid in a more general set-up than the one required in the actual problem. So, while the solution to the problem appears naturally as a specialization of variables in an appropriate determinant identity, far more valuable seems to be the underlying identity.

It turns out that this plan works great if, adapted to the problem, a coordinate system in the plane is chosen so that key quantities can be circularly permuted. Also, it is better to work with complex numbers, rather than real ones. After a brief approach to lines and circles in a plane via determinant theory we will exemplify the method on four problems: the Steiner line, a variant of Euler's nine-point circle, the Johnson-Tzitzeica circles, and a certain geometry problem, presented at the 52nd International Mathematical Olympiad, Amsterdam 2011. Interestingly enough, this last problem was solved during the contest by only nine students [3], another reason to give more credit to analytic geometry in general.

It is worth pointing out that several analytic approaches could be considered, based of the type of representation of lines and circles preferred, such as those using homogeneous coordinates, barycentric or trilinear. It is only a matter of personal preference that we use 'the complex-number method'.

2. Lines and Circles via Determinants

Let E be a Euclidean plane. After choosing a coordinate system, E will be identified with $\mathbf{R}^2$, and then with $\mathbf{C}$, the complex number field. In applications, the passage from E to $\mathbf{C}$ will be specific to the problem at hand.

If a_1, a_2 are two distinct complex numbers, elementary properties of determinants show that the line through a_1 and a_2, $l(a_1, a_2)$, consists in all the complex numbers z such that

$$\det \begin{bmatrix} z & \overline{z} & 1 \\ a_1 & \overline{a}_1 & 1 \\ a_2 & \overline{a}_2 & 1 \end{bmatrix} = 0. \tag{1}$$

Consequently, three points a_1, a_2, a_3, are non-collinear if and only if (1) does not hold, when $z = a_3$.

Similarly, given three non-collinear points a_1, a_2, and a_3, the unique circle

determined by them, $c(a_1, a_2, a_3)$, has equation

$$\det \begin{bmatrix} z\bar{z} & z & \bar{z} & 1 \\ a_1\bar{a}_1 & a_1 & \bar{a}_1 & 1 \\ a_2\bar{a}_2 & a_2 & \bar{a}_2 & 1 \\ a_3\bar{a}_3 & a_3 & \bar{a}_3 & 1 \end{bmatrix} = 0. \tag{2}$$

Obviously, when a_1, a_2, a_3 are viewed as the vertices of a triangle, $c(a_1, a_2, a_3)$ is its associated circumcircle.

Properties of determinants show that the circle $c(a_1, a_2, a_3)$ has center

$$\alpha = -\frac{\det \begin{bmatrix} a_1\bar{a}_1 & a_1 & 1 \\ a_2\bar{a}_2 & a_2 & 1 \\ a_3\bar{a}_3 & a_3 & 1 \end{bmatrix}}{\det \begin{bmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{bmatrix}} \quad \text{and radius} \quad r = \frac{\left|\det \begin{bmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{bmatrix}\right|}{\left|\det \begin{bmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{bmatrix}\right|}. \tag{3}$$

More precisely, the two formulas (3) can be obtained by formally identifying the (first row) expansion of the determinant in (2) with the appropriately scaled version of the center-radius equation of a circle, given by

$$\left((z - \alpha)(\bar{z} - \bar{\alpha}) - r^2\right) \det \begin{bmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{bmatrix} = 0.$$

Notice also that the numerator in the formula of r is the absolute value of a Vandermonde determinant, and equals $|a_1 - a_2||a_2 - a_3||a_3 - a_1|$.

The expression of r itself is a manifestation of the principle we try to unveil in this note. Indeed, it can be obtained as a specialization of the following identity of determinants:

$$\begin{aligned} \det \begin{bmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{bmatrix} \det \begin{bmatrix} \zeta_1^2 & \zeta_1 & 1 \\ \zeta_2^2 & \zeta_2 & 1 \\ \zeta_3^2 & \zeta_3 & 1 \end{bmatrix} + \\ \det \begin{bmatrix} z_1\zeta_1 & z_1 & \zeta_1 \\ z_2\zeta_2 & z_2 & \zeta_2 \\ z_3\zeta_3 & z_3 & \zeta_3 \end{bmatrix} \det \begin{bmatrix} z_1 & \zeta_1 & 1 \\ z_2 & \zeta_2 & 1 \\ z_3 & \zeta_3 & 1 \end{bmatrix} = \\ \det \begin{bmatrix} z_1\zeta_1 & z_1 & 1 \\ z_2\zeta_2 & z_2 & 1 \\ z_3\zeta_3 & z_3 & 1 \end{bmatrix} \det \begin{bmatrix} z_1\zeta_1 & \zeta_1 & 1 \\ z_2\zeta_2 & \zeta_2 & 1 \\ z_3y_3 & \zeta_3 & 1 \end{bmatrix}. \end{aligned} \tag{4}$$

In (4), z_i, ζ_i, $i = 1, 2, 3$, have the meaning of independent complex variables, and then the expression for r is easily obtained by setting $z_i = a_i$ and $\zeta_i = \bar{a}_i$ in it.

The main reason to work with $\mathbf{C}$ instead of $\mathbf{R}^2$ is the powerful interplay between lines and circles and the group of Möbius transformations on the Riemann sphere, $\mathbf{C}_\infty = \mathbf{C} \cup \infty$,

$$\mathbf{C}_\infty \ni z \longmapsto \frac{az+b}{cz+d} \in \mathbf{C}_\infty, \quad a,b,c,d \in \mathbf{C}, \ ad - bc \neq 0.$$

It is well-known [1] that lines and circles in $\mathbf{C}$, when conveniently identified with 'circles in $\mathbf{C}_\infty$', are preserved by the Möbius transformations.

This is just one of the many useful things relating geometry and Möbius transformations. Recall, for instance, that the unique Möbius transformation sending three distinct complex numbers a_1, a_2 and a_3 to $1, 0$ and ∞, respectively, denoted $z \longmapsto (z, a_1, a_2, a_3)$, the cross-ratio of z, a_1, a_2 and a_3, has the property that when $a_1, a_2,$ and a_3 are collinear, then z and z^* are symmetrical with respect to $l(a_1, a_2, a_3)$ if and only if

$$(z^*, a_1, a_2, a_3) = \overline{(z, a_1, a_2, a_3)}. \tag{5}$$

3. The Steiner Line

Steiner. *In the plane of a triangle $A_1 A_2 A_3$ consider an arbitrary point M. Let M_1, M_2, and M_3 be the points symmetrical to M, relative to the sides $A_2 A_3$, $A_3 A_1$, and $A_1 A_2$, respectively. Then the points M_1, M_2, and M_3, are collinear if and only if M belongs to the circumcircle of triangle $A_1 A_2 A_3$ (Figure 1).*
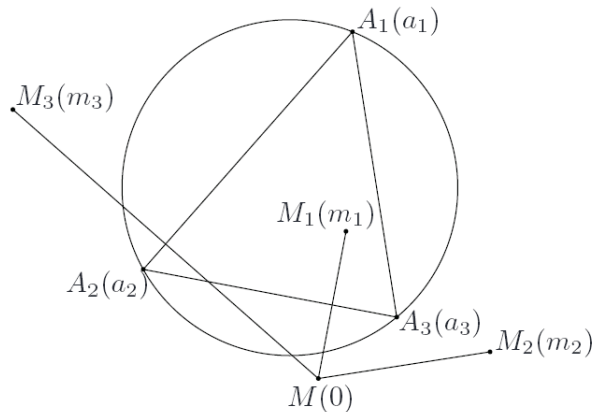

Figure 1: A Steiner (non)-line.

Choose a coordinate system with origin in M. Then M has (complex) coordinate 0, and A_1, A_2, and A_3, have coordinates, say, a_1, a_2, and a_3,

respectively. As coordinates of the vertices of a triangle they are mutually distinct. Also, if m_1 is the coordinate of M_1, equation (5) shows that

$$(m_1, a_2, a_3, \infty) = \overline{(0, a_2, a_3, \infty)}, \text{ which gives } m_1 = \frac{a_3\bar{a}_2 - a_2\bar{a}_3}{\bar{a}_2 - \bar{a}_3},$$

with similar expressions for $M_2(m_2)$ and $M_3(m_3)$. By (1) and (2) it suffices to show that

$$\det \begin{bmatrix} m_1 & \bar{m}_1 & 1 \\ m_2 & \bar{m}_2 & 1 \\ m_3 & \bar{m}_3 & 1 \end{bmatrix} = 0 \quad \text{if and only if} \quad \det \begin{bmatrix} a_1\bar{a}_1 & a_1 & \bar{a}_1 \\ a_2\bar{a}_2 & a_2 & \bar{a}_2 \\ a_3\bar{a}_3 & a_3 & \bar{a}_3 \end{bmatrix} = 0. \quad (6)$$

Now let us formally replace a_i by z_i, and $\bar{a}_i$ by ζ_i, $i = 1, 2, 3$, in the second determinant appearing in (6). As before, for each i, z_i and ζ_i should be interpreted as independent complex variables. In addition, assume the z_i's are pairwise distinct, and so are the ζ_i's. These replacements suggest, at the level of m_i's, the following substitutions:

$$m_1 \longrightarrow \alpha_1 := \frac{z_3\zeta_2 - z_2\zeta_3}{\zeta_2 - \zeta_3}, \quad \text{and} \quad \bar{m}_1 \longrightarrow \beta_1 := \frac{z_2\zeta_3 - z_3\zeta_2}{z_2 - z_3}, \quad \text{etc.}$$

The claim follows now from the following determinant identity:

$$\det \begin{bmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{bmatrix} \det \begin{bmatrix} \zeta_1^2 & \zeta_1 & 1 \\ \zeta_2^2 & \zeta_2 & 1 \\ \zeta_3^2 & \zeta_3 & 1 \end{bmatrix} \det \begin{bmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{bmatrix} +$$

$$\det \begin{bmatrix} z_1\zeta_1 & z_1 & \zeta_1 \\ z_2\zeta_2 & z_2 & \zeta_2 \\ z_3\zeta_3 & z_3 & \zeta_3 \end{bmatrix} \left(\det \begin{bmatrix} z_1 & \zeta_1 & 1 \\ z_2 & \zeta_2 & 1 \\ z_3 & \zeta_3 & 1 \end{bmatrix} \right)^2 = 0.$$

4. A Variant of Euler's Nine-Point Circle

Euler. *Let H be the orthocenter of a triangle $A_1A_2A_3$. Let P_1, P_2, and P_3 be the points symmetrical to H, relative to the sides A_2A_3, A_3A_1, and A_1A_2, respectively. Similarly, let Q_1, Q_2, and Q_3 be symmetrical to H relative to the mid-points of those sides. Then the nine points A_1, A_2, A_3, P_1, P_2, P_3, Q_1, Q_2, and Q_3, are concyclic (Figure 2).*

Choose a coordinate system with origin in H, so H has coordinate 0. Assign then to A_1, A_2, and A_3, coordinates a_1, a_2, and a_3, respectively. Since H is the orthocenter of the triangle $A_1A_2A_3$ we have $HA_1 \perp A_2A_3$, $HA_2 \perp A_3A_1$, and $HA_3 \perp A_1A_2$, which analytically mean, respectively

$$\det \begin{bmatrix} a_1 & \bar{a}_1 & 0 \\ -a_2 & \bar{a}_2 & 1 \\ -a_3 & \bar{a}_3 & 1 \end{bmatrix} = 0, \det \begin{bmatrix} a_2 & \bar{a}_2 & 0 \\ -a_3 & \bar{a}_3 & 1 \\ -a_1 & \bar{a}_1 & 1 \end{bmatrix} = 0, \det \begin{bmatrix} a_3 & \bar{a}_3 & 0 \\ -a_1 & \bar{a}_1 & 1 \\ -a_2 & \bar{a}_2 & 1 \end{bmatrix} = 0. \quad (7)$$
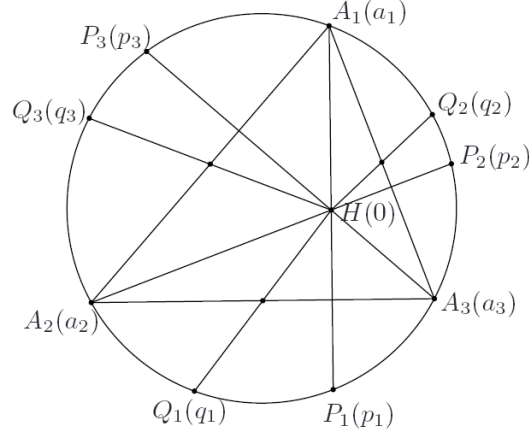
Figure 2: A variant of Euler's nine-point circle.

Evidently, the last two determinant equations in (7) could have been obtained from the first one by circular permutations.

Now the coordinate of Q_1 is clearly $q_1 := a_2 + a_3$, and by circular permutations, we have similar expressions for the coordinates q_2 and q_3 of Q_2 and Q_3. The coordinate p_1 of P_1, is as before

$$p_1 = \frac{a_3\bar{a}_2 - a_2\bar{a}_3}{\bar{a}_2 - \bar{a}_3},$$

with similar expressions for $P_2(p_2)$ and $P_3(p_3)$.

By (2), it suffices to show that in the presence of (7), q_1, q_2 and q_3, on one hand, and p_1, p_2 and p_3, on the other hand, belong to the circle $c(a_1, a_2, a_3)$. We will only show that q_1 and p_1 belong to that circle, with all the others following similarly.

Making the now familiar replacements $a_i \longrightarrow z_i$, and $\bar{a}_i \longrightarrow \zeta_i$, $i = 1, 2, 3$, the determinants appearing in the three equations (7), become

$$\det \begin{bmatrix} z_1 & \zeta_1 & 0 \\ -z_2 & \zeta_2 & 1 \\ -z_3 & \zeta_3 & 1 \end{bmatrix} := \xi_1, \quad \det \begin{bmatrix} z_2 & \zeta_2 & 0 \\ -z_3 & \zeta_3 & 1 \\ -z_1 & \zeta_1 & 1 \end{bmatrix} := \xi_2, \quad \det \begin{bmatrix} z_3 & \zeta_3 & 0 \\ -z_1 & \zeta_1 & 1 \\ -z_2 & \zeta_2 & 1 \end{bmatrix} := \xi_3$$

Our claim will follow from the following two identities, the first one applying to q_1, and the second to p_1:

$$\det \begin{bmatrix} (z_2 + z_3)(\zeta_2 + \zeta_3) & z_2 + z_3 & \zeta_2 + \zeta_3 & 1 \\ z_1\zeta_1 & z_1 & \zeta_1 & 1 \\ z_2\zeta_2 & z_2 & \zeta_2 & 1 \\ z_3\zeta_3 & z_3 & \zeta_3 & 1 \end{bmatrix} = \frac{1}{2} \det \begin{bmatrix} z_1 & \zeta_1 & \xi_2 - \xi_3 \\ z_2 & \zeta_2 & \xi_2 \\ z_3 & \zeta_3 & -\xi_3 \end{bmatrix},$$

$$\det \begin{bmatrix} -\dfrac{(z_3\zeta_2 - z_2\zeta_3)^2}{(z_2 - z_3)(\zeta_2 - \zeta_3)} & \dfrac{z_3\zeta_2 - z_2\zeta_3}{\zeta_2 - \zeta_3} & -\dfrac{z_3\zeta_2 - z_2\zeta_3}{z_2 - z_3} & 1 \\ z_1\zeta_1 & z_1 & \zeta_1 & 1 \\ z_2\zeta_2 & z_2 & \zeta_2 & 1 \\ z_3\zeta_3 & z_3 & \zeta_3 & 1 \end{bmatrix} =$$

$$\frac{1}{2}\det \begin{bmatrix} z_1 & \zeta_1 & \xi_2 - \xi_3 \\ z_2 & \zeta_2 & \xi_2 \\ z_3 & \zeta_3 & -\xi_3 \end{bmatrix}.$$

It is somewhat surprising that the two left-hand-side determinants in the identities above are equal.

5. The Johnson-Tzitzeica Circles

Johnson-Tzitzeica. *Three circles of equal radii r intersect at a point I, and let A_1, A_2, and A_3 be the other three points where they intersect pairwise. Show that the circumcircle of triangle $A_1A_2A_3$ has the same radius r [2, 4] (Figure 3).*
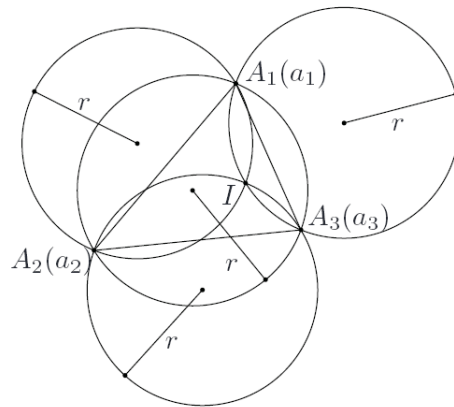

Figure 3: The Johnson-Tzitzeica circles.

With the coordinate system centered in I, let A_i have coordinate a_i, $i = 1, 2, 3$. Any one of the three circles appears as circumcircle to the triangle with vertices two of the points A_i, and I. By (3), the circumcircle of triangle

$A_1 A_2 I$ has radius ρ_3 such that

$$\rho_3^2 = \dfrac{\det \begin{bmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \det \begin{bmatrix} \bar{a}_1^2 & \bar{a}_1 & 1 \\ \bar{a}_2^2 & \bar{a}_2 & 1 \\ 0 & 0 & 1 \end{bmatrix}}{-\left(\det \begin{bmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)^2} = -\dfrac{a_1 a_2 (a_1 - a_2)\bar{a}_1 \bar{a}_2 (\bar{a}_1 - \bar{a}_2)}{(a_1 \bar{a}_2 - a_2 \bar{a}_1)^2},$$

with similar expressions for ρ_1^2 and ρ_2^2.

Also, the circumcircle of triangle $A_1 A_2 A_3$ has radius ρ, such that

$$\rho^2 = \dfrac{\det \begin{bmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{bmatrix} \det \begin{bmatrix} \bar{a}_1^2 & \bar{a}_1 & 1 \\ \bar{a}_2^2 & \bar{a}_2 & 1 \\ \bar{a}_3^2 & \bar{a}_3 & 1 \end{bmatrix}}{-\left(\det \begin{bmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{bmatrix}\right)^2}.$$

Making the usual replacements z_i and ζ_i for a_i and $\bar{a}_i$, $i = 1, 2, 3$, denote now by ξ_i, η_i, θ_i, ξ, η, and θ the following determinants:

$$\xi_3 := \det \begin{bmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \eta_3 := \det \begin{bmatrix} \zeta_1^2 & \zeta_1 & 1 \\ \zeta_2^2 & \zeta_2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \theta_3 := \det \begin{bmatrix} z_1 & \zeta_1 & 1 \\ z_2 & \zeta_2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{etc.}$$

and $\xi := \det \begin{bmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{bmatrix}$, $\eta := \det \begin{bmatrix} \zeta_1^2 & \zeta_1 & 1 \\ \zeta_2^2 & \zeta_2 & 1 \\ \zeta_3^2 & \zeta_3 & 1 \end{bmatrix}$, $\theta := \det \begin{bmatrix} z_1 & \zeta_1 & 1 \\ z_2 & \zeta_2 & 1 \\ z_3 & \zeta_3 & 1 \end{bmatrix}$

Trivially,

$$\xi = \xi_1 + \xi_2 + \xi_3, \quad \eta = \eta_1 + \eta_2 + \eta_3, \quad \text{and} \quad \theta = \theta_1 + \theta_2 + \theta_3. \qquad (8)$$

From (8), the Johnson-Tzitzeica statement follows as a specialization of either one of the determinant identities

$$\frac{\theta_1^2}{\xi_1} + \frac{\theta_2^2}{\xi_2} + \frac{\theta_3^2}{\xi_3} = \frac{\theta^2}{\xi} \quad \text{or} \quad \frac{\theta_1^2}{\eta_1} + \frac{\theta_2^2}{\eta_2} + \frac{\theta_3^2}{\eta_3} = \frac{\theta^2}{\eta}, \qquad (9)$$

in the presence of the hypothesis of equal radii, $\rho_1 = \rho_2 = \rho_3 = r$. In (9) obvious restrictions should be imposed on the complex variables z_i, ζ_i, $i = 1, 2, 3$, such that the denominators involved do not vanish.

6. An Extension of an IMO Problem

IMO 2011. *Let $A_1A_2A_3$ be a triangle with center of gravity G and with circumcircle Γ of circumcenter O and circumradius r. Let l be a line tangent to Γ at a point T, and let d be the line parallel to l and at distance $r/3$ from O. Finally, let l_1, l_2, and l_3 be the lines obtained by reflecting l in the lines A_2A_3, A_3A_1, and A_1A_2, respectively (Figure 4).*

a) If the triangle $A_1A_2A_3$ is acute, then the lines l_1, l_2, and l_3 determine a triangle whose circumcircle Γ_l is such that Γ is tangent interior to Γ_l.

b) If the triangle $A_1A_2A_3$ is right with, say, right angle at vertex A_1, then l_2 and l_3 are parallel, and l_1 is tangent to Γ at the point symmetric to T with respect to A_2A_3.

c) If the triangle $A_1A_2A_3$ is obtuse then Γ and Γ_l are tangent exterior if O and G are on the same side of d, and Γ_l is tangent interior to Γ, if O and G are on opposite sides of d (Figure 4). When G is on d then Γ_l degenerates to a point on Γ.
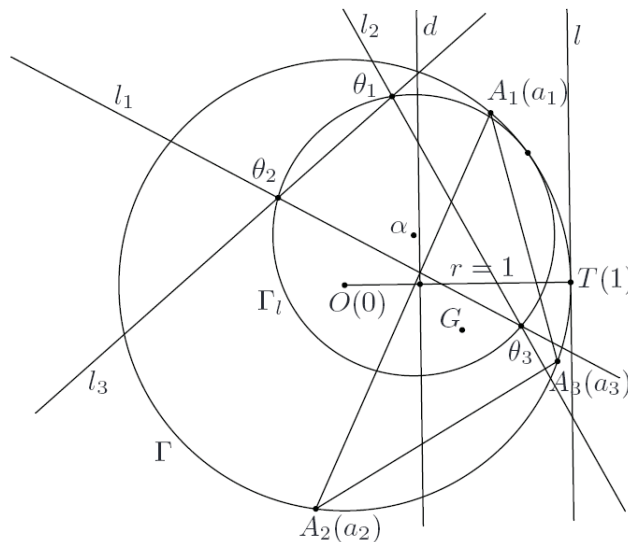

Figure 4: The IMO problem, case *c*), interior tangency.

The case *a*) was presented at the International Mathematical Olympiad, Amsterdam 2011, as Problem 6. The main difficulty in solving it synthetically consists in understanding the geometry of the two-circle point of tangency. This difficulty disappears when seeking an analytic solution, given the simple characterization of tangent circles in terms of their radii and the distance between their centers.

We can choose a coordinate system such that O has coordinate 0, distances

are scaled so $r = 1$, and T has coordinate 1. Then Γ has equation $z\overline{z} = 1$, l has equation $z + \overline{z} = 2$, and A_i, $i = 1, 2, 3$, will have coordinates some distinct complex numbers a_i of modulus 1, respectively.

In all the calculations that follow we will make crucial use of the fact that $\overline{a}_i = \dfrac{1}{a_i}$, $i = 1, 2, 3$. Using the formula (5) we see that l_3 has equation

$$z + a_1^2 a_2^2 \overline{z} - (a_1 + a_2)(1 + a_1 a_2) + 2a_1 a_2 = 0, \tag{10}$$

and similar equations hold for l_1 and l_2.

Consequently, say, the lines l_2 and l_3 intersect if and only if $a_2 + a_3 \neq 0$. If $a_2 + a_3 = 0$ then l_2 and l_3 are parallel lines, and A_2 and A_3 are diametrically opposite on Γ, i.e., the triangle $A_1 A_2 A_3$ is right, with the right angle at vertex A_1. Then l_1 has equation $z + a_2^4 \overline{z} - 2a_2^2 = 0$, which is tangent to Γ, at the point with coordinate a_2^2. A moment thought shows that this point is the reflection of T in $A_2 A_3$. This proves b).

Assume further that any two of the three lines l_1, l_2, and l_3 intersect, i.e., $a_1 + a_2 \neq 0$, $a_2 + a_3 \neq 0$, and $a_3 + a_1 \neq 0$. Then the lines l_2 and l_3 intersect at a point whose coordinate is

$$\theta_1 := \frac{(a_1 - 2)a_1 a_2 a_3 + (a_1 a_2 + a_2 a_3 + a_3 a_1)}{a_2 + a_3}. \tag{11}$$

There are similar expressions for the coordinates of the other two intersection points, θ_2, and θ_3. By equation (1) we see that

$$\det \begin{bmatrix} \theta_1 & \overline{\theta}_1 & 1 \\ \theta_2 & \overline{\theta}_2 & 1 \\ \theta_3 & \overline{\theta}_3 & 1 \end{bmatrix} =$$

$$\frac{(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)}{(a_1 + a_2)(a_2 + a_3)(a_3 + a_1)} \left(a_1 + a_2 + a_3 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 2 \right),$$

and so the points θ_1, θ_2, and θ_3 are non-collinear, and define indeed the circle $c(\theta_1, \theta_2, \theta_3) = \Gamma_l$, if and only if $a_1 + a_2 + a_3 + \dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} - 2 \neq 0$, which is equivalent with the center of gravity G of the triangle $A_1 A_2 A_3$ not falling on d. If G belongs to d the circle degenerates to a point, since $\theta_1 = \theta_2 = \theta_3 = -a_1 a_2 a_3$. By (3), the circle Γ_l has then center α and radius ρ such that

$$\alpha = -\frac{\det \begin{bmatrix} \theta_1 \overline{\theta}_1 & \theta_1 & 1 \\ \theta_2 \overline{\theta}_2 & \theta_2 & 1 \\ \theta_3 \overline{\theta}_3 & \theta_3 & 1 \end{bmatrix}}{\det \begin{bmatrix} \theta_1 & \overline{\theta}_1 & 1 \\ \theta_2 & \overline{\theta}_2 & 1 \\ \theta_3 & \overline{\theta}_3 & 1 \end{bmatrix}} = \frac{(a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3)^2}{(a_1 + a_2)(a_2 + a_3)(a_3 + a_1)}, \tag{12}$$

$$
\rho^2 = \frac{\det \begin{bmatrix} \theta_1^2 & \theta_1 & 1 \\ \theta_2^2 & \theta_2 & 1 \\ \theta_3^2 & \theta_3 & 1 \end{bmatrix} \det \begin{bmatrix} \bar{\theta}_1^2 & \bar{\theta}_1 & 1 \\ \bar{\theta}_2^2 & \bar{\theta}_2 & 1 \\ \bar{\theta}_3^2 & \bar{\theta}_3 & 1 \end{bmatrix}}{-\left(\det \begin{bmatrix} \theta_1 & \bar{\theta}_1 & 1 \\ \theta_2 & \bar{\theta}_2 & 1 \\ \theta_3 & \bar{\theta}_3 & 1 \end{bmatrix} \right)^2}
$$

and (13)

$$
= \left(\frac{(a_1 + a_2 + a_3 - 2)a_1 a_2 a_3 + (a_1 a_2 + a_2 a_3 + a_3 a_1)}{(a_1 + a_2)(a_2 + a_3)(a_3 + a_1)} \right)^2.
$$

Noticing now that the quantities

$$
\sigma := a_1 + a_2 + a_3 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 2 \quad \text{and} \quad \pi := \left(1 + \frac{a_1}{a_2}\right)\left(1 + \frac{a_2}{a_3}\right)\left(1 + \frac{a_3}{a_1}\right)
$$

are real, $\pi \neq 0$, while the quantity

$$
\epsilon := (a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3)\overline{(a_1 a_2 + a_2 a_3 + a_3 a_1 - a_1 a_2 a_3)}
$$

is real, non-negative, from (12) and (13) we conclude that

$$
\alpha \bar{\alpha} = \left(\frac{\epsilon}{\pi}\right)^2, \quad \rho^2 = \left(\frac{\sigma}{\pi}\right)^2, \quad \text{and most importantly} \quad \frac{\epsilon}{\pi} + \frac{\sigma}{\pi} = 1. \qquad (14)
$$

The first two equations (14) show that the circle Γ_l has radius $\rho = \dfrac{|\sigma|}{|\pi|}$, while the distance between the centers of the circles Γ and Γ_l is $|\alpha| = \dfrac{\epsilon}{|\pi|}$.

The last of the equations (14) holds the key to the solution of our problem in cases $a)$ and $c)$ as soon as the geometric meaning of the signs of the the real quantities σ and π is decoded.

Looking at its expression, it is clear that σ is positive when O and G are on opposite sides of d, negative when they are on the same side, and vanishes when G belongs to d.

The interpretation of the sign of π is subtler: First of all, the triangle $A_1 A_2 A_3$ has an acute, respectively obtuse, angle at, say, vertex A_1 if and only if $\omega_1 := (a_2 - a_1)(\bar{a}_3 - \bar{a}_1) + (a_3 - a_1)(\bar{a}_2 - \bar{a}_1) > 0$, respectively < 0. If follows that the triangle itself is acute or obtuse depending on the sign of the product $\omega_1 \omega_2 \omega_3$. An easy calculation shows that

$$
\omega_1 \omega_2 \omega_3 = -\pi \omega \bar{\omega},
$$

where

$$
\omega := i\left(1 - \frac{a_1}{a_2}\right)\left(1 - \frac{a_2}{a_3}\right)\left(1 - \frac{a_3}{a_1}\right) \neq 0.
$$

As a result, triangle $A_1 A_2 A_3$ is acute if $\pi < 0$ and obtuse if $\pi > 0$.

Now, the conclusion in cases $a)$ and $c)$ follows easily from the facts above, since two circles are tangent interior/exterior if and only if the distance between their centers equals the difference/sum of their radii.

7. Conclusions

The plane geometry of lines and circles translates neatly into determinant identities, via complex analytic geometry. The readers are encouraged to run their favorite geometry problems through this machinery.

In the process an obvious question will emerge: How to find the appropriate determinant identities? Clearly, the main clues are contained in the actual statement of a given problem. For the most direct approach, if many defining elements in the problem are points on a given circle then the coordinate system should be centered at the center of the circle, and the unit distance should be taken to be its radius, like in the IMO problem. This choice introduces the least number of variables. Indeed, since lines and circles as defined by equations (1) and (2) contain complex variables and their conjugates, when a quantity belongs to the unit circle, its conjugate, being equal to its reciprocal, does not require a new variable. Also, it is important not to make coordinate choices which interfere with the symmetry and cyclicity of par quantities.

Beyond that, the sought after identities must relate the 'hypothesis determinants' to the 'conclusion determinants'. Finding them is pretty much a trial and error exercise, greatly helped by reader intuition and creativity. Usually, the 'correct' identities afford elegant verifications, via determinant properties. However, it pays to know that a computer algebra system capable of carrying out symbolic calculations, like Maple or Mathematica, can help test, verify, or even discover, potential identities.

References

[1]   J. Conway, *Functions of One Complex Variable, 2nd Edition*, Springer, New York, (1978).

[2]   R. Johnson, *A Circle Theorem*, Amer. Math. Monthly, **23**, No. 5, 161-162, (1916).

[3]   R. Todor, *A Computational Solution of Question* 6 *from IMO 2011*, Gaz. Mat. B, **CXVII**, No. 1, 1-7, (2012).

[4]   G. Tzitzeica, *Problems in Geometry, 6th Edition* (*Romanian*), Editura Tehnică, Bucharest, (1961).

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